# Junior Problems

**J583.** Let $m$ and $n$ be positive integers. Prove that

$$27^{2m+n+1}+27^{m+2n+1}-27^{m+n+1}+1$$

has a factor greater than $6\cdot 27^{\min(m,n)}$.

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

**J584.** Let $x, y, z$ be rational numbers such that

$$3x^2+2022yz-2016zx,\ 3y^2+2022xz-2016xy,\ z^2+674xy-672yz$$

are all squares of rational numbers. Prove that $x=y=z=0$.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

**J585.** Let $a, b, c > 1$ be real numbers such that

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}=1.$$

Prove that

$$abc+44\geq 9(a+b+c).$$

*Proposed by Marius Stănean, Zalău, Romania*

**J586.** Let $ABC$ be a triangle with $AC=BC$ and altitudes $AD, BE, CF$. The circle with diameter $BD$ cuts $AB$ in $M$ and $BE$ in $N$. Line $MN$ cuts $AC$ in $Q$ and $CF$ in $P$. Let $S$ denote the midpoint of segment $DC$. Show that $SQP$ is an isosceles triangle.

*Proposed by Mihaela Berindeanu, Bucharest, Romania*

**J587.** Let $a, b, c$ be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a^2}{a^2+bc}+\frac{b^2}{b^2+ca}+\frac{c^2}{c^2+ab}+\frac{a^3+b^3+c^3+9abc}{(a+b)(b+c)(c+a)}\geq 3.$$

When does the equality occur?

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**J588.** Find all nonnegative integers $x, y, z$ such that

$$4^x+3^y=z^2.$$

*Proposed by Mihaela Berindeanu, Bucharest, Romania*

# Senior Problems

**S583.** Solve in integers the equation

$$(2x^2 - 10x + 50)(2y^2 - 10y + 50) = 2022^2.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**S584.** Prove that in any triangle $ABC$,

$$(b+c)m_a + (c+a)m_b + (a+b)m_c \geq 3\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

*Proposed by Marius Stănean, Zalău, Romania*

**S585.** Let $ABC$ be a scalene triangle with circumcircle $\Gamma$ and let $N$ be the center of its nine-point circle. Line $AN$ intersects circle $\Gamma$ at $D$. Let $N_1$ be the center of the nine-point circle of $\triangle BCD$. Prove that $A, N, N_1, D$ are collinear and $AD = 2NN_1$.

*Proposed by Todor Zaharinov, Sofia, Bulgaria*

**S586.** Prove that in any triangle,

$$(s^2 + r^2 + 10Rr)(4R + r) \leq 8Rs^2.$$

*Proposed by Mihaly Bencze and Neculai Stanciu, Romania*

**S587.** Diagonals $AC$ and $BD$ of a convex quadrilateral $ABCD$ meet at $E$. Points $M$ and $N$ are the midpoints of sides $AB$ and $CD$, respectively. Segment $MN$ meets diagonals $AC$ and $BD$ at $P$ and $Q$, respectively. Prove that

$$\frac{PQ}{MN} = \frac{|[BCE] - [ADE]|}{[ABCD]},$$

were $[XYZ]$ denotes the area of $XYZ$.

*Proposed by Waldemar Pompe, Warsaw, Poland*

**S588.** Find all triples $(a, b, c)$ of nonnegative integers such that

$$2^a 3^b + 7 = c^3.$$

*Proposed by Prodromos Fotiadis, Nikiforos High School, Drama, Greece*

# Undergraduate Problems

**U583.** Let $k \geq 1$ be a fixed integer and let

$$P_n(x) = x^n(x^k - x^{k-1} - \cdots - x - 1) - 1.$$

Prove that each polynomial $P_n(x)$ has a single positive root, $r_n$, and the sequence $r_1, r_2, \ldots, r_n$ is decreasing.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

**U584.** Let $m, n, p$ be positive integers greater than 1. Let $A$ be a $p \times p$ real matrix such that $A^m B = BA^m$ and $A^n B = BA^n$ for all $p \times p$ real matrices $B$. Prove that if $\det(A) \neq 0$ and $\gcd(m, n) = 1$ then $AB = BA$ for all $p \times p$ real matrices $B$.

