Junior Problems

J553. Solve in real numbers the equation

$$\left(x^2 - 2\sqrt{2}x\right)\left(x^2 - 2\right) = 2021.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J554. Let x, y, z be positive real numbers such that $x + y + z = xyz$. Prove that

$$\frac{1}{1+x^2} + \frac{1}{1+y^2} + \frac{1}{1+z^2} \geq \frac{1}{1+xy} + \frac{1}{1+yz} + \frac{1}{1+zx}$$

Proposed by An Zhenping, Xianyang Normal University, China

J555. Let a, b, c be real numbers such that

$$\frac{1}{a+2+\sqrt{a^2+8}} + \frac{1}{b+2+\sqrt{b^2+8}} + \frac{1}{c+2+\sqrt{c^2+8}} \leq \frac{1}{2}.$$

Prove that $a + b + c \geq 3$. When does the equality occur?

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J556. Let ABC be a triangle with circumcircle Γ and circumcenter O. The tangents in B and C to Γ intersect in D and AD intersects Γ in E. The parallel through A to BC intersects Γ in F. Prove that EF bisects side BC.

Proposed by Mihaela Berindeanu, Bucharest, România

J557. Let $ABCD$ be a parallelogram with $AB \neq AD$ and $\angle BAD > 90°$. We denote by M, N, P the orthogonal projections of A on BC, CD, BD, respectively, and let O be the intersection of diagonals AC and BD. Prove that points M, N, O, P lie on a circle.

Proposed by Mihai Miculiţa, Oradea, România

J558. Let $ABCDE$ be a convex pentagon with $BC = CD$, $DE = EA$ and $\angle BCD + \angle DEA = 180°$. Knowing that $\angle BCD = \alpha$ and $EC = a$, determine the area of the pentagon.

Proposed by Waldemar Pompe, Warsaw, Poland

Senior Problems

S553. Solve in real numbers the equation

$$\left(x^3-3x\right)^2+\left(x^2-2\right)^2=4.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S554. Let a, b, c, x, y, z be positive real numbers such that

$$\frac{x}{b^2+c^2+x}+\frac{y}{c^2+a^2+y}+\frac{z}{a^2+b^2+z}\geq 1.$$

Prove that

$$ab+bc+ca\leq x+y+z.$$

Proposed by An Zhenping, Xianyang Normal University, China

S555. Let ABC be a scalene triangle. We construct externally to $\triangle ABC$ the isosceles triangles XAB, YAC and ZBC such that: $\angle AXB=\angle AYC=90°$ and $\angle ZBC=\angle ZCB=\angle BAC$. Knowing that BY, CX and AZ are concurrent, find $\angle BAC$.

Proposed by Mihaela Berindeanu, Bucharest, România

S556. Let a, b, c be positive real numbers such that $a+b\leq 3c$. Find the maximum possible value of

$$\left(\frac{a}{6b+c}+\frac{a}{b+6c}\right)\left(\frac{b}{6c+a}+\frac{b}{c+6a}\right).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S557. Let a, b, c be positive real numbers such that $ab+bc+ca=3$. Prove that

$$\frac{8a^2}{b+c+2}+\frac{8b^2}{c+a+2}+\frac{8c^2}{a+b+2}+33\geq 13(a+b+c).$$

Proposed by Marius Stănean, Zalău, România

S558. Let ABC be a scalene triangle and let N be the center of its nine-point circle. Let A_1 be the symmetric of A with respect to N. Knowing that A_1 lies on the circumcircle of $\triangle ABC$, evaluate $\angle BAC$.

Proposed by Todor Zaharinov, Sofia, Bulgaria

Undergraduate Problems

U553. Let A be an $n \times n$ matrix such that $A^4 = I_n$. Prove that $A^2 + (A + I_n)^2$ and $A^2 + (A - I_n)^2$ are invertible.

Proposed by Adrian Andreescu, University of Texas at Dallas

U554. Evaluate

$$\int_a^b \left\{ \frac{x^2+1}{x^2-x+1} \right\} \mathrm{d}x$$

in terms of a and b where $a < 0 < b$ and $\{t\}$ denotes the fractional part of t.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U555. Let $f : [0, +\infty) \longrightarrow [0, \infty)$ be a differentiable function such that $f(x)e^{f(x)} = x$, for all $x \geq 0$. Evaluate

$$\int_0^e f(x)\, dx$$

Proposed by Prithvijit Chatraborty, Kolkata, India

U556. Find the volume of the solid obtained by rotating a unit cube about an axis connecting opposite vertices.

Proposed by Li Zhou, Polk State College

U557. Evaluate

$$\int_2^3 xe^x \left(\ln x + 1\right)\, dx.$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

U558. For every polynomial $P(x) = c_0 + c_1 x + \cdots + c_n x^n$ define its reciprocal $\tilde{P}(x)$ by $\tilde{P}(x) = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$. Let $f(x) = a_r x^{d_r} + \cdots + a_0 x^{d_0}$ be a polynomial with integer coefficients and $n = d_r > d_{r-1} > \cdots > d_0 = 0$. Let $g(x) = b_s x^{e_s} + \cdots + b_0$ be a polynomial with positive integer coefficients and $n = e_s > e_{s-1} > \cdots > e_0 = 0$. Prove that if $f(x)\tilde{f}(x) = g(x)\tilde{g}(x)$ and $a_0 = a_1 = \cdots = a_r = 1$, then $r = s$ and $b_0 = b_1 = \cdots = b_s = 1$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

Olympiad Problems

O553. Let ABC be a triangle with $AB = AC$ and let M be the midpoint of BC. Circle ω is tangent to BC at M and lies outside triangle ABC. Circle Ω passes through A, is internally tangent to ω, and its center lies on AM. Circle γ is internally tangent to circle Ω, touches segment BC and the extention of line AC. Through A tangents to ω and γ are drawn intersecting segment MC at points K and L, respectively. Prove that the inradius of triangle ABL is twice the inradius of triangle AKC.

Proposed by Waldemar Pompe, Warzaw, Poland

O554. Let a, b, c, d be real numbers such that $|a|$, $|b|$, $|c|$, $|d| \geq 1$ and

$$a + b + c + d + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} = 0.$$

Prove that

$$a + b + c + d \leq 2\sqrt{2}.$$

Proposed by Marius Stănean, Zalău, România

O555. Let $ABCD$ be a square of side length 1. Point X lies on the smaller arc DA of the circumcircle of square $ABCD$. Let r_1, r_2, r_3, r_4 be the inradii of triangles XDA, XAB, XBC, XCD, respectively. Determine all possible values of

$$\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}$$

as X varies on the smaller arc DA of the circumcircle of square $ABCD$.

Proposed by Waldemar Pompe, Warzaw, Poland

O556. Let a, b, c be the sidelengths of a triangle ABC. Prove that

$$\left(a^2 - bc\right) \cos \frac{B - C}{2} + \left(b^2 - ca\right) \cos \frac{C - A}{2} + \left(c^2 - ab\right) \cos \frac{A - B}{2} \geq 0.$$

Proposed by Marius Stănean, Zalău, România

O557. Evaluate

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{2k+1} \binom{n}{2k}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O558. Let $\{x\}$ be the fractional part of the real number x. Prove that for all positive integers n there are pairwise distinct rational numbers $x_1, \ldots, x_n > n$ such that $\{x_i x_j\} \in \left(\frac{1}{2}, \frac{5}{6}\right)$ for $1 \leq i, j \leq n$.

Proposed by Titu Andreescu, Dallas, USA and Navid Safaei, Tehran, Iran