Junior Problems

J547. Find all primes p such that

$$\frac{2^{p+2}-1}{p}$$

is a prime.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J548. Let a, b, c, x, y be positive real numbers such that $x + y = 1$. Prove that

$$\sqrt{\frac{a^3}{xa+yb}} + \sqrt{\frac{b^3}{xb+yc}} + \sqrt{\frac{c^3}{xc+ya}} \geq a + b + c.$$

Proposed by Mircea Becheanu, Canada

J549. Let a, b, c be positive real numbers. Prove that

$$\frac{b+c}{a^2} + \frac{c+a}{b^2} + \frac{a+b}{c^2} - \frac{9}{a+b+c} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J550. Let a, b, c be real numbers with $a, b \leq c$, such that $abc = 1$ and $ab + bc + ca = 0$. Find the greatest real number k such that

$$|a + b| \geq k|c|.$$

Proposed by Ayashi Jain, Gurgaon, Haryana, India

J551. Let $ABCD$ be a square and let M be a point on side CD. The lines AM and BD intersect in E. The perpendicular in E on AM intersects BC in N and AN intersects BD in F. Let K be the intersection point of EN and FM. Prove that AK is perpendicular to MN.

Proposed by Mircea Becheanu, Canada

J552. Let x, y, z be positive real numbers with $xy + xz + yz + xyz = 4$. Prove that

$$2\left(\sqrt{x+1} + \sqrt{y+1} + \sqrt{z+1}\right) \leq 3\sqrt{(x+1)(y+1)(z+1)}.$$

Proposed by Mihaela Berindeanu, Bucharest, România

Senior Problems

S547. Let a and b be postive real numbers less than 2 such that $ab = 2$. Solve in real numbers the equation

$$4(x^2 + ax + b)(x^2 + bx + a) + a^3 + b^3 = 9.$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

S548. Let a, b, c, d be nonnegative real numbers such that $a + b + c + d = 10$. Prove that

$$6a + 2ab + abc + abcd \leq 96.$$

Proposed by An Zhenping, Xianyang Normal University, China

S549. Let a, b, c be positive real numbers such that $a + b + c + abc = 4$. Prove that

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \leq \sqrt{1 + 4a - a^2} + \sqrt{1 + 4b - b^2} + \sqrt{1 + 4c - c^2} \leq ab + bc + ca + 3.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S550. Let a, b, c be positive real numbers. Prove that

$$\sqrt{a^2 + 2ab} + \sqrt{b^2 + 2bc} + \sqrt{c^2 + 2ca} \geq \sqrt{3ab} + \sqrt{3bc} + \sqrt{3ca}.$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S551. Let a, b, c be the side lengths of a triangle with inradius r and circumradius R. Prove that

$$\frac{R}{r} + (1 + \sqrt{5}) \geq (3 + \sqrt{5}) \cdot \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

When does equality hold?

Proposed by Marius Stănean, Zalău, România

S552. Find all triangles ABC with $AB = 8$ for which there is an interior point P such that $PB = 5$, PC, AC, BC is an arithmetic sequence with common difference 2 and $\angle BPC = 2\angle BAC$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U547. Let a, b, c, d be real numbers such that all solutions to the equation

$$x^5 + ax^4 + bx^3 + cx^2 + dx + 1022 = 0$$

are real numbers less than -1. Pove that $a + c < b + d$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U548. Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1 + \tan^n x}$$

where n is a positive integer.

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U549. Evaluate

$$\sum_{n=1}^{\infty} \frac{4n - 1}{n^2(2n - 1)^2}$$

Proposed by Toyesh Prakash Sharma, St.C.F. Andrews School, Agra, India

U550. Let

$$f_n(x) = (x^2 - x + 1)(x^4 - x^2 + 1)(x^8 - x^4 + 1)\cdots(x^{2^n} - x^{2^{n-1}} + 1).$$

Prove that for $|x| < 1$

$$\frac{1}{3} < \lim_{n\to\infty} f_n(x) \leq \frac{4}{3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U551. Let $P(x) = a_0 + a_1x + \cdots + a_dx^d$ be a polynomial with positive coefficients such that $a_k^2 > 9a_{k-1}a_{k+1}$, for all $k = 1, \ldots, d - 1$. Prove that $P(x)$ has d distinct real roots.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran

U552. Find all polynomials $P(x)$ with real coefficients for which

$$P(P(a + b)) - 4abP(a + b) + 4a^2b^2 \geq P(a^2) + P(b^2) \geq P(a^2 + b^2) - P(\sqrt{2}ab).$$

Proposed by Karthik Vedula, James S. Rickards High School, Tallahassee, USA

Olympiad Problems

O547. Let a, b, c be the side lengths of a triangle and let R and r be the circumradius and inradius, respectively. Prove that:

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{17r}{18R}\geq\frac{11}{9}.$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

O548. Let $m, n, p \geq 2$ be positive integers. Find the number of $n \times p$ matrices with entries in the set $\{1, 2, \ldots, m\}$ such that every element of the matrix is distinct from its row and column neighbors.

Proposed by Mircea Becheanu, Canada

O549. Let ABC be a triangle. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C}\geq\frac{7}{4}\left(\frac{R}{r}+\frac{r}{R}\right)-\frac{19}{8}\geq\frac{1}{16}\left(21\frac{R}{r}-10\right)\geq\frac{R}{r}.$$

(An improvement of inequality S544.)

Proposed by Marius Stănean, Zalău, România

O550. Let ABC be a triangle. Incircle with radius r touches BC at D. Point X lies inside angle BAC and outside triangle and satisfies the following conditions:

$$BD\cdot BX=CD\cdot CX \quad\text{and}\quad \tan\frac{1}{2}\angle CXB=\frac{r}{BC}.$$

Prove that X lies on the A-excircle.

Proposed by Dominik Burek, Krakow, Poland

O551. Let ABC be a triangle and let Δ be its area. Prove that

$$a(s-a)\cos\frac{B-C}{4}+b(s-b)\cos\frac{C-A}{4}+c(s-c)\cos\frac{A-B}{4}\geq 2\sqrt{3}\Delta.$$

Proposed by An Zhenping, Xianyang Normal University, China

O552. Let ABC be a triangle with incenter I. The incircle is tangent to BC, CA, AB at points D, E, F, respectively. Denote by A_1, B_1, C_1 the orthocenters of the triangles AEF, BFD, CDE, respectively.
(1) Prove that circle (DB_1C_1) passes through the foot of the altitude from A of triangle ABC.
(2) Prove that circles (DB_1C_1), (EC_1A_1), (FA_1B_1) have a common point and this point is the Feuerbach point of triangle ABC.

Proposed by Dong Luu, Hanoi National University of Education, Vietnam