Junior Problems

J529. Let a and b be positive real numbers. Prove that for any $x \geq \max(a, b)$

$$(x^2+a^2)(x^2+b^2) \geq 8\sqrt{ab}x(x-a)(x-b).$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J530. Let a, b, c be positive real numbers. Prove that

$$\frac{a+b}{\sqrt{2(a^2+b^2)}}+\frac{b+c}{\sqrt{2(b^2+c^2)}}+\frac{c+a}{\sqrt{2(c^2+a^2)}}+\frac{3(a^2+b^2+c^2)}{2(ab+bc+ca)} \geq \frac{9}{2}.$$

Proposed by Marius Stănean, Zalău, România

J531. Solve in real numbers the system of equations:

$$\begin{cases} x^5y^5+1=y^5 \\ x^9y^9+1=y^9. \end{cases}$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J532. Let a and b be real numbers such that $ab \geq \frac{1}{3}$. Prove that

$$\frac{1}{3a^2+1}+\frac{1}{3b^2+1} \geq \frac{2}{3ab+1}.$$

Proposed by An Zhenping, Xianyang Normal University, China

J533. Find the range of the expression

$$\frac{a-b}{c}$$

where a, b, c are the side-lengths of a triangle with $\angle A = 90^\circ$ and $c \leq b$.

Proposed by Shiva Oswal, Stanford Online High School, USA

J534. Find the greatest prime p such that $3p$ has three digits and divides $31^{31}+32^{155}$.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

Senior Problems

S529. Prove that if the number $\overline{abcbca}$ is divisible by 37 then so is

$$(a-b)^2+(b-c)^2+(c-a)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S530. Let ABC be a triangle with area K. Prove that

$$a(s-a)\cos\frac{B-C}{4}+b(s-b)\cos\frac{C-A}{4}+c(s-c)\cos\frac{A-B}{4}\geq 2\sqrt{3}K.$$

Proposed by An Zhenping, Xianyang Normal University, China

S531. Let x,y,z be real numbers such that $-1\leq x,y,z\leq 1$ and $x+y+z+xyz=0$. Prove that

$$x+y+z+\frac{72}{9+xy+yz+zx}\geq 8.$$

Proposed by Marius Stănean, Zalău, România

S532. Let a,b,c be positive real numbers such that $abc=1$. Prove that for any $0\leq t\leq \min\{a,b,c\}$,

$$(2a^2-6at+7t^2)(2b^2-6bt+7t^2)(2c^2-6ct+7t^2)\geq 108t^5(a-t)(b-t)(c-t).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S533. For every positive integer we denote by $\tau(a)$ the number of positive divisors of a and by $d_1(a)<d_2(a)<\cdots<d_{\tau(a)}(a)$ the increasing sequence of these divisors. Let $(x_1,x_2,\ldots,x_n)$ be a sequence of positive integers. We say that the refinement of this sequence is the sequence

$$d_1(x_1),d_2(x_1),\ldots,d_{\tau(x_1)}(x_1),d_1(x_2),\ldots,d_{\tau(x_2)}(x_2),\ldots,d_1(x_n),\ldots,d_{\tau(x_n)}(x_n).$$

For example, the refinement of the sequence $(1,4,7,10)$ is the sequence $(1,1,2,4,1,7,1,2,5,10)$. Let p be prime number. Starting with the sequence $(1,p,p^2,\ldots,p^p)$ we refine it 2020 times in a row. Evaluate how many times each prime power p^i appears in the last sequence.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

S534. Let ABC be a triangle with incircle tangent to sides BC, CA, AB at points D, E, F, respectively. Suppose that K, L, M are midpoints of BF, BD, BC, respectively. The line which passes through D and is parallel to the bisector of the angle CBA intersects KL at P. Prove that $PM\parallel EF$ if and only if $BC=3BD$.

Proposed by Dominik Burek, Krakow, Poland

Undergraduate Problems

U529. Let $P(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + 1$ be a polynomial whose roots $x_1, x_2, \ldots, x_n$ are real and positive. Prove that for any positive real number t

$$(t^2 - tx_1 + x_1^2)(t^2 - tx_2 + x_2^2)\cdots(t^2 - tx_n + x_n^2) \geq 2^n t^{\frac{n}{2}} |P(t)|.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U530. Evaluate

$$\lim_{n\to\infty} \frac{\sin 1 + \left(\sin \frac{1}{2}\right)^2 + \cdots + \left(\sin \frac{1}{n}\right)^n}{\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}}$$

Proposed by Florin Rotaru, Focşani, România

U531. Evaluate

$$\int_0^1 \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{\sqrt[3]{1+x} + \sqrt[3]{1-x}}\, dx.$$

Proposed by Alessandro Ventullo, Milan, Italy

U532. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt[n]{1^2 3^2 \cdots (2n+1)^2}}{1 + 3 + \cdots + (2n+1)}$$

Proposed by Mircea Becheanu, Canada

U533. Evaluate

$$\int_1^2 (1 + \ln x) x^x \mathrm{d}x.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U534. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x)f(x+y) \geq f(x)^2 + xy$$

for all $x, y \in \mathbb{R}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O529. Let a and b be odd positive integers. Prove that for any positive integer n there is a positive integer m such that 2^n divides at least one of the numbers $a^m b^2 - 1$ and $b^m a^2 - 1$.

Proposed by Navid Safaei, Sharif University of Science, Tehran, Iran

O530. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\sqrt{4a^2 - a + 1} + \sqrt{4b^2 - b + 1} + \sqrt{4c^2 - c + 1} \geq 2(a + b + c)$$

Proposed by An Zhenping, Xianyang Normal University, China

O531. Let ABC be an acute triangle and A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. The orthogonal projection of A_1 onto AB is denoted by A' and the orthogonal projection of A_1 onto AC is denoted by A''. The intersection of the tangents in A' and A'' to the circumcircle of $\Delta A_1 A' A''$ is denoted by X. Points Y and Z are constructed similarly. Prove that lines XA_1, YB_1, ZC_1 are concurrent.

Proposed by Mihaela Berindeanu, Bucharest, România

O532. Points D and E lie on sides AB and BC of triangle ABC, respectively. Assume the incenter I of triangle ABC lies inside quadrilateral $ADEC$ and $BD > AD$. Let r be the inradius of triangle ABC and suppose the distance x from point I to line DE satisfies

$$\frac{x}{r} = \frac{BD - AD}{AB}\,.$$

Prove that the inradius of triangle BDE is equal to the radius of a circle tangent to segment AD and to rays AC and DE.

Proposed by Waldemar Pompe, Warszaw, Poland

O533. Let a, b, c, d be positive real numbers. Prove that

$$\frac{bcd}{a^2} + \frac{acd}{b^2} + \frac{abd}{c^2} + \frac{abc}{d^2} \geq 2\sqrt{a^2 + b^2 + c^2 + d^2}.$$

Proposed by An Zhenping, Xianyang Normal University, China

O534. Let n be a positive integer. An arithmetic sequence is called n-small if it consists of integers and if its common difference has at most n positive divisors. Prove that there is $c > 0$ such that for any $N \geq 1$ the set $\{1^n, 2^n, 3^n, ...\}$ shares at most $cN^{1/n}$ elements with any n-small arithmetic sequence of length N.

Proposed by Titu Andreescu, USA and Gabriel Dospinescu, France