# Junior Problems

**J511.** Let $a, b, c$ be positive real numbers such that $a+b+c \leq 3\sqrt[3]{3abc}$. Prove that

$$9(a+b+c)^3 < \left(a+3b+3c+\frac{2bc}{a}\right)\left(3a+b+3c+\frac{2ca}{b}\right)\left(3a+3b+c+\frac{2ab}{c}\right)$$

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

**J512.** Let $ABC$ be a triangle with $AC = BC$ and let $a$ be a positive real number. Consider two variable circles with radii $r_1$ and $r_2$ lying inside the triangle such that the first circle is tangent to segments $AB$, $AC$, the second one to segments $AB$, $BC$, and $r_1 + r_2 = a$. Prove that the common external tangents to these circles, different from line $AB$, are all tangent to a fixed circle.

*Proposed by Waldemar Pompe, Warsaw, Poland*

**J513.** Let $a, b, c$ be distinct positive real numbers such that $ab+bc+ca = 1$. Prove that

$$\sum_{cyc} \frac{(a+b)(a+c)-bc}{(b-c)(b^3-c^3)} \geq \left(\sum_{cyc} \frac{a}{|b-c|}\right)^2$$

*Proposed by Adrian Andreescu, University of Texas at Dallas, USA*

**J514.** Let $a, b, c$ be nonnegative real numbers such that

$$(a^2-a+1)(b^2-b+1)(c^2-c+1) = 1.$$

Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \leq 27.$$

*Proposed by Marius Stănean, Zalău, Romania*

**J515.** Let $a, b, c$ be complex numbers such that $a+b+c = 3$. Prove that any two of the equalities

$$a^3-9 = bc(a-9);\ \ b^3-9 = ca(b-9);\ \ c^3-9 = ab(c-9)$$

imply the third.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**J516.** Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a}{\sqrt{b+c}} + \frac{b}{\sqrt{c+a}} + \frac{c}{\sqrt{a+b}} \geq \sqrt{\frac{3(a+b+c)}{2}}$$

*Proposed by Mircea Becheanu, Montreal, Canada*

# Senior Problems

**S511.** Let $a, b, c$ be positive real numbers such that $ab + bc + ca = 3$. Prove that

$$(\sqrt{a} + \sqrt{b} + \sqrt{c} + 1)^2 \leq 2(a+b)(b+c)(c+a)$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**S512.** Let $a, b, c$ be positive real numbers. Prove that

$$\frac{a^3}{\sqrt{b^2 - bc + c^2}} + \frac{b^3}{\sqrt{c^2 - ca + a^2}} + \frac{c^3}{\sqrt{a^2 - ab + b^2}} \geq a^2 + b^2 + c^2.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**S513.** Solve the equation

$$16\{x\}(x + 2020\{x\}) = [x]^2,$$

where $[x]$ and $\{x\}$ are the greatest integer less than or equal to $x$ and the fractional part of $x$, respectively.

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**S514.** Let $a, b, c$ be positive numbers. Prove that

$$\frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} \geq \frac{4\left(a^2 + b^2 + c^2\right)}{ab + bc + ba} + 2.$$

*Proposed by An Zhenping, Xianyang Normal University, China*

**S515.** Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that

$$\left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\right)^6 \geq 27(a+2)(b+2)(c+2)$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**S516.** Let $a$, $b$, $c$ be nonnegative real numbers such that

$$(a+b)(b+c)(c+a) = 2.$$

Prove that

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) + 8a^2b^2c^2 \leq 1.$$

*Proposed by Marius Stănean, Zalău, Romania*

# Undergraduate Problems

**U511.** Let $A$ and $B$ be $n \times n$ matrices such that $AB = A + B$. Prove that

$$\operatorname{rank}(A^2) + \operatorname{rank}(B^2) \le 2\operatorname{rank}(AB).$$

*Proposed by Konstantinos Metaxas, Athens, Greece*

**U512.** Consider the polynomial $P(x) = x^6 + 4x^5 + 8x^4 + 12x^3 + 16x^2 + 16x + 8$. Evaluate

$$\int \frac{x^9 + 16x}{P(x)P(-x)}\,dx$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**U513.** Let $n$ be a positive integer. Prove that the polynomial

$$X^n + 2\binom{n}{1}X^{n-1} + \cdots + 2\binom{n}{k}X^{n-k} + \cdots + 2\binom{n}{n-1}X + 1$$

is divisible by $X^2 + 5X + 1$, if and only if $n = 3$.

