Junior Problems

J541. Solve in positive real numbers the system of equations

$$\begin{cases}(x-\sqrt{xy})(x+3y)=8\left(9+8\sqrt{3}\right)\\(y-\sqrt{xy})(3x+y)=8\left(9-8\sqrt{3}\right).\end{cases}$$

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

J542. Let $ABCD$ be a unit square. Points M and N lie on sides BC and CD, respectively, such that $\angle MAN = 45^\circ$. Prove that

$$1 \le MC + NC \le 4 - 2\sqrt{2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J543. Let a and b be positive real numbers. Prove that

$$|a^5 - b^5| = ab \max\left(a^3,\, b^3\right)$$

if and only if

$$|a^3 - b^3| = ab \min(a,\, b).$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J544. Let a, b, c, x, y, z be positive real numbers such that $x+y+z=3$. Prove that

$$\frac{a}{a+2bx}+\frac{b}{b+2cy}+\frac{c}{c+2az}\ge 1.$$

Proposed by An Zhenping, Xianyang Normal University, China

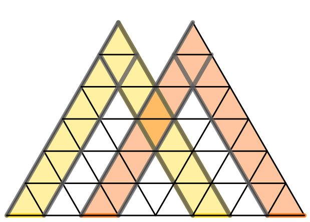
J545. Let a, b, c be distinct positive real numbers such that

$$\left(a+\frac{b^2}{a-b}\right)\left(a+\frac{c^2}{a-c}\right)=4a^2.$$

Prove that $a^2 > bc$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

J546. For $m \ge n \ge 0$, let AM_n^m be the *AwesomeMath figure of degree* $(m,\, n)$, formed by two equilateral triangles of side m, overlapping in an equilateral triangle of side n. Assume that the triangles are subdivided into equilateral triangles of side 1. Count the number of parallelograms in AM_n^m.



Proposed by Li Zhou, Polk State College, Winter Haven, USA

Senior Problems

S541. Prove that for each positive integer n the number

$$3^{3^{n+1}+3}+3^{3^n+2}+1$$

is composite.

Proposed by Adrian Andreescu, University of Texas at Dallas, USA

S542. Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Points D and E are taken on side BC such that $\angle DAB = \angle EAC$. Lines AD and BI intersect in F, lines AE and CI intersect in G, and lines BC and FG intersect in P. Prove that $AP \perp AI$.

Proposed by Mihai Miculiţa, Oradea, România

S543. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{2abc}{ab+bc+ca} \geq \frac{11}{3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S544. Let ABC be a triangle. Prove that

$$\frac{\cos A}{\sin^2 A}+\frac{\cos B}{\sin^2 B}+\frac{\cos C}{\sin^2 C} \geq \frac{R}{r}.$$

Proposed by An Zhenping, Xianyang Normal University, China

S545. Let x, y, z be nonnegative real numbers such that $x^2+y^2+z^2+xyz=4$. Prove that

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2} \geq \frac{1}{4}+\frac{4}{(x+y)(y+z)(z+x)}.$$

Proposed by Marius Stănean, Zalău, România

S546. Solve in real numbers the system of equations

$$\begin{aligned} x^3-2xyz+y^3 &= \frac{1}{2} \\ y^3-2xyz+z^3 &= 1 \\ z^3-2xyz+x^3 &= -\frac{3}{2}. \end{aligned}$$

Proposed by Titu Andreescu, University of Texas at Dallas, USA

Undergraduate Problems

U541. Let R be a (not necessarily commutative) ring which contains $\mathbb{Q}$ as a subring and in which every non-invertible element is a divisor of zero. Assume that x and y are elements of R such that $xy = yx$ and $x^m = y^n = 1$, where m and n are relatively prime positive integers. Prove that $1 + x + y$ is invertible in R.

Proposed by Mircea Becheanu, Canada

U542. Evaluate

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \left(\frac{1}{\sqrt{2}} + \frac{1}{1+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{2}} \right)$$

Proposed by Toyesh Prakash Sharma, Agra, India

U543. Let n be a positive integer. Evaluate

$$\lim_{x\to 0} \frac{1}{x^{n+1}} \left(\int_0^x e^{t^n} \mathrm{d}t - x \right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U544. Find all real numbers x such that the sequence $(\cos 2^n x)_{n\geq 1}$ converges.

Proposed by Mihaela Berindeanu, Bucharest, România

U545. Prove that

$$\int_e^{4e} \frac{dx}{\ln x - \ln 2} \geq \frac{90e}{34\ln 2 + 15}$$

Proposed by Olimjon Jalilov, Tashkent, Uzbekistan

U546. Let p be an odd prime and let $n > 2$ be an integer. For any permutation f of the set $\{1, 2, \ldots, n\}$, $I(f)$ denote the number of inversions of f. Let A_j denotes the number of permutations f such that $I(f) \equiv j \pmod{p}$, for all $0 \leq j \leq p-1$. Prove that $A_1 = A_2 = \cdots = A_{p-1}$ if and only if $p \leq n$.
(Note: An inversion of f is a pair (i, j) such that $i > j$ and $f(i) < f(j)$.)

Proposed by Shubhrajit Bhattacharya, Chennai Mathematical Institute, India

Olympiad Problems

O541. Let a, b, c be the side-lengths triangle and let S be its area. Let R and r be the circumradius and the inradius of the triangle, respectively. Prove that

$$\cot^2 A + \cot^2 B + \cot^2 C \geq \frac{1}{5}\left(31 - 52\frac{r}{R}\right).$$

Proposed by Titu Andreescu, USA and Marius Stănean, România

O542. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$\frac{1}{x^3 + y^3 + z^3} + \frac{24}{xy + yz + zx} \geq 81.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O543. Let ABC be a triangle. Point M is the midpoint of side AB and D lies inside the triangle. Let E be the reflection of D with respect to M. Inside triangle ABC a point P is chosen such that DP and AC are parallel and $\angle CBP = \angle DAC$. Prove that $\angle ACP = \angle BCE$.

Proposed by Waldemar Pompe, Warsaw, Poland

O544. Find all triples of positive integers $(a,\, b,\, p)$, with p prime, such that

$$\frac{2^a + 2^b}{a + b} = a^p + b^p.$$

Proposed by Karthik Vedula, James S.Rickards High School, Tallahassee, USA

O545. Let a and b be integers with $a > 2$ and $\gcd(a,\, b) = 1$. Prove that for any positive integer n there are infinitely many positive integers k such that $(ak + b)^n$ divides $\binom{2k}{k}$.

Proposed by Navid Safaei, Sharif University of Technology , Tehran, Iran

O546. Let a, b, c be real numbers such that $a^2 + b^2 + c^2 = 6$. Find all possible values of the expression:

$$\left(\frac{a+b+c}{3} - a\right)^5 + \left(\frac{a+b+c}{3} - b\right)^5 + \left(\frac{a+b+c}{3} - c\right)^5.$$

Proposed by Marius Stănean, Zalău, România