*Proposed by Mircea Becheanu, Canada*

**U585.** Evaluate

$$\sum_{n=1}^{\infty} \left[ n^2 \left( \zeta(2) - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2} \right) - n + \frac{1}{2} - \frac{1}{6n} \right].$$

*Proposed by Ovidiu Furdui and Alina Sîntămărian, Cluj-Napoca, Romania*

**U586.** Find all functions $f, g : \mathbb{Q} \to \mathbb{R}$ such that

$$f(x)f(x+y) = f(y)^2 f(x-y)^2 g(y)$$

for all $x, y \in \mathbb{Q}$.

*Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran*

**U587.** For $x, y \geq 5$ show that

$$\left(\frac{1}{x}\right)^{\frac{1}{x}} \left(\frac{1}{y}\right)^{\frac{1}{y}} \leq \left(\frac{4}{x^2+y^2}\right)^{\frac{2}{x+y}}$$

*Proposed by Toyesh Prakash Sharma, Agra College, India*

**U588.** Prove that

$$\lim_{n \to \infty} \beta^{-\frac{1}{n}}(n\pi, n\pi) = 4^{\pi},$$

where $\beta(x, y)$ is the Euler integral of the first kind.

*Proposed by Ankush Kumar Parcha, India*

# Olympiad Problems

**O583.** Let $a, b, c$ be real numbers. Prove that

$$a^3 + b^3 + c^3 - 3abc \leq (a^2 + b^2 + c^2 + 2)^{3/2} - 3(a + b + c),$$

with equality if and only if $ab + bc + ca = 1$.

*Proposed by Florin Pop, USA and Gigi Stoica, Canada*

**O584.** Let $ABCD$ be a circumscriptible quadrilateral and let $\{O\} = AC \cap BD$. Let $r_1, r_2, r_3, r_4$ be the inradii and $R_1, R_2, R_3, R_4$ be the radii of $O$-excircles of triangles $AOB$, $BOC$, $COD$, $DOA$, respectively. Prove that

$$\frac{AB}{1 - \frac{r_1}{R_1}} + \frac{CD}{1 - \frac{r_3}{R_3}} = \frac{BC}{1 - \frac{r_2}{R_2}} + \frac{DA}{1 - \frac{r_4}{R_4}}.$$

*Proposed by Marius Stănean, Zalău, Romania*

**O585.** Prove that in any triangle $ABC$

$$\frac{9}{16}\left(\frac{12r^2}{R^2} - 1\right) \leq \sum_{cyc} \cos A \sin B \sin C \leq \frac{9}{4}\left(\frac{3}{4} - \frac{r^2}{R^2}\right)$$

*Proposed by Marian Ursărescu, Roman, Romania*

**O586.** Diagonals $AC$ and $BD$ of convex quadrilateral $ABCD$ intersect at point $E$. Triangles $ABP$ and $CDQ$ are constructed outside of the quadrilateral $ABCD$, such that

$$\angle PAB = \angle DAE, \ \ \angle PBA = \angle CBE$$

$$\angle QDC = \angle ADE, \ \ \angle QCD = \angle BCE.$$

Prove that $P$, $E$, $Q$ are collinear.

*Proposed by Waldemar Pompe, Warsaw, Poland*

**O587.** Let $a, b, c, d$ be positive real numbers. Prove that

$$22a + 25b + 30c + 30d \geq 360\sqrt[3]{\frac{abcd}{2a + 5b + 10c + 30d}}.$$

When does equality hold?

*Proposed by An Zhenping, Xianyang Normal University, China*

**O588.** Let $a, b, c, d$ be positive real numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} + \frac{1}{1+d} = 1.$$

Prove that

$$ab + ac + ad + bc + bd + cd + 18 \geq 6(a + b + c + d).$$

*Proposed by Marius Stănean, Zalău, Romania*