Junior Problems

J475. Let ABC be a triangle with $\angle B$ and $\angle C$ acute and different from $45°$. Let D be the foot of the altitude from A. Prove that $\angle A$ is right if and only if

$$\frac{1}{AD - BD} + \frac{1}{AD - CD} = \frac{1}{AD}$$

Proposed by Adrian Andreescu, University of Austin at Texas, USA

J476. Let x, y, z be positive numbers such that $x + y + z \geq S$. Prove that

$$6\left(x^3 + y^3 + z^3\right) + 9xyz \geq S^3.$$

Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain

J477. Find the area of a kite $ABCD$ with $AB - CD = \left(\sqrt{2} + 1\right)\left(\sqrt{3} + 1\right)$ and $11\angle A = \angle C$.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J478. Prove that in any triangle ABC the following inequality holds:

$$4\left(\ell_a^2 + \ell_b^2 + \ell_c^2\right) \leq (a + b + c)^2.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J479. Let a, b, c be nonzero real numbers, not all equal, such that

$$\left(\frac{a^2}{bc} - 1\right)^3 + \left(\frac{b^2}{ca} - 1\right)^3 + \left(\frac{c^2}{ab} - 1\right)^3 = 3\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} - \frac{bc}{a^2} - \frac{ca}{b^2} - \frac{ab}{c^2}\right).$$

Prove that $a + b + c = 0$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J480. Let m, n be integers greater than 1. Find the number of ordered systems $(a_1, a_2, \ldots, a_m)$ where a_i are nonnegative integers less than n and such that

$$a_1 + a_3 + \cdots \equiv a_2 + a_4 + \ldots \mod(n+1).$$

Proposed by Mircea Becheanu, Montreal, Canada

Senior Problems

S475. Let a and b be positive real numbers such that

$$\frac{a^3}{b^2}+\frac{b^3}{a^2}=5\sqrt{5ab}.$$

Prove that

$$\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}=\sqrt{5}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S476. Prove that in any triangle ABC,

$$4\cos\frac{A+\pi}{4}\cos\frac{B+\pi}{4}\cos\frac{C+\pi}{4}\geq\sqrt{\frac{r}{2R}}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S477. Let $ABCD$ be a kite inscribed in a circle such that

$$2AB^2+AC^2+2AD^2=4BD^2.$$

Prove that $\angle A=4\angle C$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S478. Let a,b,c be positive numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+4\left(\frac{a}{2a+b+c}+\frac{b}{a+2b+c}+\frac{c}{a+b+2c}\right)\geq\frac{9}{2}$$

Proposed by Titu Zvonaru, Comăneşti, Romania

S479. Let $a_1,a_2,\ldots,a_n$ be nonnegative real numbers and let $(i_1,i_2,\ldots,i_n)$ be a permutation of the numbers $(1,2,\ldots,n)$ such that $i_k\neq k$, for all $k=1,2,\ldots,n$. Prove that

$$a_1^na_{i_1}+a_2^na_{i_2}+\cdots+a_n^na_{i_n}\geq a_1a_2\ldots a_n\left(a_1+a_2+\cdots+a_n\right).$$

Proposed by Mircea Becheanu, Montreal, Canada

S480. Let a,b,c be positive real numbers. Prove that

$$\frac{a\left(a^3+b^3\right)}{a^2+ab+b^2}+\frac{b\left(b^3+c^3\right)}{b^2+bc+c^2}+\frac{c\left(c^3+a^3\right)}{c^2+ca+a^2}\geq\frac{2}{3}\left(a^2+b^2+c^2\right).$$

Proposed by An Zhenping, Xianyang Normal University, China

Undergraduate Problems

U475. Evaluate

$$\lim_{x\to 0}\frac{\sin(x\sin x)+\sin(x\sin(x\sin x))}{x\sin(\sin x)+\sin(\sin(x\sin x))}$$

Proposed by Mircea Becheanu, Montreal, Canada

U476. Evaluate

$$\int\frac{x(x+1)(4x-5)}{x^5+x-1}dx$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U477. Evaluate

$$\lim_{n\to\infty}\frac{\frac{\pi^2}{6}-\sum_{k=1}^{n}\frac{1}{k^2}}{\log\left(1+\frac{1}{n}\right)}$$

Proposed by Tiago Landim de Sousa Leão, University of Pernambuco, Brasil

U478. Let n be a positive integer. Prove that

$$\prod_{k=1}^{n}\left(1+\tan^4\frac{k\pi}{2n+1}\right)$$

is a positive integer that is the sum of two perfect squares.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U479. Evaluate

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{n+m}\frac{x^{2(n+m)}}{(2n+2m)!}.$$

Proposed by Àngel Plaza, University of Las Palmas de Gran Canaria, Spain

U480. Let

$$A=\begin{pmatrix}4 & -3 & 2\\ 15 & -10 & 6\\ 10 & -6 & 3\end{pmatrix}$$

Find the least possible n for which one entry of A^n is 2019.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Olympiad Problems

O475. Let a, b, c be nonnegative real numbers such that $\frac{a}{b+c} \geq 2$. Prove that

$$(ab+bc+ca)\left[\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right] \geq \frac{49}{18}.$$

Proposed by Marius Stănean, Zalău, Romania

O476. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$\left(a^2-ab+b^2\right)\left(b^2-bc+c^2\right)\left(c^2-ca+a^2\right)+11abc \leq 12.$$

Proposed by An Zhenping, Xianyang Normal University, China

O477. Solve in integer numbers the equation

$$\left(x^2-3\right)\left(y^3-2\right)+x^3=2\left(x^3y^2+2\right)+y^2.$$

Proposed by Konstantinos Metaxas, Athens, Greece

O478. Let $ABCDEF$ be a cyclic hexagon inscribed in a circle of radius 1. Suppose that the diagonals AD, BE, CF are concurrent in the point P. Prove that

$$AB+CD+EF \leq 4.$$

Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain

O479. Let $ABCD$ be a quadrilateral with $AB=CD=4$, $AD^2+BC^2=32$ and $\angle(ABD)+\angle(BDC)=51^\circ$. If $BD=\sqrt{6}+\sqrt{5}+\sqrt{2}+1$, find AC.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

O480. At a party, some people shake hands. The followings are known:

- Each person shakes hands with exactly 20 persons.
- For each pair of persons who shake hands with each other, there is exactly one other person who shakes hands with both of them.
- For each pair of persons who do not shake hands with each other, there are exactly six other persons who shake hands with both of them.

Determine the number of people in the party.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran