Junior Problems

J487. Let $ABCD$ be a cyclic kite. Prove that $3\angle A = \angle C$ or $\angle A = \angle 3C$ if and only if

$$\frac{AC}{BD} - \frac{BD}{AC} = \frac{1}{\sqrt{2}}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J488. Let a and b be positive real numbers such that $ab = a + b$. Prove that

$$\sqrt{1+a^2} + \sqrt{1+b^2} \geq \sqrt{20 + (a-b)^2}$$

Proposed by An Zhenping, Xianyang Normal University, China

J489. Prove that in any triangle ABC

$$8r(R-2r)\sqrt{r(16R-5r)} \leq a^3 + b^3 + c^3 - 3abc \leq 8R(R-2r)\sqrt{(2R+r)^2 + 2r^2}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J490. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3}{1+ab^2} + \frac{b^3}{1+bc^2} + \frac{c^3}{1+ca^2} \geq \frac{3abc}{1+abc}$$

Proposed by An Zhenping, Xianyang and Li Xin, Wugong, China

J491. Find all triples (x, y, z) of positive integers such that

$$5(x^2 + 2y^2 + z^2) = 2(5xy - yz + 4zx)$$

and at least one of x, y, z is a prime.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J492. Let $n > 1$ be an integer and let a, b, c be positive real numbers such that $a^n + b^n + c^n = 3$. Prove that

$$\frac{1}{a^{n+1}+n} + \frac{1}{b^{n+1}+n} + \frac{1}{c^{n+1}+n} \geq \frac{3}{n+1}$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

Senior Problems

S487. Find all primes $a \geq b \geq c \geq d$ such that

$$a^2 + 2b^2 + c^2 + 2d^2 = 2(ab + bc - cd + da).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S488. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} + \frac{1}{3}\left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}\right) \geq 2.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

S489. Find all pairs (m, n) of positive integers with $m + n = 2019$ for which there is a prime p such that

$$\frac{4}{m+3} + \frac{4}{n+3} = \frac{1}{p}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S490. Prove that there is a real function f for which there is no a real function g such that $f(x) = g(g(x))$ for all $x \in \mathbb{R}$.

Proposed by Pavel Gadzinski, Bielsko-Biala, Poland

S491. Prove that in any acute triangle ABC the following inequality holds:

$$\frac{1}{\left(\cos\frac{A}{2} + \cos\frac{B}{2}\right)^2} + \frac{1}{\left(\cos\frac{B}{2} + \cos\frac{C}{2}\right)^2} + \frac{1}{\left(\cos\frac{C}{2} + \cos\frac{A}{2}\right)^2} \geq 1.$$

Proposed by Florin Rotaru, Focşani, România

S492. Find the greatest real constant C such that the inequality

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) - (abc - 1)^2 \geq C(a + b + c)^2$$

holds for all positive real numbers a, b, c.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Undergraduate Problems

U487. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that the following conditions hold simultaneously:

(a) $f(f(x)) = x$ for all $x \in \mathbb{R}$,

(b) $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$,

(c) $\lim\limits_{x\to\infty} f(x) = -\infty$.

Proposed by Vincelot Ravoson, Lycée Henry IV, Paris, France

U488. Let a and b be positive real numbers. Evaluate

$$\int_{a^{-b}}^{a^b} \frac{\arctan x}{x} dx.$$

Proposed by Michele Caselli, University of Modena, Italy

U489. Find all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(f(f(x))) - 3f(f(x)) + 3f(x) - x = 0$$

for all $x \in \mathbb{R}$.

Proposed by Titu Andreescu, USA, and Marian Tetiva, România

U490. Find the greatest real number k such that the inequality

$$\frac{a^2}{b} + \frac{b^2}{a} \geq \frac{2(a^{k+1} + b^{k+1})}{a^k + b^k}$$

holds for all positive real numbers a and b.

Proposed by Nguyen Viet Hung and Vo Quoc Ba Can, Hanoi, Vietnam

U491. Find all polynomials P with complex coefficients such that

$$P(a) + P(b) = 2P(a+b),$$

whenever a and b are complex numbers satisfying $a^2 + 5ab + b^2 = 0$.

Proposed by Titu Andreescu, USA and Mircea Becheanu, Canada

U492. Let C be an arbitrary positive real number and let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1^2 + x_2^2 + \ldots + x_n^2 = n$. Prove that

$$\sum_{i=1}^{n} \frac{x_i}{x_i + C} \leq \frac{n}{C+1}.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria. Spain

Olympiad Problems

O487. Find all n and all distinct positive integers $a_1, a_2, \ldots, a_n$ such that

$$\binom{a_1}{3} + \cdots + \binom{a_n}{3} = \frac{1}{3}\binom{a_1 + \cdots + a_n - n}{2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O488. Let $m, n > 1$ be integers with m even. Find the number of ordered systems $(a_1, a_2, \ldots, a_m)$ of integers such that:

(i) $0 \le a_1 \le a_2 \le \cdots \le a_m \le n$,

(ii) $a_1 + a_3 + \cdots \equiv a_2 + a_4 + \cdots \bmod (n+1)$.

Proposed by Virgil Domocos, Montreal, Canada

O489. In triangle ABC, $\angle A \ge \angle B \ge 60^\circ$. Prove that

$$\frac{a}{b} + \frac{b}{a} \le \frac{1}{3}\left(\frac{2R}{r} + \frac{2r}{R} + 1\right)$$

and

$$\frac{a}{c} + \frac{c}{a} \ge \frac{1}{3}\left(7 - \frac{2r}{R}\right)$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O490. Let ABC be a triangle with incenter I and $A-$excenter I_A. The line through I perpendicular to BI meets AC at X, while the line through I perpendicular to CI meets AB at Y. Prove that X, I_A, Y are collinear if and only if $AB + AC = 3BC$.

Proposed by Tovi Wen, USA

O491. If a, b, c are real numbers greater than -1 such that $a + b + c + abc = 4$, prove that

$$\sqrt[3]{(a+3)(b+3)(c+3)} + \sqrt[3]{(a^2+3)(b^2+3)(c^2+3)} \ge 2\sqrt{ab+bc+ca+13}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O492. Let a, b, c, x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$\sqrt{abcxyz} \le \frac{4}{27}.$$

Proposed by Marius Stănean, Zalău, România