Junior Problems

J499. Let a, b, c, d be positive real numbers such that

$$a(a-1)^2 + b(b-1)^2 + c(c-1)^2 + d(d-1)^2 = a + b + c + d.$$

Prove that

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (d-1)^2 \leq 4.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J500. Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that

$$\frac{1}{5a^2 - 2a + 1} + \frac{1}{5b^2 - 2b + 1} + \frac{1}{5c^2 - 2c + 1} + \frac{1}{5d^2 - 2d + 1} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

J501. In a convex quadrilateral $ABCD$, M and N are the midpoints of diagonals AC and BD, respectively. The intersection of the diagonals lies on segments CM and DN, while points P and Q lie on segment AB and satisfy

$$\angle PMN = \angle BCD \quad \text{and} \quad \angle QNM = \angle ADC\,.$$

Prove that lines PM and QN meet at a point lying in line CD.

Proposed by Waldemar Pompe, Warsaw, Poland

J502. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{c(a^2 + bc)} + \frac{b^3}{a(b^2 + ca)} + \frac{c^3}{b(c^2 + ab)} \geq \frac{3}{2}$$

Proposed by Konstantinos Metaxas, Athens, Greece

J503. Solve in positive integers the equation

$$\min(x^4 + 8y, 8x + y^4) = (x + y)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J504. Let ABC be a triangle with circumcenter O, E be an arbitrary point on AC and F a point on AB such that B lies between A and F. Let K be the circumcenter of the triangle AEF. Denote by D the intersection of lines BC and EF, M the intersection of lines AK and BC, and N the intersection of lines AO and EF. Prove that points A, D, M, N are concyclic.

Proposed by Mihai Miculiţa, Oradea, România

Senior Problems

S499. Let a and b be distinct real numbers. Prove that $27ab\left(\sqrt[3]{a}+\sqrt[3]{b}\right)^3=1$ if and only if $27ab(a+b+1)=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S500. Let a,b,c be pairwise distinct real numbers. Prove that

$$\left(\frac{a-b}{b-c}-2\right)^2+\left(\frac{b-c}{c-a}-2\right)^2+\left(\frac{c-a}{a-b}-2\right)^2\geq 17.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S501. Solve the equation $[x]\{8x\}=2x^2$, where $\lfloor a\rfloor$ and $\{a\}$ are the greatest integers less than or equal to a and the fractional part of a, respectively.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S502. Find all positive integers n such that

$$a+b+c\,|\,a^n+b^n+c^n-nabc$$

for all positive integers a,b,c.

Proposed by Oleg Muskarov, Sofia, Bulgaria

S503. Solve in positive integers the equation

$$101x^3-2019xy+101y^3=100.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S504. Let $a\geq b\geq c\geq 0$ be real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+ca^2+\frac{3}{8}abc\leq\frac{27}{8}$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U499. Let a, b, c be positive real numbers not greater than 2. The sequence $(x_n)_{n\geq 0}$ is defined by $x_0 = a, x_1 = b, x_2 = c$ and

$$x_{n+1} = \sqrt{x_n + \sqrt{x_{n-1} + x_{n-2}}}$$

for all $n \geq 2$. Prove that $(x_n)_{n\geq 0}$ is convergent and find its limit.

Proposed by Mircea Becheanu, Canada and Nicolae Secelean, România

U500. Evaluate

$$\lim_{n\to\infty} \tan \pi\sqrt{4n^2 + n}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

U501. Let $a_1, a_2, \ldots, a_n \geq 1$ be real numbers such that $a_1 a_2 \ldots a_n = 2^n$. Prove that

$$a_1 + \cdots + a_n - \frac{2}{a_1} - \cdots - \frac{2}{a_n} \geq n.$$

Proposed by Marin Chirciu, Piteşti, România

U502. Find all pairs (p, q) of primes such that pq divides

$$(20^p + 1)(7^q - 1).$$

Proposed by Alessandro Ventullo, Milan, Italy

U503. Let $m < n$ be positive integers and let a and b be real numbers. It is known that for every positive real number c the polynomial

$$P_c(x) = bx^n - ax^m + a - b - c$$

has exactly m roots strictly inside the unit circle. Prove that the polynomial

$$Q(x) = mx^n + nx^m - m + n$$

has exactly $m - \gcd(m, n)$ roots lying strictly inside the unit circle.

Proposed by Navid Safaei, Sharif Institute of Technology, Tehran, Iran

U504. Evaluate

$$\int \frac{x^2 + 1}{(x^3 + 1)\sqrt{x}}\, dx$$

Proposed by Titu Andreescu, University of Texas a Dallas, USA

Olympiad Problems

O499. For each positive integer d find the interval $I \subset \mathbb{R}$ of largest length such that for any choice of $a_0, a_1, \ldots, a_{2d-1} \in I$ the polynomial

$$P(x) = x^{2d} + a_{2d-1}x^{2d-1} + \cdots + a_1 x + a_0$$

has no real root.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O500. In triangle ABC, $\angle A \leq \angle B \leq \angle C$. Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{R}{r} + \frac{r}{R} + \frac{1}{2}$$

and

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c} \leq \frac{7}{2} - \frac{r}{R}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O501. Let x, y, z be real numbers such that $-1 \leq x, y, z \leq 1$ and $x + y + z + xyz = 0$. Prove that

$$x^2 + y^2 + z^2 + 1 \geq (x + y + z \pm 1)^2.$$

Proposed by Marius Stănean, Zalău, România

O502. Let $ABCDE$ be a convex pentagon and let M be the midpoint of AE. Suppose that $\angle ABC + \angle CDE = 180^\circ$ and $AB \cdot CD = BC \cdot DE$. Prove that

$$\frac{BM}{DM} = \frac{AB}{AC} . \frac{CE}{DE}$$

Proposed by Khakimboy Egamberganov, ICTP, Trieste, Italy

O503. Prove that in any triangle ABC,

$$\left(\frac{a+b}{m_a + m_b}\right)^2 + \left(\frac{b+c}{m_b + m_c}\right)^2 + \left(\frac{c+a}{m_c + m_a}\right)^2 \geq 4.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O504. Let G be a connected graph such that all degrees are at least 2 and there are no even cycles. Prove that G has a spanning subgraph such that the degree of each vertex is 1 or 2. Prove that the conclusion does not hold if we drop the 'no even cycles' condition. (A spanning subgraph of G is a subgraph which contains all vertices of G.)

Proposed by Radu Bumbăcea, Bucharest, România