Junior Problems

J523. Let a, b, c be real numbers. Prove that

$$(a-1)^2+(b-1)^2+(c-1)^2 \geq \frac{ab+bc+ca}{2}-3.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J524. Prove that

$$\sum_{1\leq i<j\leq n} \frac{i+2ij+j}{\sqrt{(i+1)(j+1)}} < \frac{n\left(n^2-1\right)}{2}$$

Proposed by Mihaly Bencze, Braşov, România

J525. Prove that positive integers a, b, c are consecutive in some order if and only if

$$a^3+b^3+c^3=3(a+b+c+abc).$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J526. Let ABC be a triangle with circumcenter O, incenter I, and excenters I_a, I_b, I_c. Prove that

$$OI^2+(OI_a)^2+(OI_b)^2+(OI_c)^2=12R^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J527. ind all n for which $2\ldots225$ (n twos) is a perfect square.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J528. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\sum_{cyc} \frac{a^2+b^2}{a+b+2} \geq \frac{3(a+b+c-1)}{4}$$

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S523. Let a, b, c in $[1, 8]$. Prove that

$$\left(2-\frac{a}{b^2}\right)\left(2-\frac{b}{c^2}\right)\left(2-\frac{c}{a^2}\right) \le abc.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S524. Find all systems (x, y, z, t) of real numbers such that

$$x(y+z+t)^2 = y(x+z+t)^2 = z(x+y+t)^2 = t(x+y+z)^2.$$

Proposed by Mircea Becheanu, Montreal, Canada

S525. Find the maximum of $(x-2)(y+1)$ over all real numbers x and y satisfying $3x^2+4xy+5y^2=1$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S526. Let x, y, z be nonnegative real numbers. Prove that

$$\sqrt{3}\left(x^3+y^3+z^3\right) \ge \sqrt{\prod_{cyc}\left(2x^2-xy+2y^2\right)} \ge \sqrt{3}\sum_{cyc} xy(x+y) - 3\sqrt{3}xyz.$$

Proposed by Marius Stănean, Zalău, România

S527. Let ABC be a triangle with $AB = AC$ and $\angle A = 90^\circ$. Points E and F are given inside the angle $\angle BAC$ such that $\angle EAF = \angle EAB + \angle FAC$ and $BE \parallel CF$. Prove that

$$EF^2 = BE^2 + CF^2.$$

Proposed by Mihai Miculiţa, Oradea, România

S528. Let a, b, c be positive real numbers such that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{11}{a+b+c}.$$

Find the minimum of

$$\left(a^4+b^4+c^4\right)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U523. Prove that for each nonegative integer n,

$$\prod_{k=0}^{10^n-1}\left(\frac{4(2k+1)}{2k^2-2k+1}+1\right)$$

is a positive integer whose sum of digits is 5.

Proposed by Titu Andreescu, Unversity of Texas at Dallas, USA

U524. Prove that for all integers $n > 2$,

$$2-\frac{1}{n!}<\left(1+\frac{1}{2!}\right)\left(1+\frac{2}{3!}\right)\cdots\left(1+\frac{n-1}{n!}\right)<3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U525. Let $P(x)$ be a polynomial of degree d with integer coefficients and let $d_1, \ldots, d_k$ be distinct integers. Prove that for any positive integer $k \le d$ there are unique integers $a_1, \ldots, a_k$, not all zero, with $\gcd(a_1, \ldots, a_k) = 1$ and such that the polynomial $a_1 P(x+d_1) + \cdots + a_k P(x+d_k)$ has degree $d-k+1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U526. Evaluate

$$\int_0^2 \frac{1}{t}\arctan\frac{3t}{t^2+4}dt$$

Proposed by Olimjon Jalilov, Tashkent, Uzbekistan

U527. Find the smallest constant C such that

$$\sum_{k=1}^{n}\frac{k}{k^4+4}<C$$

holds for all positive integer n.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U528. Consider the double sequence

$$s(m,\, n)=\frac{m-1}{n}\sum_{k=1}^{n}\frac{\left\lfloor \frac{3k}{n}(m-1)\right\rfloor+3}{\left\lfloor 1+\frac{k}{n}(m-1)\right\rfloor!}$$

Find $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} s(m,\, n)$ and $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} s(m,\, n)$.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia

Olympiad Problems

O523. Let Γ_1, Γ_2 be two disks of center O_1, O_2 and radius r_1, r_2, respectively. Line O_1O_2 intersects the circumference of Γ_1 at a point A and $d = O_1O_2 - (r_1 + r_2) \geq 0$. Prove that for any points $M \in \Gamma_1$ and $N \in \Gamma_2$ the inequality $MN \geq MA$ holds true if and only if

$$r_1 r_2^2 \leq d(d + r_2)(d + 2r_2).$$

Proposed by Oleg Muskarov, Sofia, Bulgaria

O524. Let x, y, z be positive real numbers such that $x^4 + y^4 + z^4 = 3$. Prove that

$$\sqrt{\frac{yz}{7-2x}} + \sqrt{\frac{zx}{7-2y}} + \sqrt{\frac{xy}{7-2z}} \leq \frac{3\sqrt{5}}{5}$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

O525. For any two number sets A and B define their sum as $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$, and their difference by $A - B = \{a - b \mid a \in A \text{ and } b \in B\}$. Let $P = \{2, 3, 5, 7, 11, \ldots\}$ be the set of primes and $S = \{0, 1, 8, 16, 27, 64, 81, 125, 216, 256, \ldots\}$ be the set of perfect cubes and fourth powers. Prove whether or not there are infinitely many positive integers not in

$$(P + S) \cup (P - S) \cup (S - P).$$

Proposed by Li Zhou, Polk State College, USA

O526. Let x, y, z be nonnegative real numbers such that $xy + yz + zx = 3$. Prove that

$$\left(x^2 + y^2 + z^2 + 1\right)^3 \geq \left(x^3 + y^3 + z^3 + 5xyz\right)^2.$$

Proposed by Marius Stănean, Zalău, România

O527. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a nonconstant function such that

$$\max\{f(x + y), f(x - y)\} = f(x)f(y)$$

for all $x, y \in \mathbb{R}$. Prove that $f(x) \geq 1$ for all $x \in \mathbb{R}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O528. Find all functions $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(1)|f(m)$ and

$$f(mn)f(\gcd(m, n)) = \gcd(m, n)f(m)f(n)$$

for all $m, n \in \mathbb{N}$.

Proposed by Besfort Shala, University of Primorska, Koper, Slovenia