Junior Problems

J517. Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $a_1 = 1$, $a_2 = 2$ and

$$\frac{a_{n+1}^3 + a_{n-1}^3}{9a_n} + a_{n+1}a_{n-1} = 3a_n^2.$$

Find a_n in a closed form.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J518. Find all real numbers x, y, z such that

$$x(y+z)^2 = y(z+x)^2 = z(x+y)^2 = 108.$$

Proposed by Mircea Becheanu, Montreal, Canada

J519. Let x, y, z be positive numbers such that $xyz(x+y+z)$=3. Prove that

$$\left(2x^2 - xy + 2y^2\right)\left(2y^2 - yz + 2z^2\right)\left(2z^2 - zx + 2x^2\right) \geq 27.$$

Proposed by Hoang Le Nhat Tung, Hanoi, Vietnam

J520. Find all positive integers n for which $2^{3n-1}5^{n+1} + 96$ is a perfect square.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J521. Is it possible to write the integers $1, 2, \ldots, 2020$ in a row so that the sum of any eleven neighboring numbers is divisible by 5?

Proposed by Li Zhou, Polk State College, Florida, USA

J522. Let a, b, c be nonnegative real numbers. Prove that

$$\left(a^2 + 4b^2\right)\left(b^2 + 4c^2\right)\left(c^2 + 4a^2\right) \geq 64abc(2a-b)(2b-c)(2c-a).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Senior Problems

S517. Let a, b, c be real numbers such that

$$a^3+b^3+c^3-1=3(a-1)(b-1)(c-1).$$

Prove that $a+b+c \leq 2$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S518. Let ABC be a triangle with $BC=a$, $AB=AC=b$ and $a^3-b^3=3ab^2$. Calculate $\angle BAC$.

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

S519. Prove that in any triangle ABC

$$2\sqrt{3} \leq \operatorname{cosec} A+\operatorname{cosec} B+\operatorname{cosec} C \leq \frac{2\sqrt{3}}{9}\left(1+\frac{R}{r}\right)^2.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S520. Let a, b, c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a} \geq \frac{2r}{R}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S521. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{8a^3}{(b+c)^3}+\frac{b+c}{a}\right)\left(\frac{8b^3}{(c+a)^3}+\frac{c+a}{b}\right)\left(\frac{8c^3}{(a+b)^3}+\frac{a+b}{c}\right) \geq$$

$$\frac{143(a+b)(b+c)(c+a)}{8abc}-116.$$

Proposed by Marius Stănean, Zalău, Romania

S522. Let $a_1, \ldots, a_n$ and $x_1, \ldots, x_n$, $(n \geq 2)$, be positive real numbers such that

$$\prod_{i=1}^{n} a_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} x_i = n.$$

Prove that

$$\sum_{i=1}^{n} \frac{1}{(n-1)a_i x_i + 1} \geq 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U517. We say that the polynomial $a_nx^n + \cdots + a_1x + a_0$ with real coefficients is powerful if

$$|a_n| + \cdots + |a_1| = |a_0|.$$

Prove that if $P(x)$ is a polynomial with nonzero real coefficients of degree d, such that $P(x)(x-1)^s(x+1)^t$ is powerfull for some nonnegative integers s and t, then either $P(x)$ or $(-1)^dP(-x)$ has non increasing coefficients.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U518. Evaluate

$$\lim_{n\to\infty}\left(\frac{1^2}{n^3+n^2+1}+\frac{2^2}{n^3+n^2+2}+\cdots+\frac{n^2}{n^3+n^2+n}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U519. Let k be a fixed integer. Evaluate

$$\sum_{n=k+1}^{\infty}\frac{1}{n(n^2-1^2)^2(n^2-2^2)^2\dots(n^2-k^2)^2}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U520. Evaluate

$$\lim_{x\to 0}\frac{(1-\cos x)(1-\cos 2x)\cdots(1-\cos nx)}{\sin^{2n}x}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U521. Find all automorphisms of the group $S_3\times\mathbb{Z}_3$.

Proposed by Mircea Becheanu, Montreal, Canada

U522. Let $f:(0,\infty)\longrightarrow(0,\infty)$ be a continuous function. For any positive integer n we denote $t_n = n\sqrt[n]{n}$. Evaluate

$$\lim_{n\to\infty}\int_{t_n}^{t_{n+1}} f\left(\frac{x}{n}\right)dx.$$

Proposed by Florin Rotaru, Focşani,Romania

Olympiad Problems

O517. Prove that for any positive real numbers a, b, c

$$\sqrt{\frac{2ab}{a^2+b^2}}+\sqrt{\frac{2bc}{b^2+c^2}}+\sqrt{\frac{2ca}{c^2+a^2}}+\frac{3\left(a^2+b^2+c^2\right)}{ab+bc+ca}\geq 6.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O518. Let p be a prime congruent to 3 modulo 4. Alice and Bob have just filled in the cells of a table of size $2\times p$ (2 rows, p columns). They proceeded thus: for every $m\in\{0,1,\ldots,p-1\}$, in the m-th cell of the first row of the table Alice wrote the remainder that m^2 leaves when it is divided by p; for every $n\in\{0,1,\ldots,p-1\}$, in the n-th cell of the second row of the table Bob wrote the remainder that n^4 leaves when it is divided by p. Prove that both rows of Alice and Bob's table contain the same numbers with the same multiplicities.

Proposed by José Hernández Santiago, Matemáticas UAGro

O519. Let a, b, c be positive numbers such that $a+b+c=ab+bc+ca$. Prove that

$$\frac{3}{1+a}+\frac{3}{1+b}+\frac{3}{1+c}-\frac{4}{(1+a)(1+b)(1+c)}\geq 4$$

Proposed by An Zhenping, Xianyang Normal University, China

O520. Let x, y, z be positive integers such that

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}=6$$

and $\gcd(z,x)=1$. Find the maximum value of $x+y+z$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

O521. In the triangle ABC we denote by m_a, m_b, m_c the lengths of its medians and by w_a, w_b, w_c the lengths of the angle bisectors. Prove that

$$\frac{m_a}{w_a}+\frac{m_b}{w_b}+\frac{m_c}{w_c}\leq 1+\frac{R}{r}$$

Proposed by Dragoljub Milošević, Gornji Milanovac, Serbia

O522. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{27abc}{4\left(a^3+b^3+c^3\right)}\geq\frac{21}{4}.$$

Proposed by Marius Stănean, Zalău, Romania