Junior Problems

J469. Let a and b be distinct real numbers. Prove that

$$(3a+1)(3b+1) = 3a^2b^2 + 1$$

if and only if

$$\left(\sqrt[3]{a} + \sqrt[3]{b}\right)^3 = a^2b^2.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J470. Solve in real numbers the equation

$$\left(x^3 - 2\right)^3 + \left(x^2 - 2\right)^2 = 0.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

J471. Find all real numbers a for which the equation

$$\left(\frac{x}{x-1}\right)^2 + \left(\frac{x}{x+1}\right)^2 = a$$

has four distinct real roots.

Proposed by Adrian Andreescu, University of Texas at Austin, USA

J472. Let a, b, c be positive numbers such that $ab + bc + ca = 1$. Prove that

$$a\sqrt{b^2+1} + b\sqrt{c^2+1} + c\sqrt{a^2+1} \geq 2.$$

Proposed by An Zhenping, Xianyang Normal University, China

J473. Let a, b, c be distinct real numbers. Prove that

$$\left(\frac{a}{b-a}\right)^2 + \left(\frac{b}{c-b}\right)^2 + \left(\frac{c}{a-c}\right)^2 \geq 1.$$

Proposed by Anish Ray, Institute of Mathematics, Bhubaneswar, India

J474. Let k be a positive integer. Suppose x and y are positive integers such that for every positive integer n, $n > k$,

$$x^{n-k} + y^n \mid x^n + y^{n+k}.$$

Prove that $x = y$.

Proposed by Valentio Iverson, Medan, North Sumatra, Indonesia

Senior Problems

S469. Let $ABCD$ be a kite with $\angle A = 5\angle C$ and $AB \cdot BC = BD^2$. Find $\angle B$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

S470. Let x, y, z be positive real numbers such that $xyz(x+y+z) = 4$. Prove that

$$(x+y)^2 + 3(y+z)^2 + (z+x)^2 \geq 8\sqrt{7}.$$

Proposed by An Zhenping, Xianyang Normal University, China

S471. Prove that the following inequality holds for all positive real numbers a, b, c:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9(a+b+c)}{ab+bc+ca} \geq 8\left(\frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c}{c^2+ab}\right).$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

S472. Let ABC be a triangle with $\angle B$ and $\angle C$ acute and let D be the foot of the altitude from A. Prove that $\angle A$ is right if and only if

$$\frac{BD}{AB^2} + \frac{CD}{AC^2} = \frac{2}{BC}.$$

Proposed by Adrian Andreescu, University of Texas at Austin, USA

S473. Let a, b, c be positive real numbers. Prove that

$$(a-b)^4 + (b-c)^4 + (c-a)^4 \leq 6\left(a^4 + b^4 + c^4 - abc(a+b+c)\right).$$

Proposed by Nicuşor Zlota, Focşani, Romania

S474. Let a, b, c, d be real numbers such that $a^2 + b^2 + c^2 + d^2 = 12$. Prove that

$$a^3 + b^3 + c^3 + d^3 + 9(a+b+c+d) \leq 84.$$

Proposed by Marius Stănean, Zalău, Romania

Undergraduate Problems

U469. Let $x > y > z > t > 1$ be real numbers. Prove that

$$(x-1)(z-1)\ln y \ln t > (y-1)(t-1)\ln x \ln z.$$

Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain

U470. Let n be a positive integer. Evaluate

$$\lim_{x\to 0} \frac{1-\cos^n x \cos nx}{x^2}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

U471. Let $f(x) = ax^2 + bx + c$, where $a < 0 < b$ and $b\sqrt[3]{c} \geq \frac{3}{8}$. Prove that

$$f\left(\frac{1}{\Delta^2}\right) \geq 0,$$

where $\Delta = b^2 - 4ac$.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U472. If $f, g, h : \mathbb{R} \longrightarrow \mathbb{R}$ are derivatives, find whether or not the function $max\{f, g, h\}$ is the derivative of a function.

Proposed by Dorin Andrica, Babeş-Bolyai University, Cluj-Napoca, Romania

U473. For each continuous function $f : [0,1] \longrightarrow [0,\infty)$, let

$$I_f = \int_0^1 (2f(x) + 3x)\, f(x)\, dx$$

and

$$J_f = \int_0^1 (4f(x) + x)\sqrt{xf(x)}\, dx.$$

Find the minimum of $I_f - J_f$ over all such functions f.

Proposed by Titu Andreescu, University of Texas at Dallas, USA

U474. Let $f : [0,1] \longrightarrow \mathbb{R}$ be a differentiable function such that $f(1) = 0$ and

$$\int_0^1 x^n f(x) dx = 1.$$

Prove that

$$\int_0^1 \left(f'(x)\right)^2 dx \geq (2n+3)(n+1)^2.$$

When does the equality occur?

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

Olympiad Problems

O469. Find the greatest constant k such that the following inequality holds for all positive real numbers a and b:

$$\frac{1}{a^3} + \frac{1}{b^3} + \frac{k}{a^3+b^3} \geq \frac{16+4k}{(a+b)^3}.$$

Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam

O470. Let a, b, c, x, y, z be nonnegative real numbers such that $a \geq b \geq c$, $x \geq y \geq z$, and

$$a + b + c + x + y + z = 6.$$

Prove that

$$(a+x)(b+y)(c+z) \leq 6 + abc + xyz.$$

Proposed by Marius Stănean, Zalău, Romania

O471. Let a, b, c be positive real numbers such that $a^2 + b^2 + c^2 + abc = 4$. Prove that for all real numbers x, y, z, the following inequality holds:

$$ayz + bzx + cxy \leq x^2 + y^2 + z^2.$$

Proposed by An Zhenping, Xianyang Normal University, China

O472. Let ABC be an acute triangle and let A_1, B_1, C_1 be the tangency points of the incircle of ABC and BC, CA, AB, respectively. The circumcircles of triangles BB_1C_1 and CC_1B_1 intersect BC at A_2 and A_3, respectively. The circumcircles of $\triangle\ AB_1A_1$ and BA_1B_1 intersect AB at C_2 and C_3, respectively. The circumcircles of triangles AC_1A_1 and CC_1A_1 intersect AC at B_2 and B_3, respectively. Lines A_2B_1 and A_3C_1 meet at point A', lines B_2A_1 and B_3C_1 meet at B', and C_2A_1 and C_3B_1 meet at C'. Prove that lines A_1A', B_1B', and C_1C' are concurrent.

Proposed by Mihaela Berindeanu, Bucharest, Romania

O473. Let x, y, z be positive real numbers such that $x^6 + y^6 + z^6 = 3$. Prove that

$$x + y + z + 12 \geq 5\left(x^6y^6 + y^6z^6 + z^6x^6\right).$$

Proposed by Hoan Le Nhat Tung, Hanoi, Vietnam

O474. Let $P(x) = a_dx^d + a_{d-1}x^{d-1} + \cdots + a_2x^2 + a_0$ be a polynomial with positive integer coefficients of degree $d \geq 2$. We define the sequence $(b_n)_{n\geq 1}$, where $b_1 = a_0$ and $b_n = P(b_{n-1})$, for all $n \geq 2$. Prove that for all $n \geq 2$, there is a prime p such that $p \mid b_n$ and p does not divide $b_1 \cdots b_{n-1}$.

Proposed by Navid Safaei, Sharif University of Technology, Tehran, Iran