*Proposed by Mircea Becheanu, Montreal, Canada*

**U514.** Let $A, B, C \in (-\pi/2, \pi/2)$ be such that

$$x^3 + \frac{1}{6}x^2 - \frac{1}{3}x - \frac{1}{15} = (x - \sin A)(x - \sin B)(x - \sin C).$$

Evaluate $391T^4 - 216T^2 - 32T$, where $T = |\tan A \tan B| + |\tan B \tan C| + |\tan C \tan A|$.

*Proposed by Li Zhou, Polk State College, USA*

**U515.** Prove that for every positive integer $n$,

$$\left(1 + \frac{1}{1^2} + \frac{1}{2^2}\right)\left(1 + \frac{1}{2^2} + \frac{1}{3^2}\right)\cdots\left(1 + \frac{1}{n^2} + \frac{1}{(n+1)^2}\right) < \frac{37}{5}.$$

*Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam*

**U516.** Evaluate

$$\lim_{n\to\infty}\left(\frac{1 + \frac{1}{2} + \cdots + \frac{1}{n}}{\ln n}\right)^{\ln n}$$

*Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

# Olympiad Problems

**O511.** Find all 5-tuples $(v, w, x, y, z)$ of integers satisfying the system of equations

$$\begin{aligned} x^2 + xy - 2yz + 3zx &= 2020 \\ y^2 + yz - 2zx + 3xy &= v \\ z^2 + zx - 2xy + 3yz &= w \\ v + w &= 5. \end{aligned}$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**O512.** Let $ABC$ be a triangle. Points $P$ and $Q$ lie on sides $BC$ and $AB$, respectively, such that

$$\angle PAB = \angle CQA = 90^\circ - \frac{1}{2}\angle ACB.$$

Prove that if $AP = 2CP$, then the inradii of triangles $ACQ$ and $BCQ$ are equal.

*Proposed by Waldemar Pompe, Warsaw, Poland.*

**O513.** Let $x, y, z$ be positive real numbers such that

$$(x + y + z)^9 = 9^5 x^3 y^3 z^3.$$

Prove that

$$\left(\frac{3(x+y)}{z} + \frac{z^2}{xy} + 2\right)\left(\frac{3(y+z)}{x} + \frac{x^2}{yz} + 2\right)\left(\frac{3(z+x)}{y} + \frac{y^2}{zx} + 2\right) < 3^7.$$

*Proposed by Titu Andreescu, University of Texas at Dallas, USA*

**O514.** Find all pairs of integers $(m, n)$ such that $6(m+1)(n-1)$, $(m-1)(n+1)+6$ and $(m+2)(n-2)$ are simultaneously perfect cubes.

*Proposed by Alessandro Ventullo, Milan, Italy*

**O515.** Let $a$ and $b$ be real numbers. Find the extreme values of the expression

$$\frac{(1-a)(1-b)(1-ab)}{(1+a^2)(1+b^2)}.$$

*Proposed by Marius Stănean, Zalău, Romania*

**O516.** Let $ABC$ be a triangle and let $\Delta$ be its area. Prove that

$$a^2 \tan\frac{A}{2} + b^2 \tan\frac{B}{2} + c^2 \tan\frac{C}{2} \geq 2\frac{R}{r}\Delta$$

*Proposed by An Zhenping, Xianyang Normal University, China*