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MathemAttic

No. 27

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2021**.

MA131. *Proposed by Ed Barbeau.*

Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the smallest $m+1$ integers is equal to the sum of the largest m integers.

MA132. *Proposed by Nguyen Viet Hung.*

Find all pairs (x, y) of positive integers satisfying the equation

$$x^2 - 2x + 29 = 7^x y.$$

MA133. If the perimeter of an isosceles right-angled triangle is 8, what is its area?

MA134.

> 80 students responded to a survey about sports they played.
>
> 30 played basketball.
>
> 26 played rugby.
>
> 28 played hockey.
>
> 12 played basketball and rugby.
>
> 8 played hockey and rugby.
>
> x played basketball and hockey only.
>
> 4 played all 3 sports.
>
> Twice as many played none of the 3 sports as played basketball and hockey only.

If a student is picked at random from the whole group, what is the probability that the student plays only 1 of the 3 sports?

MA135. *Sent in by Ed Barbeau, from correspondence with Harold Reiter.*

Solve the alphametic
$$SETA - ATES = EAST$$
where $S > E > T > A$ are digits in the 4-digit numbers.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2021**.

MA131. *Proposé par Ed Barbeau.*

Déterminer tous les ensembles comprenant un nombre impair $2m + 1$ d'entiers positifs consécutifs, où m est un entier, $m \geq 1$, sachant qu'en plus la somme des $m + 1$ plus petits éléments de l'ensemble égale la somme des m plus grands éléments.

MA132. *Proposé par Nguyen Viet Hung.*

Déterminer toutes les paires d'entiers positifs, (x, y), vérifiant
$$x^2 - 2x + 29 = 7^x y.$$

MA133. Si le périmètre d'un triangle isocèle rectangle est de 8, quelle en est sa surface ?

MA134.

80 étudiants répondent un sondage visant connatre leurs activités sportives.

30 jouent au basketball.

26 jouent au rugby.

28 jouent au hockey.

12 jouent au basketball et au rugby.

8 jouent au hockey et au rugby.

> x jouent au basketball et au hockey, et rien d'autre.
>
> 4 jouent au basketball, au rugby, et au hockey.
>
> Enfin, le nombre d'étudiants jouant aucun sport est le double du nombre d'étudiants qui jouent au basketball et au hockey, et rien d'autre.

Si un étudiant est choisi au hasard, quelle est la probabilité qu'il joue exactement 1 des sports ?

MA135. *Envoyé par Ed Barbeau, de la correspondance avec Harold Reiter.*

Résoudre le cryptarithme

$$SETA - ATES = EAST$$

où $S > E > T > A$ sont les chiffres des nombres à 4 chiffres en question.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(2), p. 72–73.

MA106. Suppose

$$N = 1 + 11 + 101 + 1001 + 10001 + \cdots + 1000\cdots01,$$

where there are 50 zeros in the last term. When N is written as a single integer in decimal form, find the sum of its digits.

Originally Question 27 of 2012 University of Cape Town Mathematics Competition (Grade 12).

We received 15 submissions of which 12 were correct and complete. We present the solution by Amir Ali Fayazi.

Let $A = 1{+}10{+}10^2{+}...{+}10^{51}$ and $B = \overbrace{1 + \cdots + 1}^{51}$. It is easy to see that $N = A{+}B$. By assumption, $A = \overbrace{111\ldots11}^{52}$ and $B = 51$. Hence, we infer that $N = \overbrace{111\ldots11}^{50}62$. This yields that the sum of digits of N is equal to $50 \times 1 + 6 + 2 = 58$.

MA107. A wooden cube is painted red on five of its six sides and then cut into identical small cubes, of which 52 have exactly two red sides. How many small cubes have no red sides?

Originally Question 27 of 2013 University of Cape Town Mathematics Competition (Grade 12).

We received 9 submissions of which 6 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

Denote the number of small cubes along an edge of the original cube by n (so there are a total of n^3 small cubes). For concreteness, assume that the unpainted face of the original cube is the top face. A small cube has exactly two red sides if it is on one of the four vertical edges, but is not one of the four corner cubes on the bottom face (there are $4(n-1)$ of these) or if it is on one of the edges of the bottom face, but is not one of the four corner cubes (there are $4(n-2)$ of these). Thus $4(n-1)+4(n-2) = 52$ and hence $n = 8$. A small cube has no paint on it if it is in the interior of the cube (there are $6^3 = 216$ of these) or it is in the interior of the top (unpainted) face (there are $6^2 = 36$ of these). This gives a total of 252 unpainted small cubes.

For the record, there are four cubes with three faces painted and 204 cubes with one face painted.

MA108. Suppose that a, b, c and d are positive integers that satisfy the equations

$$ab + cd = 38, \quad ac + bd = 34, \quad ad + bc = 43.$$

What is the value of $a + b + c + d$?

Originally Question 29 of 2013 University of Cape Town Mathematics Competition (Grade 12).

We received 13 correct solutions and one incomplete submission. Six of the solvers gave the solution below.

Adding the equations $ac + bd = 34$ and $ad + bc = 43$ yields $(a + b)(c + d) = 77$. Since each factor on the left side must exceed 1, one of the factors must be 7 and the other 11. Hence $a + b + c + d = 7 + 11 = 18$.

Editor's comment. Strictly speaking, this solution should perhaps be regarded as incomplete, as it is implicitly assumed the system is viable. The number 38 plays no role and the numbers 34 and 43 are involved insofar as their sum is 77. They could have been replaced by other numbers for which the system has no solutions in positive integers. So a complete solution should indicate that the situation is possible. However, it might be that the solution given was intended by the poser of the problem, so we can allow some latitude.

More than half of the solvers made the effort to find all the positive integer solutions to the system. The more or less efficient ways of narrowing down and checking possibilities were not sufficiently edifying to include here. They found that (a, b, c, d) has to be one of $(2, 5, 7, 4)$, $(4, 7, 5, 2)$, $(5, 2, 4, 7)$, $(7, 4, 2, 5)$. Some noted that the equations remain unchanged under the permutations $(ab)(cd)$, $(ac)(bd)$, $(ad)(bc)$.

A different choice of integers for the right side of the equations makes things more interesting. Suppose that we ask for the sum $a+b+c+d$ when a, b, c, d are positive integers for which $ab + cd = 34$, $ac + bd = 46$ and $ad + bc = 31$. In this case, we find that there are at least two solutions (a, b, c, d) to the system, $(1, 6, 4, 7)$ and $(2, 5, 3, 8)$, both of which yield 18 as the desired sum.

Competitors in the essay contest introduced in issue 5 might consider the situation when the three numbers 38, 34 and 43 are replaced by p, q and r. When does the system admit a solution? How many values of the sum $a + b + c + d$ are possible?

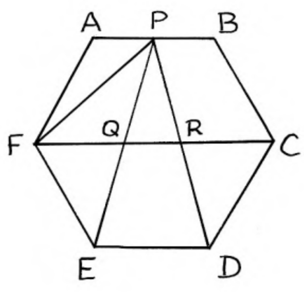
MA109. Ten equal spheres are stacked to form a regular tetrahedron. How many points of contact are there between the spheres?

Originally Question 26 of 2016 University of Cape Town Mathematics Competition (Grade 12).

We received 7 submissions, of which 6 were correct and complete. We present the solution by Richard Hess, modified by the editor.

Each of the four corner spheres touches three other spheres and each of the six edge spheres touches six other spheres. This gives a total of 48 touches. Each point of contact is counted twice, therefore the number of points of contact is 24.

MA110. In the figure, $ABCDEF$ is a regular hexagon and P is the midpoint of AB.



Find the ratio

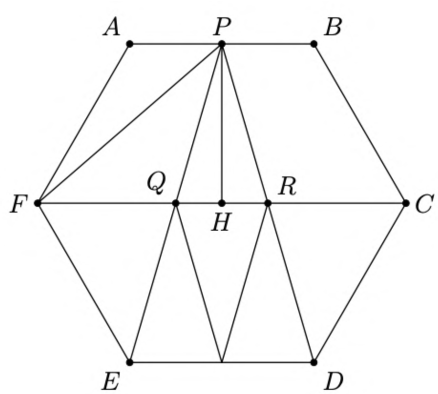
$$\frac{Area(DEQR)}{Area(FPQ)}.$$

Originally Question 28 of 2012 University of Cape Town Mathematics Competition (Grade 12).

We received 8 solutions, of which 5 were correct. We present the solution by Dominique Mouchet, modified by the editor.

Sans perte de généralité, on peut supposer que les côtés de l'hexagone sont de longueur 1. Soit H le pied de la perpendiculaire à la droite FC passant par le point P. Comme PH est la hauteur du triangle équilatéral ABH, on peut utiliser le théorème de Pythagore pour montrer que $PH = \frac{\sqrt{3}}{2}$. De plus, comme PH correspond à la moitié de la hauteur de $\triangle PED$ abaissée de P sur ED, on peut se servir des similitudes des triangles pour montrer que

$$QR = \frac{ED}{2} = \frac{1}{2}.$$



En reliant maintenant les points Q et R au point milieu de ED, on obtient une subdivision du $\triangle PED$ en quatre triangles congrus. L'un d'eux étant $\triangle PQR$. Si l'on se sert des crochets pour désigner l'aire, on a alors

$$[DEQR] = 3 \cdot [PQR] = 3\frac{\frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{2} = \frac{3\sqrt{3}}{8}.$$

Par un simple calcul, on obtient

$$FQ = \frac{FC - QR}{2} = \frac{2 - \frac{1}{2}}{2} = \frac{3}{4},$$

de sorte que

$$[FQP] = \frac{FQ \cdot PH}{2} = \frac{\frac{3}{4} \cdot \frac{\sqrt{3}}{2}}{2} = \frac{3\sqrt{3}}{16}.$$

Ainsi,

$$\frac{[DEQR]}{[FPQ]} = \frac{\left(\frac{3\sqrt{3}}{8}\right)}{\left(\frac{3\sqrt{3}}{16}\right)} = 2.$$

PROBLEM SOLVING VIGNETTES

No.18

Shawn Godin

Fun with Mental Math

I recently reread a couple of articles by Martin Gardner [2], specifically, Chapter 6 *Calculating Prodigies* and Chapter 7 *Tricks of Lightning Calculators*. Nowadays, the need to be able to do calculations in our head isn't that great. Most people have quick access to a calculator at most times, in their phone, and calculators are used extensively in school mathematics classes. However, being able to do some mental calculations is a good exercise and allows us to become closer to our friends, the numbers. Mental calculations may allow us to recognize some patterns and possibly even spot an error when something is miskeyed into our calculator. In this issue we will look at some fun and impressive calculations that you can do in your head with a little bit of practice.

1 2 is smaller than 5

In general, when doing mental calculations, it is easier to deal with smaller numbers. To test this theory out, I want you to do two different computations. Time yourself for each one to see which one is quicker. Your two computations are:

1. 8317×5
2. $96510 \div 2$

Which one did you do more quickly? If you are pretty good with mental calculations, they might have been close. If so, try this: have a friend pick two random numbers with a large number of digits. Time yourself finding the product of the first and 5 and then add a 0 to the end of the other number and time yourself dividing it by 2.

What is the point of these computations? Since we generally write numbers in base $10 = 2 \times 5$, then we can write $2 = 10 \div 5$ and $5 = 10 \div 2$. Thus, our first example can be rewritten as

$$8317 \times 5 = 8317 \times (10 \div 2) = 8317 \times 10 \div 2 = 83170 \div 2$$

in other words, we can turn a problem of multiplying any number by 5 into a division by 2. Thus, the second computation I gave for you is equivalent to 9651×5. There are a couple of advantages to this: division by 2 is done quite easily in your head, and if you are trying to impress someone you can write down the answer *from left to right*! Similarly, you could change multiplication by 2 into division by 5 but, other than writing the answer from left to right, there isn't much of an advantage to this method.

We saw that we can change multiplication by 5 (or 2) into division by 2 (or 5). Similarly, we can turn division by 5 into multiplication by 2. So now $6874 \div 5$ becomes $687.4 \times 2 = 1374.8$.

2 Squares and then some

A fairly well known "trick" for squaring a two-digit number that ends in 5 is to multiply the tens digit by the next integer, write 25 after it and you have your answer. This is one of the first tricks explained by mathemagician Art Benjamin in his book on mental math [1]. (If you are interested in mental calculations, this book is a must read). For example, to calculate 65^2, since $6 \times 7 = 42$, our result is $65^2 = 4225$. Why does this work? A number that we have described is of the form $10d + 5$ for some positive integer d, less than 10. Squaring yields

$$\begin{aligned}(10d+5)^2 &= 100d^2 + 100d + 25\\ &= 100d(d+1) + 25\end{aligned}$$

The $100d(d+1)$ yields the product of the tens digit, d with the next integer, $d+1$ followed by two zeros since it is multiplied by 100. Hence, when the 25 is added, we have our result. Notice, even though I stated that d was a positive integer less than 10 there is nothing in the derivation that would change if d was just a positive integer. Thus, since $24 \times 25 = 600$, the rule gives $245^2 = 60025$, which can easily be verified.

This method can be extended to multiplying two two-digit numbers with the same tens digit and whose units digits add to 10, like 43 and 47. Doing the calculation we get $43 \times 47 = 2021$, where $4 \times 5 = 20$ and $3 \times 7 = 21$. It looks like our first rule is a special case of a more general rule. We can show that the new rule works as well. Our numbers would be $10t + u$ and $10t + (10 - u)$, where t is the common tens digit, u is the units digit of one number and hence $10 - u$ is the units digit of the other. Multiplying the two numbers yields

$$\begin{aligned}(10t+u) \times [10t + (10-u)] &= 100t^2 + 100t - 10ut + 10ut + u(10-u)\\ &= 100t(t+1) + u(10-u)\end{aligned}$$

where $100t(t+1)$ is the product of the common tens digit with the next integer, followed by two zeros, and $u(10-u) < 100$ is the product of the two units digits. So to calculate 84×86, since $8 \times 9 = 72$ and $4 \times 6 = 24$, the result is 7224.

Again, there is nothing that requires t to be a single digit so we can extend the idea. Since $30 \times 31 = 930$ and $8 \times 2 = 16$, we can deduce that $308 \times 302 = 93016$.

Back to squaring, what if the number doesn't end with a 5? It turns out we can use an algebraic identity that *should* be familiar to most high school students:

$(a-b)(a+b) = a^2 - b^2$. Rearranging we can get

$$a^2 = (a-b)(a+b) + b^2.$$

How is this helpful? If we chose b such that one of the numbers $a-b$ or $a+b$ is "nice", our computation becomes easier.

For example, to calculate 97^2, choosing $b = 3$ we get

$$\begin{aligned} 97^2 &= (97+3)(97-3) + 3^2 \\ &= 100 \times 94 + 3^2 \\ &= 9409 \end{aligned}$$

3 *Special* products

Let's take a look at some products that you can perform in your head that take advantage of the number that we are multiplying by. For example, determine the following product 45×37. Now, perform the following algorithm:

- add the digits of 45: $4 + 5 = 9$,
- create the four-digit number with first and last digits the same as 45 and the middle two the sum from the last step: 4995,
- divide the last number by three: $4995 \div 3$.

Hopefully, you will have calculated $45 \times 37 = 1665 = 4995 \div 3$.

Why does this work? Notice that $37 \times 3 = 111$. When we multiply by 111 using the traditional algorithm, we end up with three copies of 45 that have been shifted over (which corresponds to 45, 450 and 4500). When we add them, two pairs of 4 and 5 get added together, and the last ones "stand alone".

$$\begin{array}{cccc} & & 4 & 5 \\ \times & 1 & 1 & 1 \\ \hline & & 4 & 5 \\ & 4 & 5 & \\ 4 & 5 & & \\ \hline 4 & 9 & 9 & 5 \end{array}$$

By choosing a number whose digits sum to less than 10 we don't have to worry about a carry. As such, we can picture the number 4995 and do the division by three in our head, as before. Similar to our first tricks, this one turns a multiplication into a division.

$$45 \times 37 = 45 \times 111 \div 3.$$

We can still use this rule when the digits of the number being multiplied by 37 sum to ten or more. In this case, we will have carries to deal with. We will only look at the units digit of the sum, and then do as we would in our other algorithm except add 1 to the first two digits. For example, to do 84×37, since $8+4=12$ we would calculate $9324 \div 3 = 3108$ (where $9 = 8+1$ and $3 = 2+1$ come from carries).

We can accomplish even more impressive feats using $143 \times 7 = 1001$. Then, to multiply any three-digit number by 143 we divide the six-digit number formed by concatinating two copies of the three digit number together and dividing the result by 7. For example, to calculate 837×143 you would do $837\ 837 \div 7 = 119\ 691$ which can be done in your head quite quickly, with practice. On top of that, if you write

$$\underline{\qquad} \times 143 =$$

and have a friend fill in the blank, and then instruct them to calculate it so they can check your answer, you may be done before they can type it into the calculator.

A plethora of special numbers can be found by looking at the factorizations of numbers that would be easy to multiply by and hoping they have a factor that is easy to divide by in your head. For example, $10101 = 7 \times 1443$, so multiplication of a two-digit number by 1443 is accomplished by dividing the six-digit number formed by concatinating three copies of the number to be multiplied by seven. For example, $29 \times 1443 = 292929 \div 7 = 41847$.

As we have done elsewhere, we can turn division into multiplication as well. Using our last example, if we asked someone to multiply any two-digit number by 1443 and give us the answer, you could do the division in your head very quickly. Suppose they give you number 109 668. If their number was x, then we know that $1443x = 109\ 668$ and hence $7 \times (1443x) = 10101x$. Since their original number was two-digit we know that this result will be a six-digit number with their number repeated three times. Therefore, if we start the product $109\ 668 \times 7$ in our head, we can stop after the first two digits, and we will have their number.

Have fun developing algorithms of your own!

4 Problems

Here are a few exercises based on what we have discussed:

1. Determine a quick method for multiplying and dividing by 125.
2. Determine an algorithm for multiplying based on $15\ 873 \times 7 = 111\ 111$. (You can do a similar one based on $37037 \times 3 = 111111$, but if you have shown the first one, this may not be that impressive.)
3. Determine an algorithm for multiplying based on $142\ 857\ 143 \times 7 = 1\ 000\ 000\ 001$. (Discussed in [2]).
4. Determine an algorithm for multiplying 8 335 by a three-digit number discovered by Gardner and talked about in [2].

5. Determine an algorithm for dividing by 87 143. That is, you should be able to get someone to multiply any number up to four-digits by 87 143 and you should be able to determine the original number, in your head.

References

[1] Benjamin, Art and Shermer, Michael, *Secrets of Mental Math*, Three Rivers Press, New York, 2006.

[2] Gardner, Martin, *Mathematical Carnival*, Mathematical Association of America, Washington, 1989.

The MathemAttic Article Contest

In issue 5 of Volume 47, the editorial staff at MathemAttic announced an article writing contest. For the competition we are looking for expository articles in mathematics that would be of interest to the readers of MathemAttic. We will publish a number of the strongest papers in MathemAttic next year. There will also be a few prizes from the CMS available for exceptional article.

We are particularly interested in hearing from students (high school or university), but we will accept articles from anybody (prizes will be limited to students). If you are a student, please provide us with your grade, age, and school. A word on credit: make sure you (briefly) acknowledge anybody who helped you significantly with research or with the overall presentation.

The contest deadline will be November 1, 2021.
Please email your submissions to `MathemAttic@cms.math.ca` with "MA Article Contest" in the subject line.

For more details, check issue 5 announcement.

OLYMPIAD CORNER

No. 395

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **November 30, 2021**.

OC541. In a convex quadrilateral $ABCD$, suppose $\angle ABC = \angle ACD$ and $\angle ACB = \angle ADC$. Assume that the center O of the circle circumscribed to the triangle BCD is different from point A. Prove that triangle OAC is a right triangle.

OC542. Let $x_1, x_2, \ldots, x_n$ be positive integers. Assume that in their decimal representations no x_i "is an extension" of another x_j. For instance, 123 is an extension of 12, 459 is an extension of 4, but 134 is not an extension of 123. Prove that

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 3.$$

OC543. There are 50 cards in a box with the first 100 positive integers written on them. That is, the first card has number 1 on one side and number 2 on the other side, the second card has number 3 on one side and number 4 on the other, and so on up to the 50-th card which has number 99 on one side and 100 on the other side. Eliza takes four cards out of the box and calculates the sum of the eight numbers written on them. How many distinct sums can Eliza get?

OC544. Prove that if $n \geq 2$ is an integer, then there exist invertible matrices $A_1, A_2, \ldots, A_n \in \mathcal{M}_2(\mathbb{R})$ with nonzero entries such that

$$A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1} = (A_1 + A_2 + \cdots + A_n)^{-1}.$$

OC545. Solve in real numbers the system of equations

$$\begin{cases} x^2y + 2 &= x + 2yz \\ y^2z + 2 &= y + 2zx \\ z^2x + 2 &= z + 2xy \end{cases}$$

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2021**.

OC541. Un quadrilatère convexe $ABCD$ est tel que $\angle ABC = \angle ACD$ et $\angle ACB = \angle ADC$. Supposer que A n'est pas égal à O, centre du cercle circonscrit du triangle BCD. Démontrer que le triangle OAC est rectangle.

OC542. Soient $x_1, x_2, \ldots, x_n$ des entiers positifs. Supposer que dans leurs représentations décimales, aucun x_i est un prolongement d'un autre x_j. Par exemple, 123 est un prolongement de 12 ; 459 est un prolongement de 4 ; 134 n'est pas un prolongement de 123. Démontrer que

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} < 3.$$

OC543. Dans une boîte se trouvent 50 cartes sur lesquelles sont inscrits les entiers de 1 à 100, d'une façon un peu spéciale : la première carte a l'entier 1 d'un côté et 2 de l'autre, la deuxième carte a l'entier 3 d'un côté et 4 de l'autre, et ainsi de suite jusqu'à la cinquantième carte qui a l'entier 99 d'un côté et 100 de l'autre. Élise retire quatre cartes de la boîte et calcule la somme des huit entiers qu'elle y retrouve. Combien de sommes distinctes Élise peut-elle obtenir ?

OC544. Soit $n \geq 2$ un entier. Démontrer qu'il existe des matrices inversibles $A_1, A_2, \ldots, A_n \in \mathcal{M}_2(\mathbb{R})$ ayant des entrées non nulles, de façon à ce que

$$A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1} = (A_1 + A_2 + \cdots + A_n)^{-1}.$$

OC545. Déterminer toute solution réelle du système d'équations

$$\begin{cases} x^2y + 2 & = & x + 2yz \\ y^2z + 2 & = & y + 2zx \\ z^2x + 2 & = & z + 2xy \end{cases}$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(2), p. 82–83.

OC516. Pasha placed numbers from 1 to 100 in the cells of the square 10×10, each number exactly once. After that, Dima considered all sorts of squares, with the sides going along the grid lines, consisting of more than one cell, and painted in green the largest number in each such square (one number could be coloured many times). Is it possible that all two-digit numbers are painted green?

Originally from 2019 Caucasus Mathematical Olympiad, Senior, First Day, Question 1.

We received 7 correct submissions. We present a typical solution.

We show that this is impossible. First, note that if a number is the largest in some $n \times n$ square with $n \geqslant 2$, then the number must also be the largest in some 2×2 square inside the $n \times n$ square. Therefore, when painting the numbers, we can consider only the 2×2 squares. The count of 2×2 squares is $9^2 = 81$, as each one is uniquely determined by its cell in the bottom left corner. Hence, at most 81 numbers can be painted green. However, we have $100 - 1 - 9 = 90$ two-digit numbers. In conclusion, not all two-digit numbers can be painted green.

OC517. Denote by $\mathbb{N}$ the set of positive integers $1, 2, 3, \ldots$ Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $n! + f(m)!$ divides $f(n)! + f(m!)$ for all m, $n \in \mathbb{N}$.

Originally from 2019 XXVI Macedonian Mathematical Olympiad FON University, Problem 4.

We received 6 submissions of which 4 were correct and complete. We present a typical solution.

Let $P(m, n)$ denote the statement that $n! + f(m)! \mid f(n)! + f(m!)$. Also, let p denote a prime number.

$P(1, 1)$ implies that $1 + f(1)! \mid f(1)! + f(1)$ and $1 + f(1)! \mid f(1) - 1$. Since $|f(1) - 1| < f(1)! + 1$, we deduce $f(1) = 1$.

$P(1, n)$ yields $n! + 1 \mid f(n)! + 1$. This implies $n! \leq f(n)!$ and $n \leq f(n)$.

$P(1, p - 1)$ yields $(p - 1)! + 1 \mid f(p - 1)! + 1$. However, by Wilson's Theorem, $p \mid (p - 1)! + 1$ and hence $p \mid f(p - 1)! + 1$. We can see that $f(p - 1) \geq p$ is not possible, as this would imply that $p|1$. Therefore, $f(p-1) < p$ and $f(p-1) = (p-1)$ for any prime number p.

$P(m, p-1)$ yields $(p-1)!+f(m)! \mid (p-1)!+f(m!)$ and so, $(p-1)!+f(m)! \mid f(m!)-f(m)!$ for all primes p. We fix m and take arbitrarily large prime p to conclude that $f(m!) = f(m)!$ for all $m \in \mathbb{N}$.

$P(m,n)$ can be rewritten as $n! + f(m)! \mid f(n)! + f(m)!$ and that in turn implies $n! + f(m)! \mid f(n)! - n!$ for all $m, n \in \mathbb{N}$. We fix n and take arbitrarily large m to conclude $f(n)! = n!$ and $f(n) = n$.

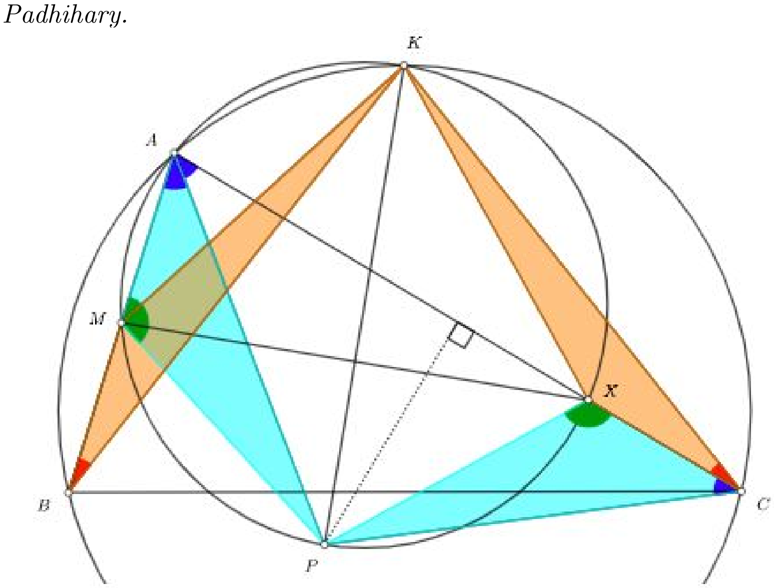
It can be easily checked that the identity function, $f(n) = n$, satisfies the problem conditions.

OC518. In a triangle ABC with $AB \neq AC$ let M be the midpoint of AB, let K be the midpoint of the arc BAC in the circumcircle of ABC, and let the perpendicular bisector of AC meet the bisector of the angle BAC at P. Prove that A, M, K, P are concyclic.

Originally from 2020 Caucasus Mathematical Olympiad, Senior, Second Day, Question 7.

We received 7 submissions of which 6 were correct and complete. We present three solutions.

Solution 1, submitted independently by UCLan Cyprus Problem Solving Group and Taes Padhihary.



Let ω be the circumcircle of the triangle AMP and let X be the other point of intersection of AC with ω. Then $\angle PMA = \angle PXC$. Since PA is on the perpendicular bisector of AC, then

$$\angle PCX = \angle PAC = \angle BAC/2 = \angle MAP.$$

So the triangles PMA and PXC are similar. Since $PA = PC$, the two triangles are equal. It follows that $MB = MA = XC$. Since K is on the perpendicular bisector of BC, then $KB = KC$. Since B, A, K, C are concyclic, then

$$\angle KBM = \angle KBA = \angle KCA = \angle KCX.$$

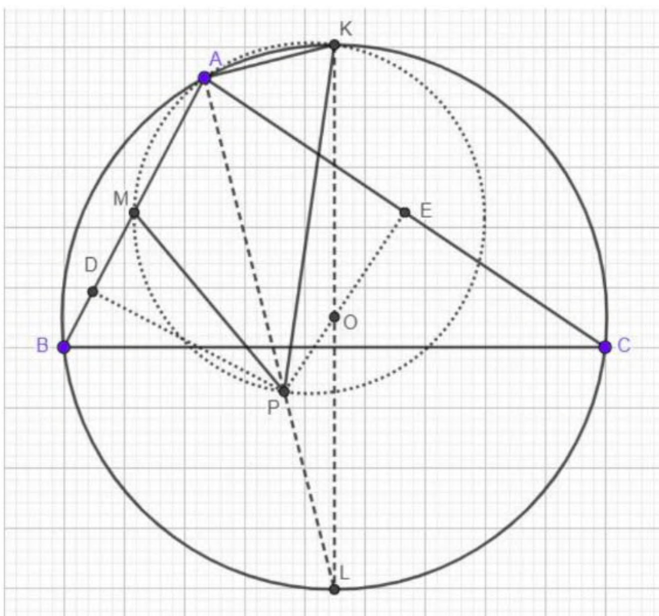
It follows that the triangles KMB and KXC are equal.

We now see that $\angle KMA = \angle KXA$. So K also belongs on ω and therefore A, M, K, P are concyclic, as required.

Solution 2, by Theo Koupelis.

Let O be the centre and R be the radius of the circumcircle of triangle ABC. Let L be the midpoint of arc BC. Then LOK is a diameter, $\angle LAK = \angle PAK = 90°$, and $\angle ALK = \frac{|B-C|}{2}$. Therefore

$$\begin{aligned} AK &= 2R \cdot \sin \frac{|B-C|}{2} = \frac{2R}{\sin \frac{A}{2}} \cdot \sin \frac{|B-C|}{2} \cos \frac{B+C}{2} \\ &= \frac{2R}{2 \sin \frac{A}{2}} \cdot |\sin B - \sin C| = \frac{|b-c|}{2 \sin \frac{A}{2}}. \end{aligned}$$



Let D be the foot of the perpendicular from P on AB. Then, because P is on the angle bisector of $\angle BAC$, we have $AD = AE = b/2$ and $MD = \frac{|b-c|}{2}$. Also, $DP = EP = AP \sin \frac{A}{2}$. Thus,

$$\frac{MD}{DP} = \frac{|b-c|}{2 \cdot AP \sin \frac{A}{2}} = \frac{AK}{AP}.$$

Therefore the right triangles MDP and KAP are similar. Thus $\angle AKP = \angle DMP$, and A, M, K, P are concyclic.

Solution 3, submitted independently by Corneliu Manescu-Avram and Ivko Dimitrić.

Choose a complex system of coordinates with the circumcircle of triangle ABC as unit circle. Let $A(a^2)$, $B(b^2)$, $C(c^2)$, where a, b, c are complex numbers with $|a| = |b| = |c| = 1$. Denote the complex coordinates of the other points by the same small letters. Then $k = bc$ and $m = (a^2 + b^2)/2$.

The equation of the perpendicular bisector of AC is $z = a^2c^2\bar{z}$ and the equation of the angle bisector of $\angle BAC$ is $z - a^2bc\bar{z} = a^2 - bc$, so that

$$p = \frac{c(a^2 - bc)}{c - b}.$$

From

$$\overline{\frac{(a^2 - m)(k - p)}{(a^2 - p)(k - m)}} = \overline{\frac{c(a^2 - b^2)}{b(c^2 - a^2)}} = \frac{c(a^2 - b^2)}{b(c^2 - a^2)} = \frac{(a^2 - m)(k - p)}{(a^2 - p)(k - m)}$$

we deduce that the points A, M, K, and P are concyclic.

OC519. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence:

$$x, x + 1, x + 2, x + 3, \ldots$$

Originally from 1993 Canadian Mathematical Olympiad.

We received 16 submissions, all of which were correct and complete. We present a typical solution.

Suppose there exists three distinct terms $x + i, x + j, x + k$ that form a geometric progression. Without loss of generality, we can assume $(x + i)(x + k) = (x + j)^2$. This implies $x(i + k - 2j) = j^2 - ik$. If $i + k = 2j$, then $ik = j^2$ and $i = k = j$. This is a contradiction since $x + i, x + j, x + k$ are assumed to be distinct. Thus $i + k \neq 2j$ and $x = (j^2 - ik)/(i + k - 2j)$ is rational.

Suppose the converse, namely $x = p/q$ is rational, for some integers p and q. Moreover, $q \neq 0$ and without loss of generality, we can assume $q > 0$. We distinguish the following three cases.

If $p = 0$, then $x = 0$ and $x + 1, x + 2, x + 4$ trivially form a geometric progression.

If $p > 0$ we consider $x, x + p, x + p(q + 2)$. This is a geometric progression as

$$x(x + p(q + 2)) = \frac{p}{q} \cdot \frac{p + pq(q + 2)}{q} = \left(\frac{p(q + 1)}{q}\right)^2 = (x + p)^2\,.$$

If $p < 0$, then we select n such that $x + n > 0$. Then we can find three distinct terms of $x + n, x + n + 1, x + n + 2 \dots$ which form a geometric progression. These three terms are in the original sequence, as well.

OC520. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometers driving from start; Rob makes a 90° right turn after every r kilometers driving from start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction. Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

Originally from 2009 Asian Pacific Mathematics Olympiad, Problem 5.

We received 2 correct submissions. We present the solution by Oliver Geupel.

We show that the coprime numbers ℓ and r satisfy the required condition if and only if either $\ell \equiv r \equiv 1 \pmod 4$ or $\ell \equiv r \equiv 3 \pmod 4$.

Consider the car after having driven the first ℓr kilometers of the route. It will have made r left turns and ℓ right turns. If $\ell \equiv r + 2 \pmod 4$, then it faces opposite to its direction at start; whence it will return to Argovia after driving ℓr more kilometers. If $\ell \equiv r \pm 1 \pmod 4$, then the car faces to the right or left relative to its direction at start. Then, its positions at start, after ℓr kilometers, after $2\ell r$ kilometers, and after $3\ell r$ kilometers will form the vertices of a square, and it will return to Argovia after $4\ell r$ kilometers. But if the car returns to Argovia after a multiple of ℓr kilometers, then its route is periodic, and the car cannot reach Zillis if it is sufficiently far away.

It remains to consider $\ell \equiv r \pmod 4$. Consider the route in the complex plane with unit length 1 kilometer, where the coordinates of Argovia and Zillis are 0 and a positive real number, respectively. Let i be the complex number $\sqrt{-1}$. After ℓr kilometers from start, the position of the car is

$$p = \sum_{k=0}^{\ell r-1} i^{\lfloor k/\ell \rfloor}(1/i)^{\lfloor k/r \rfloor} = \sum_{k=0}^{\ell r-1} i^{\lfloor k/\ell \rfloor - \lfloor k/r \rfloor}.$$

For $0 \le k < \ell r$, let $s_k = k - \ell\lfloor k/\ell \rfloor$ and $t_k = k - r\lfloor k/r \rfloor$.

In the case $\ell \equiv r \equiv 1 \pmod 4$, we compute

$$p = \sum_{k=0}^{\ell r-1} i^{\lfloor k/\ell \rfloor - \lfloor k/r \rfloor} = \sum_{k=0}^{\ell r-1} i^{(k-s_k)-(k-t_k)} = \sum_{k=0}^{\ell r-1} i^{t_k - s_k}.$$

The map $k \mapsto (s_k, t_k)$ constitutes a bijection between the sets $\{0, 1, \dots, \ell r - 1\}$

and $\{0,1,\ldots,\ell-1\} \times \{0,1,\ldots,r-1\}$ by the Chinese remainder theorem. Hence

$$p = \sum_{k=0}^{\ell r-1} i^{t_k - s_k} = \left(\sum_{j=0}^{r-1} i^j\right)\left(\sum_{k=0}^{\ell-1} i^{-k}\right) = 1 \cdot 1 = 1.$$

Similarly, in the case $\ell \equiv r \equiv 3 \pmod 4$, we obtain

$$\begin{aligned} p = \sum_{k=0}^{\ell r-1} i^{\lfloor k/\ell\rfloor - \lfloor k/r\rfloor} = \sum_{k=0}^{\ell r-1} i^{(s_k-k)-(t_k-k)} = \sum_{k=0}^{\ell r-1} i^{s_k - t_k} &= \left(\sum_{j=0}^{\ell-1} i^j\right)\left(\sum_{k=0}^{r-1} i^{-k}\right) \\ &= i \cdot \frac{1}{i} = 1. \end{aligned}$$

Hence, for every natural number n, after $n\ell r$ kilometers from start, the car is at position n facing towards the positive real axis. Consequently, it will eventually reach Zillis.

FOCUS ON...

No. 47

Michel Bataille

Examples of algebraic identities

Introduction

Problem **2976** [2004 : 429,432] asked for a proof of the following inequality that holds for all real numbers a, b, c:

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (ab + bc + ca)^3.$$

Kee-Wai Lau's solution was amazingly short: the inequality follows from the identity

$$\prod_{\text{cyclic}} (a^2 + ab + b^2) - (ab + bc + ca)^3$$
$$= \frac{1}{6}\left[2(ab + bc + ca)^2 \sum_{\text{cyclic}} (a - b)^2 + (a + b + c)^2 \sum_{\text{cyclic}} a^2(b - c)^2\right].$$

In comparison, my own solution – one-page long, distinguishing several cases – looked very laborious! This problem reinforced my idea of noting down the interesting identities as I met them. This number offers a selection of these identities, with the hope that they will be useful to the beginner and perhaps encourage her/him to start a collection. The chosen ones are among those appearing frequently in solutions and examples of their interventions will also be given.

Checking the identities is an exercise left to the reader.

About the polynomial $p(x, y, z) = x^3 + y^3 + z^3 - 3xyz$

The identity to be remembered is

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)[x^2 + y^2 + z^2 - (xy + yz + zx)]. \qquad (1)$$

The difference $x^2 + y^2 + z^2 - (xy + yz + zx)$ can also be written as

$$\frac{1}{2}[(x - y)^2 + (y - z)^2 + (z - x)^2] \quad \text{or}$$
$$(x + y + z)^2 - 3(xy + yz + zx) \quad \text{or}$$
$$\frac{1}{2}[3(x^2 + y^2 + z^2) - (x + y + z)^2].$$

Also (1) shows that the inequality $x^3 + y^3 + z^3 \geq 3xyz$ holds as soon as $x + y + z \geq 0$.

This versatile identity (1) is well-known and of frequent use. Here are a few examples, starting with one in solid geometry:

> Prove that the surface with equation $x^3+y^3+z^3-3xyz=1$ is a surface of revolution.

If $M_0(x_0,y_0,z_0)$ is on the surface, then the circle through M_0 whose axis is the line (ℓ) $x=y=z$ is the intersection of the sphere centered at the origin O with radius OM_0 and the plane through M_0 perpendicular to (ℓ). Thus, a point $M(x,y,z)$ of this circle satisfies

$$x^2+y^2+z^2=x_0^2+y_0^2+z_0^2 \quad \text{and} \quad x+y+z=x_0+y_0+z_0.$$

From (1), we obtain that

$$x^3+y^3+z^3-3xyz=x_0^3+y_0^3+z_0^3-3x_0y_0z_0=1$$

and M is on the surface. The result follows.

The following problem, extracted from 95.I proposed in *The Mathematical Gazette* in 2011, leads to another application.

> Let ABC and $A'B'C'$ be triangles with $a=BC, b=CA, c=AB$ and $B'C'=\sqrt{a}$, $C'A'=\sqrt{b}$, $A'B'=\sqrt{c}$. Prove that
>
> $$\sin^2\frac{1}{2}A+\sin^2\frac{1}{2}B+\sin^2\frac{1}{2}C=\cos^2 A'+\cos^2 B'+\cos^2 C'.$$

Let $s=\dfrac{a+b+c}{2}$. Since $\sin\dfrac{1}{2}A=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$ and $\cos A'=\dfrac{s-a}{\sqrt{bc}}$ (from the Law of Cosines in $\Delta A'B'C'$), we readily find that it is sufficient to prove $X=0$ where

$$X=a(s-a)^2+b(s-b)^2+c(s-c)^2-(a(s-b)(s-c)+b(s-c)(s-a)+c(s-a)(s-b)).$$

Expanding yields

$$X=-2s(a^2+b^2+c^2)+a^3+b^3+c^3+2s(ab+bc+ca)-3abc$$

and $X=0$ follows at once from (1).

Of course, (1) often appears in inequality problems. Problem 929 of *The College Mathematics Journal* required to prove that $a^3+b^3+c^3+3abc\le 6$ whenever $a,b,c>0$ and $a^2+b^2+c^2=3$. We show that the following improvement holds:

$$(a^3+b^3+c^3+3abc)(a+b+c)\le 9+3(ab+bc+ca)\le 6(a+b+c).$$

The easy proof of the right inequality is left to the reader. Setting $m=ab+bc+ca$ and using (1) we obtain that the left inequality can be written as

$$6abc(a+b+c)+(a+b+c)^2(3-m)\le 9+3m,$$

which is successively equivalent to

$$6abc(a+b+c)+(3+2m)(3-m)\le 9+3m,$$

$$6abc(a+b+c) \leq 2m^2,$$

$$0 \leq c^2(b-a)^2 + b^2(a-c)^2 + a^2(c-b)^2,$$

and the latter clearly holds. In passing, note the interesting identity:

$$2[(xy+yz+zx)^2 - 3xyz(x+y+z)] = x^2(y-z)^2 + y^2(z-x)^2 + z^2(x-y)^2.$$

The factorization

$$p(x,y,z) = (x+y+z)(x+wy+w^2z)(x+w^2y+wz),$$

where $w = \exp(2\pi i/3)$ is easily checked (using $w^3 = 1$ and $1 + w + w^2 = 0$) and is worth remembering. For example, it leads to a quick solution to problem **2481** [1999 : 430 ; 2000 : 504]:

> Suppose that A, B, C are 2×2 real commutative matrices. Prove that
>
> $$\det((A+B+C)(A^3+B^3+C^3-3ABC)) \geq 0.$$

Since A, B, C are commutative matrices, we see that

$$(A+B+C)(A^3+B^3+C^3-3ABC) = (A+B+C)^2(A+wB+w^2C)(A+w^2B+wC).$$

Because A, B, C are real and $w^2 = \overline{w}$, we have

$$\det(A+w^2B+wC) = \overline{\det((A+wB+w^2C)}$$

and therefore

$$\det((A+B+C)(A^3+B^3+C^3-3ABC)) = (\det(A+B+C))^2 |\det((A+wB+w^2C)|^2 \geq 0.$$

To close this section, we warmly recommend the nice article **[1]**, in which the reader will find plenty of applications of (1).

About $S(x,y,z) = xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x$

The two identities to be known are

$$S(x,y,z) = (x+y+z)(xy+yz+zx) - 3xyz \qquad \text{and}$$
$$S(x,y,z) = (x+y)(y+z)(z+x) - 2xyz.$$

As an immediate application, we consider problem **4196** [2016 : 445 ; 2017 : 453].

> Show that for all positive real numbers a, b and c, we have
>
> $$1 \leq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \leq 2.$$

The following solution is as short as the featured one: The central expression X writes as

$$X = \frac{a^2b + b^2c + c^2a + (ab + bc + ca)(a + b + c)}{(a + b)(b + c)(c + a)} = \frac{a^2b + b^2c + c^2a + S + 3abc}{S + 2abc}$$
$$= 1 + \frac{a^2b + b^2c + c^2a + abc}{S + 2abc},$$

where $S = S(a, b, c)$. Thus, we have $X > 1$ and also $X < 2$ since $S > a^2b+b^2c+c^2a$.

Here are two related identities:

$$S(x, y, z) - 6xyz = x(y - z)^2 + y(z - x)^2 + z(x - y)^2, \tag{2}$$

$$S(x, y, z) = x^3 + y^3 + z^3 + 2xyz + (x + y - z)(y + z - x)(z + x - y). \tag{3}$$

In some way, the first one explains why $S(x, y, z) \geq 6xyz$ for positive x, y, z independently of the arithmetic mean-geometric mean inequality! The second one suggests applications to inequalities for the triangle. An example is the following variant of solution to problem **4019** [2015 : 74 ; 2016 : 88]:

A triangle with side lengths a, b, c has perimeter 3. Prove that

$$a^3 + b^3 + c^3 + a^4 + b^4 + c^4 \geq 2(a^2b^2 + c^2a^2 + a^2b^2).$$

Let F denote the area of the triangle. We know that

$$16F^2 = 2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4) = (a+b+c)(a+b-c)(b+c-a)(c+a-b).$$

Therefore the inequality is equivalent to $a^3+b^3+c^3 \geq 16F^2$ or, since $a+b+c = 3$,

$$(a^3 + b^3 + c^3)(a + b + c) \geq 48F^2.$$

Now, from (2) and (3) we deduce that

$$a^3+b^3+c^3+(a+b-c)(b+c-a)(c+a-b) = S(a, b, c)-2abc \geq 6abc-2abc = 4abc,$$

hence

$$(a^3 + b^3 + c^3)(a + b + c) + 16F^2 \geq 4abc(a + b + c).$$

Recalling that $abc = 4RF = 4Rrs$ and $a + b + c = 2s$ where R, r and s denote the circumradius, the inradius and the semiperimeter of the triangle, respectively, we obtain

$$4abc(a + b + c) = 32sRF \geq 64rsF = 64F^2$$

(using Euler's inequality $R \geq 2r$).

Thus,

$$(a^3 + b^3 + c^3)(a + b + c) + 16F^2 \geq 64F^2$$

and the desired inequality follows.

About $D(x,y,z) = (x-y)(y-z)(z-x)$

Quite a number of identities involve $D(x,y,z)$. We start with several of them, all of degree 3.

$$x^2(y-z) + y^2(z-x) + z^2(x-y) = -D(x,y,z)$$

$$2(xy^2 + yz^2 + zx^2 - 3xyz) = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 + D(x,y,z) \quad (4)$$

and exchanging x and y,

$$2(x^2y + y^2z + z^2x - 3xyz) = x(y-z)^2 + y(z-x)^2 + z(x-y)^2 - D(x,y,z). \quad (5)$$

By difference, we obtain

$$xy^2 + yz^2 + zx^2 - (x^2y + y^2z + z^2x) = D(x,y,z) \quad (6)$$

We can now present a problem close to problem 1171 of *The College Mathematics Journal* (proposed in March 2020) offering a good opportunity to use some of the identities already seen.

> Let a, b, c be the roots of the equation $x^3 - 2x^2 - x + 1 = 0$ with $a > b > c$. Find the value of the expression $ab^2 + bc^2 + ca^2$.

Note that $a + b + c = 2, \quad ab + bc + ca = -1, \quad abc = -1.$

Let $A = ab^2 + bc^2 + ca^2$ and $B = a^2b + b^2c + c^2a$. From an identity about $S(x,y,z)$, we readily obtain

$$A + B = (a+b+c)(ab+bc+ca) - 3abc = 2(-1) - 3(-1) = 1.$$

On the other hand, a simple calculation gives

$$AB = (ab)^3 + (bc)^3 + (ca)^3 + abc(a^3 + b^3 + c^3) + 3(abc)^2.$$

From an identity about $p(x,y,z)$, we deduce that $a^3 + b^3 + c^3 = -3 + 2(4+3) = 11$ and

$$\begin{aligned}(ab)^3 + (bc)^3 + (ca)^3 &= 3(abc)^2 + (ab+bc+ca)[(ab+bc+ca)^2 - 3abc(a+b+c)] \\ &= 3 - [1 + 3 \times 2] = -4.\end{aligned}$$

Thus, $AB = -4 - 11 + 3 = -12$ and therefore A, B are the solutions to the quadratic equation $X^2 - X - 12 = 0$, which are -3 and 4. Since $A - B = (a-b)(b-c)(c-a) < 0$ (from (6) and because $a > b > c$), we conclude that $A = -3$.

For an application to inequalities, we consider **4036** [2015 : 171 ; 2016 : 182], of which we offer another solution.

> Let a, b and c be non-negative real numbers. Prove that for any real $k \geq \frac{11}{24}$ we have
>
> $$k(ab+bc+ca)(a+b+c) - (a^2c + b^2a + c^2b) \leq \frac{(3k-1)(a+b+c)^3}{9}.$$

Using an identity about $p(x,y,z)$, we can arrange the given inequality as

$$(a^3+b^3+c^3)-6(ab^2+bc^2+ca^2)+3(a^2b+b^2c+c^2a)+6abc \le 3k(a^3+b^3+c^3-3abc). \tag{7}$$

Since $a^3+b^3+c^3-3abc \ge 0$, it is enough to show (7) when $k=\frac{11}{24}$, that is, after some simple transformations,

$$a^3+b^3+c^3+16(ab^2+bc^2+ca^2) \ge 8(a^2b+b^2c+c^2a)+27abc. \tag{8}$$

With the help of (4) and (5), and, again, an identity about $p(x,y,z)$, (8) is transformed into

$$(9a+b+c)(b-c)^2+(a+9b+c)(c-a)^2+(a+b+9c)(a-b)^2 \ge 24(a-b)(b-c)(a-c). \tag{9}$$

Now, suppose without loss of generality that $c=\min(a,b,c)$ and let $\mathcal{L}(a,b,c)$ be the left-hand side of (9) and $\mathcal{R}(a,b,c)=-24D(a,b,c)$ be its right-hand side. If $a_1=a-c$, $b_1=b-c$, and $c_1=0$, then a_1,b_1,c_1 are non-negative real numbers and $a_1-b_1=a-b$, $b_1-c_1=b-c$, $a_1-c_1=a-c$, hence $\mathcal{R}(a_1,b_1,c_1)=\mathcal{R}(a,b,c)$. On the other hand, we calculate

$$\mathcal{L}(a,b,c)-\mathcal{L}(a_1,b_1,c_1)=11c((b-c)^2+(c-a)^2+(a-b)^2)$$

hence $\mathcal{L}(a,b,c) \ge \mathcal{L}(a_1,b_1,c_1)$. So it is sufficient to prove $\mathcal{L}(a_1,b_1,c_1) \ge \mathcal{R}(a_1,b_1,c_1)$. Recalling that $c_1=0$, this inequality is equivalent to $a_1^3+b_1^3+16a_1b_1^2 \ge 8a_1^2b_1$. We are done since $a_1^3+16a_1b_1^2 \ge 2\sqrt{a_1^3\cdot 16a_1b_1^2}=8a_1^2b_1$.

For the benefit of the reader, here is a list of beautiful identities involving $D(x,y,z)$:

$$x^3(y-z)+y^3(z-x)+z^3(x-y)=-(x+y+z)D(x,y,z)$$

$$x^4(y-z)+y^4(z-x)+z^4(x-y)=-(x^2+y^2+z^2+xy+yz+zx)D(x,y,z)$$

$$(y+z)(z+x)(x-y)+(z+x)(x+y)(y-z)+(x+y)(y+z)(z-x)=-D(x,y,z)$$

$$x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)=-(xy+yz+zx)D(x,y,z)$$

$$(x-y)^5+(y-z)^5+(z-x)^5=5(x^2+y^2+z^2-xy-yz-zx)D(x,y,z)$$

$$\begin{aligned}&(x-y)^7+(y-z)^7+(z-x)^7\\&\quad=7[(x-y)^2(y-z)^2+(y-z)^2(z-x)^2+(z-x)^2(x-y)^2]D(x,y,z)\end{aligned}$$

Note that the last two come in addition to $(x-y)^3+(y-z)^3+(z-x)^3=3D(x,y,z)$, which directly follows from $p(x-y,y-z,z-x)=0$. Also, checking the last one is made easier by setting $a=y-z,\ b=z-x,\ c=x-y$ (so that $a+b+c=0$).

Reference

[1] Desmond MacHale, My Favourite Polynomial, *The Mathematical Gazette*, Vol. 75, No 472, June 1991, p. 157-165

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 30, 2021**.

4661. *Proposed by Mihaela Berindeanu.*

Let ABC be a triangle with the point $M \in BC$ such that

$$MC - MB = \frac{AC^2 - AB^2}{2BC}.$$

The centroids of triangles AMB and AMC are G_1 and G_2, respectively. Prove that A, G_1, M, C are concyclic points if and only if A, B, M, G_2 are also concyclic points.

4662. *Proposed by Michel Bataille.*

Let A and B be complex $p \times p$ matrices such that $AB = BA$ and $A^3B = A$ and let m, n be integers with $m \geq n \geq 1$ and $m \neq 2n$. Show that A^mB^n is equal to a power of A or a power of AB.

4663. *Proposed by Vijay Dasari.*

Let M be any point in the plane of an acute triangle ABC with sides a, b, c. Prove that

$$\frac{AM^2}{b^2+c^2-a^2} + \frac{BM^2}{c^2+a^2-b^2} + \frac{CM^2}{a^2+b^2-c^2} \geq 1,$$

with equality when M is the orthocenter.

4664. *Proposed by Marian Cucoanes and Lorian Saceanu.*

Let $ABCDEF$ be a convex cyclic hexagon that respects the following rules:

a) The lines AD, BE, CF are concurrent;

b) $(1/3)(AF + BC + DE) = AB = CD = EF$.

Prove that $ABCDEF$ is a regular hexagon.

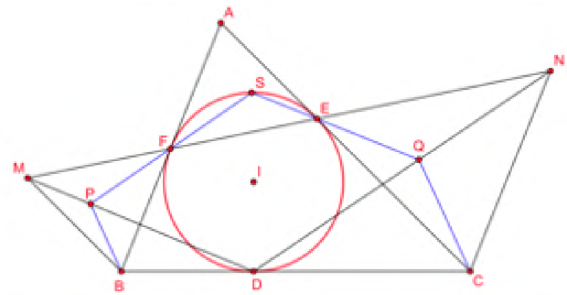
4665. *Proposed by Daniel Sitaru.*

Find

$$\lim_{n\to\infty}\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(1+\sin^n x)}dx\right).$$

4666. *Proposed by Dong Luu.*

Let ABC be a triangle and let the circle I be tangent to BC, CA and AB at points D, E and F, respectively. Let M, N be the points on the line EF such that BM is parallel to AC and CN is parallel to AB. Let P and Q be points on DM and DN, respectively such that BP is parallel to CQ. Denote by S the intersection point of PF and QE. Prove that S lies on the circle I.



4667. *Proposed by Conar Goran.*

Let $x_1, \ldots, x_n > 0$ be real numbers and $s = \sum_{i=1}^{n} x_i$. Prove

$$\prod_{i=1}^{n} x_i^{x_i} \leq \left(\frac{1}{s} \sum_{i=1}^{n} x_i^2 \right)^s .$$

When does equality occur?

4668. *Proposed by Jiahao Chen.*

Let Γ be the inscribed circle of triangle ABC, and I is the center of Γ. Suppose Γ touches BC, CA and AB at D, E and F, respectively. Let X be an arbitrary point on the smaller arc DF, and the line perpendicular to XE passing through I intersects line BX in point Y. Show that IY is the external angle bisector of the angle AYC.

4669. *Proposed by Warut Suksompong.*

For a given positive integer n, a $4n \times 4n$ table is partitioned into $16n^2$ unit squares, each of which is coloured in one of 4 given colours. A set of four cells is called *colourful* if the centers of the cells form a rectangle with sides parallel to the sides of the table, and the cells are coloured in all four different colours. Determine the maximum number of colourful sets.

4670. *Proposed by Nguyen Viet Hung.*

Let a, b, c be real numbers such that $(a+b)(b+c)(c+a) \neq 0$. Prove that

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2 + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 1.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 novembre 2021**.

4661. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle et M un point tel que $M \in BC$ et

$$MC - MB = \frac{AC^2 - AB^2}{2BC}.$$

Les centroïdes des triangles AMB et AMC sont G_1 et G_2 respectivement. Démontrer que A, G_1, M et C sont cocycliques si et seulement si A, B, M, G_2 sont cocycliques.

4662. *Proposé par Michel Bataille.*

Soient A et B des matrices complexes de taille $p \times p$ telles que $AB = BA$ et $A^3B = A$; de plus, soient m et n des entiers tels que $m \geq n \geq 1$ et $m \neq 2n$. Démontrer que A^mB^n est égal à une puissance de A ou à une puissance de AB.

4663. *Proposé par Vijay Dasari.*

Soit M un point dans le plan du triangle acutangle ABC de côtés a, b et c. Démontrer que

$$\frac{AM^2}{b^2 + c^2 - a^2} + \frac{BM^2}{c^2 + a^2 - b^2} + \frac{CM^2}{a^2 + b^2 - c^2} \geq 1,$$

l'égalité ayant lieu lorsque M est l'orthocentre de ABC.

4664. *Proposé par Marian Cucoanes et Lorian Saceanu.*

Soit $ABCDEF$ un hexagone convexe et cyclique tel que

a) les lignes AD, BE et CF sont concourantes;

b) $(1/3)(AF + BC + DE) = AB = CD = EF$.

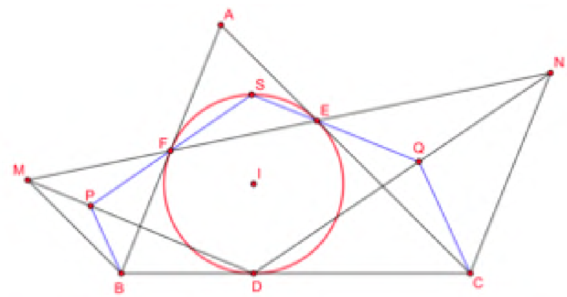
Démontrer que $ABCDEF$ est un hexagone régulier.

4665. *Proposé par Daniel Sitaru.*

Déterminer
$$\lim_{n\to\infty}\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x(1+\sin^n x)}dx\right).$$

4666. *Proposé par Dong Luu.*

Soit ABC un triangle et I le centre de son cercle inscrit, tangent à BC, CA et AB aux points D, E et F respectivement. Soient M et N des points sur la ligne EF tels que BM est parallèle à AC et CN est parallèle à AB. Enfin, soient P et Q des points sur DM et DN respectivement tels que BP est parallèle à CQ. Démontrer que S, l'intersection de PF et QE, se trouve sur I.



4667. *Proposé par Conar Goran.*

Soient $x_1,\dots,x_n>0$ des nombres reels où $s=\sum_{i=1}^n x_i$. Démontrer que
$$\prod_{i=1}^n x_i^{x_i} \le \left(\frac{1}{s}\sum_{i=1}^n x_i^2\right)^s.$$

Quand l'égalité tient-elle ?

4668. *Proposé par Jiahao Chen.*

Soit I le centre du cercle inscrit Γ pour le triangle ABC, où Γ touche BC, CA et AB en D, E et F respectivement. Soit X un point arbitraire sur le plus petit arc DF et supposer que la ligne perpendiculaire à XE et passant par I intersecte la ligne BX au point Y. Démontrer que IY est la bissectrice externe de l'angle AYC.

4669. *Proposé par Warut Suksompong.*

Pour un entier positif n, un tableau de taille $4n\times 4n$ est formé de $16n^2$ petits carrés, chacun étant coloré utilisant l'une ou l'autre des 4 couleurs disponibles. Un ensemble de 4 petits carrés est dit *flamboyant* s'ils sont colorés utilisant toutes les 4 couleurs et si leurs centres forment un rectangle avec côtés parallèles aux côtés du tableau. Déterminer le nombre maximal d'ensembles flamboyants.

4670. *Proposé par Nguyen Viet Hung.*

Soient a, b, c des nombres réels tels que $(a+b)(b+c)(c+a) \neq 0$. Démontrer que

$$\left(\frac{a}{a+b}\right)^2 + \left(\frac{b}{b+c}\right)^2 + \left(\frac{c}{c+a}\right)^2 + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 1.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(2), p. 97–100.

4611. *Proposed by Nguyen Viet Hung.*

Evaluate
$$\frac{1}{\sin^4\frac{\pi}{14}}+\frac{1}{\sin^4\frac{3\pi}{14}}+\frac{1}{\sin^4\frac{5\pi}{14}}.$$

We received 24 solutions, one of which was incorrect. We present the solution by Brian Beasley.

Let $a=\sin\frac{\pi}{14}=\cos\frac{3\pi}{7}$, $b=\sin\frac{3\pi}{14}=-\cos\frac{5\pi}{7}$, and $c=\sin\frac{5\pi}{14}=\cos\frac{\pi}{7}$. To apply the multiple-angle identity
$$\cos 7\theta=64\cos^7\theta-112\cos^5\theta+56\cos^3\theta-7\cos\theta,$$
we let
$$f(x)=64x^7-112x^5+56x^3-7x+1=(x+1)(8x^3-4x^2-4x+1)^2$$
and note that the three zeros of $8x^3-4x^2-4x+1$ are a, $-b$, and c. Then $a-b+c=\frac{1}{2}$, $-ab-bc+ca=-\frac{1}{2}$, and $abc=\frac{1}{8}$. This yields
$$\frac{1}{4}=(a-b+c)^2=(a^2+b^2+c^2)+2(-ab-bc+ca)=a^2+b^2+c^2-1,$$
so $a^2+b^2+c^2=\frac{5}{4}$. Similarly, we obtain
$$\begin{aligned}\frac{1}{4}&=(-ab-bc+ca)^2\\&=[(ab)^2+(bc)^2+(ca)^2]+2abc(-a+b-c)\\&=(ab)^2+(bc)^2+(ca)^2-\frac{1}{8},\end{aligned}$$
so $(ab)^2+(bc)^2+(ca)^2=\frac{3}{8}$. Finally, we have
$$\frac{9}{64}=[(ab)^2+(bc)^2+(ca)^2]^2=[(ab)^4+(bc)^4+(ca)^4]+2(abc)^2(a^2+b^2+c^2)$$
and hence $(ab)^4+(bc)^4+(ca)^4=\frac{13}{128}$. We conclude that
$$\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}=\frac{(ab)^4+(bc)^4+(ca)^4}{(abc)^4}=\frac{\frac{13}{128}}{(\frac{1}{8})^4}=416.$$

4612. *Proposed by Mihaela Berindeanu.*

In the convex quadrilateral $ABCD$, we have

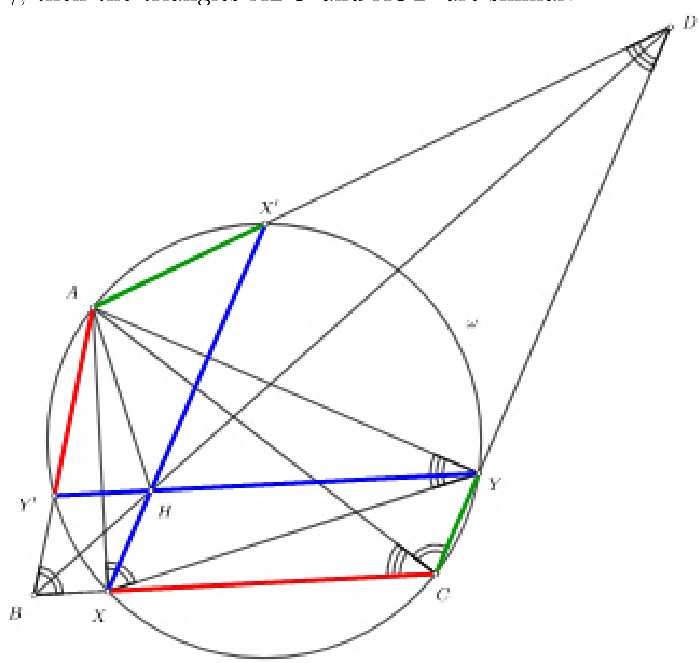
$$\measuredangle(BAC) = \measuredangle(CAD) \quad \text{and} \quad \measuredangle(CDA) = \measuredangle(BCA).$$

Denote $O \in AC$, $X \in BC$, $Y \in CD$ such that $OA = OC$, $AX \perp BC$ and $AY \perp CD$. The perpendicular line from A to XY cuts BD at Z. Show that $\overrightarrow{OZ} = \overrightarrow{OA} + \overrightarrow{OX} + \overrightarrow{OY}$.

We received 8 submissions, all of which were correct; we feature the solution by the UCLan Cyprus Problem Solving Group, modified by the editor.

We have $\overrightarrow{OA} + \overrightarrow{OX} = \overrightarrow{CO} + \overrightarrow{OX} = \overrightarrow{CX}$ and $\overrightarrow{OZ} - \overrightarrow{OY} = \overrightarrow{YZ}$. So it is enough to show that YZ is parallel and equal to CX; that is, we are to show that $XCYZ$ is a parallelogram.

Let us write α, β, γ for the angles of the triangle ABC. Since $\angle CAD = \alpha$ and $\angle CDA = \gamma$, then the triangles ABC and ACD are similar.



Since $\angle AXC = \angle AYC = 90°$, A, X, C, Y are concyclic on the circle with diameter AC, call it ω. So we get $\angle AXY = \angle ACY = \beta$ and $\angle AYX = \angle ACX = \gamma$. Thus the triangle AXY is also similar to triangle ABC.

Let H be the orthocenter of triangle AXY. Then YH (the altitude) and CX (given) are both perpendicular to AX, so they must be parallel. Similarly, XH and CY are parallel because both are perpendicular to AY. So $XCYH$ is a parallelogram, and the problem reduces to showing that $Z = H$; specifically we show that the orthocenter H of triangle AXY lies on BD. To that end we let Y'

be the other point of intersection of ω with the perpendicular from Y to AX, and X' be the other point of intersection of ω with the perpendicular from X to AY. Because the inscribed angles $YY'A$ and YXA are equal, and we know that the latter equals $\beta = \angle CBA$, we have Y' lies on the line AB. Similarly, X' lies on the line AD. Pascal's theorem applied to the inscribed hexagon $AY'YCXX'$ says that the points of intersection of AY' with CX (namely B), of $Y'Y$ with XX' (namely H), and of YC with $X'A$ (namely D) are collinear. This concludes the argument.

Editor's Comments. Walther Janous observed that the convexity of the quadrilateral $ABCD$ is not needed: we used only that the triangles ABC and ACD are similar. C.R. Pranesachar motivated his solution by referring to the familiar result that the circumcenter O and orthocenter H of an arbitrary triangle AXY satisfies

$$\overrightarrow{OA} + \overrightarrow{OX} + \overrightarrow{OY} = \overrightarrow{OH}.$$

This result was used implicitly in the first step of the featured solution.

4613. *Proposed by Daniel Sitaru.*

Let A and B be $n \times n$ real matrices with $n \in \mathbb{N}, n \geq 2$ such that $AB = BA$. Show that

$$\det(4(A^2 + B^2) + AB + 3(A + B) + I_n) \geq 0.$$

We received 13 submissions, all correct. We present the solution provided by Marie-Nicole Gras.

Multiplying the given expression by 16, we get

$$M = 64A^2 + 64B^2 + 16AB + 48A + 48B + 16I_n.$$

Since $AB = BA$, we can compute (as with real numbers):

$$\begin{aligned} M &= (8A + B + 3I_n)^2 - B^2 - 6B - 9I_n + 64B^2 + 48B + 16I_n \\ &= (8A + B + 3I_n)^2 + 63B^2 + 42B + 7I_n \\ &= (8A + B + 3I_n)^2 + (3\sqrt{7}B + \sqrt{7}I_n)^2 =: C^2 + D^2. \end{aligned}$$

In the complex numbers field $\mathbb{C}$, we have

$$\begin{aligned} \det(M) &= \det(C^2 + D^2) \\ &= \det\big((C + iD)(C - iD)\big) \\ &= \det(C + iD)\det(C - iD). \end{aligned}$$

Then, $\det(M)$ is the product of two conjugate complex numbers. Therefore it follows that $\det(M) \geq 0$.

4614. *Proposed by Florin Stanescu.*

Let k be a given natural number and let $(a_n)_{n\geq 1}$ be a sequence such that

$$\lim_{n\to\infty} \frac{1}{n^k}\left(\frac{a_1}{1}+\frac{a_2}{2}+\cdots+\frac{a_n}{n}\right)=1.$$

Prove that the sequence $\left(\dfrac{a_1+a_2+\cdots+a_n}{n^{k+1}}\right)_{n\geq 1}$ is convergent by finding its limit.

We received 7 submissions, of which 6 were correct. We present the solution by Michel Bataille.

For each $n\geq 1$, let $A_n=\frac{a_1}{1}+\frac{a_2}{2}+\cdots+\frac{a_n}{n}$. Then we have

$$\begin{aligned} a_1+a_2+\cdots+a_n &= 1\cdot A_1+2(A_2-A_1)+\cdots+(n-1)(A_{n-1}-A_{n-2})+n(A_n-A_{n-1})\\ &= -A_1-A_2-\cdots-A_{n-1}+nA_n\\ &= (n+1)A_n-(A_1+A_2+\cdots+A_n). \end{aligned}$$

It follows that

$$\frac{a_1+a_2+\cdots+a_n}{n^{k+1}}=\frac{n+1}{n^{k+1}}\cdot A_n-\frac{A_1+A_2+\cdots+A_n}{n^{k+1}}.$$

By the given condition, we have

$$\lim_{n\to\infty}\frac{n+1}{n^{k+1}}\cdot A_n=\lim_{n\to\infty}\left(\frac{n+1}{n}\cdot\frac{A_n}{n^k}\right)=1\cdot 1=1.$$

In addition, since $A_n \sim n^k>0$ as $n\to\infty$ and $\sum\limits_{n\geq 1} n^k$ is a divergent series, application of Stolz-Cesaro's Theorem gives

$$\sum_{j=1}^{n} A_j \sim \sum_{j=1}^{n} j^k.$$

Next, note that the sum $S_n=\frac{1}{n}\left(\left(\frac{1}{n}\right)^k+\left(\frac{2}{n}\right)^k+\cdots+\left(\frac{n}{n}\right)^k\right)$ is a Riemann sum for the function $f(x)=x^k$ on $[0,1]$, so $\lim_{n\to\infty}S_n=\int_0^1 x^k dx=\frac{1}{k+1}$.

Hence

$$\lim_{n\to\infty}\frac{1}{n}\left(\left(\frac{1}{n}\right)^k+\left(\frac{2}{n}\right)^k+\cdots+\left(\frac{n}{n}\right)^k\right)=\frac{1}{k+1},$$

so $\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{j=1}^n j^k=\frac{1}{k+1}$, so $\sum\limits_{j=1}^{n} A_j \sim \sum_{j=1}^n j^k \sim \frac{n^{k+1}}{k+1}$ as $n\to\infty$. Therefore

$$\lim_{n\to\infty}\frac{1}{n^{k+1}}(A_1+A_2+\cdots+A_n)=\frac{1}{k+1},$$

from which we can deduce that the sequence $\left(\dfrac{a_1+a_2+\cdots+a_n}{n^{k+1}}\right)_{n\geq 1}$ is convergent and its limit is $1-\frac{1}{k+1}=\frac{k}{k+1}$.

4615. *Proposed by Anthony Garcia.*

Let f be a twice differentiable function on $[0,1]$ such that $\int_0^1 f(x)dx = \frac{f(1)}{2}$. Prove that

$$\int_0^1 (f''(x))^2 dx \geq 30(f(0))^2.$$

We received 11 solutions, all of which were correct. We present the solution by Henry Ricardo.

The Cauchy-Schwarz inequality gives us

$$\left(\int_0^1 (x-x^2)f''(x)\,dx\right)^2 \;\leq\; \int_0^1 (x-x^2)^2 dx \cdot \int_0^1 \left(f''(x)\right)^2 dx. \tag{1}$$

Integration by parts yields

$$\begin{aligned}\int_0^1 (x-x^2)f''(x)\,dx &= -\int_0^1 (1-2x)f'(x)\,dx \\ &= f(1)+f(0)-2\int_0^1 f(x)dx \;=\; f(0),\end{aligned} \tag{2}$$

and a simple calculation results in

$$\int_0^1 (x-x^2)^2\,dx \;=\; \frac{1}{30}. \tag{3}$$

Substituting (2) and (3) in (1), we get

$$\left(f(0)\right)^2 \leq \frac{1}{30}\int_0^1 \left(f''(x)\right)^2 dx,$$

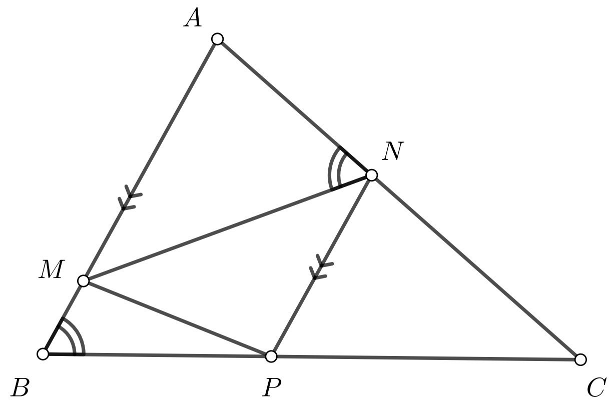
which is the desired result.

4616. *Proposed by Marius Drăgan, modified by the Editorial Board.*

For each suitable point N on side AC of ΔABC define P to be the point where the line parallel to AB meets the side BC, and M to be the point on side AB for which $\angle MNA = \angle B$. If the area of ΔABC equals 1, determine the maximum area of triangle MPN.

We received 14 solutions, all of which were correct, albeit not all interpreted the problem the same way. Five solutions (including that of the proposer) implicitly required the points M, N, and P to be on the segments AB, AC, BC, respectively, resulting in two cases for the value of the maximum. The remaining nine solutions implicitly allowed the points M, N, and P to be anywhere on the corresponding lines, resulting in a single case for the maximum. We feature one solution of each type.

Solution 1, by UCLan Cyprus Problem Solving Group.



Let a, b, c be the side lengths of the triangle. We denote the area of a triangle XYZ by $[XYZ]$.

Assume $NC = \lambda b$. Since NP is parallel to AB, then $PC = \lambda a$ and $[PNC] = \lambda^2$.

We also have $AN = (1-\lambda)b$. Since ANM is similar to ABC (angle criterion) then $AM = (1-\lambda)^2 b^2/c$ and $[AMN] = (1-\lambda)^2 b^2/c^2$.

We also have $BM = c - \frac{(1-\lambda)b^2}{c}$ and $BP = (1-\lambda)a$. So

$$[BMP] = \frac{1}{2}(BM)(BP)\sin(\angle ABC) = \frac{(BM)(BP)}{ac} = (1-\lambda)\left(1 - \frac{(1-\lambda)b^2}{c^2}\right).$$

We get

$$[MNP] = 1 - ([PNC] + [AMN] + [BMP]) = \lambda - \lambda^2 = \frac{1}{4} - \left(\frac{1}{2} - \lambda\right)^2.$$

So the area of MNP is maximized when λ is as close to $1/2$ as possible.

We observe that if λ cannot take all possible values in $[0, 1]$, then its smallest possible value occurs when $M = B$. In this case we have

$$c = AB = AM = \frac{(1-\lambda)b^2}{c}.$$

This gives $\lambda \geqslant 1 - \frac{c^2}{b^2}$.

So the maximum possible area of MPN is $1/4$ if $c \geqslant b\sqrt{2}/2$ and

$$\left(1 - \frac{c^2}{b^2}\right)\frac{c^2}{b^2} = \frac{c^2(b^2 - c^2)}{b^4}$$

otherwise.

Solution 2, by Michel Bataille.

In barycentric coordinates relative to (A, B, C), we have $N = (t : 0 : 1-t)$ for some real $t \in [0,1]$. The parallel to AB through N also passes through the point at infinity $(1 : -1 : 0)$ of AB, hence its equation is $(1-t)(x+y) = tz$. This yields $P = (0 : t : 1-t)$.

Since M is on the side AB, we have $M = (u : 1-u : 0)$ for some real $u \in [0,1]$ and since the area of ΔABC is 1, the area $[MPN]$ of ΔMPN is $|\delta|$ where

$$\delta = \begin{vmatrix} u & 0 & t \\ 1-u & t & 0 \\ 0 & 1-t & 1-t \end{vmatrix} = t(1-t)[u+1-u] = t(1-t).$$

Thus,

$$[MPN] = t(1-t) \leq \frac{1}{4}$$

with equality if and only if $t = \frac{1}{2}$.

In conclusion the desired maximum is $\frac{1}{4}$ attained when N is the midpoint of AC.

4617. *Proposed by Nermin Hodzic, Adnan Ali and Salem Malikic.*

Let a, b, c be positive real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2.$$

Show that $\max(a, b, c) \geq \sqrt[3]{9abc}$.

We received 10 submissions, of which 8 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.

By symmetry and homogeneity we may assume that $a \geqslant b \geqslant c = 1$. We then must prove that $a \geqslant \sqrt[3]{9ab}$, which is equivalent to $a^2 \geqslant 9b$.

If $b \geqslant 9$ then $a^2 \geqslant b^2 \geqslant 9b$.

Hence, we can assume $1 \leqslant b < 9$. Since $c = 1$, the constraint becomes

$$\frac{a}{1+b} + \frac{b}{1+a} + \frac{c}{a+b} = 2,$$

which is equivalent to

$$a^3 - (b+1)a^2 - (b^2+b+1)a + (b^3 - b^2 - b + 1) = 0$$

and leads us to define

$$f(x) = x^3 - (b+1)x^2 - (b^2+b+1)x + (b^3 - b^2 - b + 1)$$

so that a is a zero of $f(x)$.

If $b = 1$ then $f(x) = x(x-3)(x+1)$, whose only positive zero is when $x = 3$. So $a = 3$ and $a^2 \geqslant 9b$ as required.

If $1 < b < 9$, note that $f(0) = b^3 - b^2 - b + 1 = (b-1)^2(b+1) > 0$ and $f(b) = 1 - 3b^2 - 2b < 0$. Hence, the monic, cubic polynomial, $f(x)$, has one negative zero, one zero in the interval $(0, b)$, and one zero greater than b. Thus, a is the unique zero of $f(x)$ which is greater than b.

To see that $a^2 \geqslant 9b$, we will show that $3\sqrt{b} < a$.

Since $1 < b < 9$ we have $\sqrt{b} < 3$ and so, $b < 3\sqrt{b}$. Since $f(x) > 0$ for $x > b$ only when $x > a$, to see that $3\sqrt{b} < a$, it suffices to show that $f(3\sqrt{b}) < 0$.

We have

$$f(3\sqrt{b}) = 27b^{3/2} - 9b^2 - 9b - 3b^{5/2} - 3b^{3/2} - 3b^{1/2} + b^3 - b^2 - b + 1 = g(\sqrt{b})$$

where

$$\begin{aligned} g(x) &= x^6 - 3x^5 - 10x^4 + 24x^3 - 10x^2 - 3x + 1 \\ &= (x-1)(x^5 - 2x^4 - 12x^3 + 12x^2 + 2x - 1) \\ &= (x-1)^2(x^4 - x^3 - 13x^2 - x + 1)\,. \end{aligned}$$

Letting $h(x) = x^4 - x^3 - 13x^2 - x + 1$, it remains to show that $h(x) < 0$ for $x \in (1, 3)$. In fact,

$$h(x) = (x^3-1)(x-1) - 13x^2 = (x-1)^2(x^2+x+1) - 13x^2 < 13(x-1)^2 - 13x^2 < 0\,.$$

We note that equality holds if and only if $(a, b, c) \in \{(3t, t, t), (t, 3t, t), (t, t, 3t)\}$ for some $t > 0$.

4618. *Proposed by Cherng-tiao Perng.*

Let $\mathcal{C}$ be a nondegenerate conic and $\mathcal{L}$ be a line. Let O, P be two distinct points such that $O, P \notin \mathcal{L}$ and $P \in \mathcal{C}$. Denote the alternative intersection of OP and $\mathcal{C}$ by Q_0. Furthermore let P' be a point on OP such that $P' \notin \mathcal{L}$. For any Q on $\mathcal{C}$ other than Q_0, let

$$QP \cap \mathcal{L} = \{D\} \text{ and } DP' \cap QO = \{Q'\}.$$

Prove that when Q varies on $\mathcal{C}$, Q' moves on a fixed conic through P'.

We received 4 solutions for this problem. We present the solution by Theo Koupelis.

Without loss of generality, we consider a coordinate system where P is at the origin and the x-axis is the line OP. Then $P = (0, 0)$, $O = (x_0, 0)$, and $P' = (x_{P'}, 0)$. Also, let $Q = (x_Q, y_Q)$, and D be the point of intersection between the line PQ and $\mathcal{L}$. For $Q \not\equiv P$, the equation of the line PQ is given by $y_{PQ} = \dfrac{y_Q}{x_Q} \cdot x$. Also, let the equation of line $\mathcal{L}$ be $y_{\mathcal{L}} = mx + b$. Then the coordinates of the point D are given by

$$(x_D, y_D) = \left(\frac{bx_Q}{y_Q - mx_Q},\ \frac{by_Q}{y_Q - mx_Q} \right).$$

Now the equations for the lines DP' and OQ are given by

$$y_{DP'} = \frac{-by_Q}{x_{P'}(y_Q - mx_Q) - bx_Q} \cdot (x - x_{P'}), \quad \text{and} \quad y_{OQ} = \frac{y_Q}{x_Q - x_0} \cdot (x - x_0),$$

and therefore the coordinates of their intersection point Q' are

$$(x_{Q'}, y_{Q'}) = \left(x_0 + \frac{b(x_{P'} - x_0)(x_Q - x_0)}{x_{P'}(y_Q - mx_Q) - bx_0},\ \frac{by_Q(x_{P'} - x_0)}{x_{P'}(y_Q - mx_Q) - bx_0} \right). \tag{1}$$

We can now write the coordinates of the point Q in terms of the coordinates of the points Q', P', P, and O as

$$\begin{aligned} x_Q &= \frac{-bx_0 \cdot x_{Q'} - x_0 x_{P'} \cdot y_{Q'} + bx_0 x_{P'}}{mx_{P'} \cdot x_{Q'} - x_{P'} \cdot y_{Q'} + b(x_{P'} - x_0) - mx_0 x_{P'}}, \\ y_Q &= \frac{-x_0(b + mx_{P'}) \cdot y_{Q'}}{mx_{P'} \cdot x_{Q'} - x_{P'} \cdot y_{Q'} + b(x_{P'} - x_0) - mx_0 x_{P'}}. \end{aligned}$$

Simplifying we have

$$(x_Q, y_Q) = \left(\frac{ax_{Q'} + \bar{b}y_{Q'} + c}{dx_{Q'} + ey_{Q'} + f},\ \frac{gy_{Q'}}{dx_{Q'} + ey_{Q'} + f} \right), \tag{2}$$

where $a = -bx_0$, $\bar{b} = -x_0 x_{P'}$, $c = bx_0 x_{P'}$, $d = mx_{P'}$, $e = -x_{P'}$, $f = b(x_{P'} - x_0) - mx_0 x_{P'}$, and $g = -x_0(b + mx_{P'})$ are constants.

Because the point Q belongs to a conic, its coordinates (x_Q, y_Q) satisfy an equation of the form

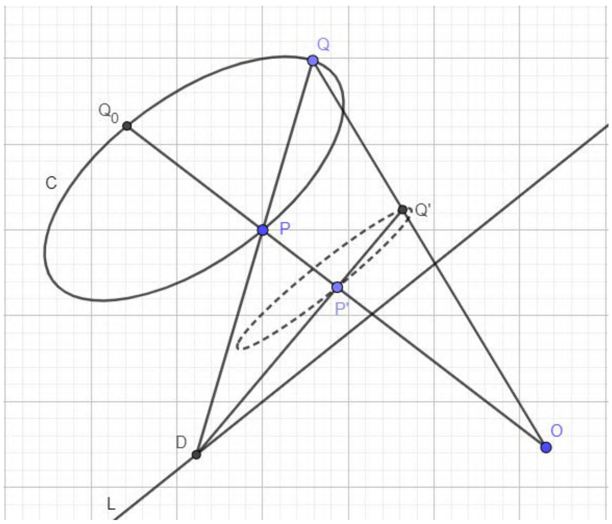
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{3}$$

where A, B, C, D, E, and F are given constants. Of course, in our case, because $P = (0, 0)$ is on the same conic, we have $F = 0$. By direct substitution into (3), we see now that using the transformation shown in (2) the coordinates $(x_{Q'}, y_{Q'})$ also satisfy an equation of the form

$$A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0,$$

where the coefficients A', B', C', D', E', and F' depend on the constants A, B, C, D, E, m, b, x_0, and $x_{P'}$. Therefore, when Q varies on $\mathcal{C}$, Q' moves on a fixed conic. This new conic does go through point P'; indeed, when $Q \to P$, then $x_Q \to 0$ and $y_Q \to 0$ and therefore from (1) we get $(x_{Q'}, y_{Q'}) \to (x_{P'}, 0)$ and thus $Q' \to P'$.

An example is shown below.



4619. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequences a_n and b_n such that $a_n = \sum_{k=1}^{n} \dfrac{1}{k^2}$ and $b_n = \sum_{k=1}^{n} \dfrac{1}{(2k-1)^2}$.

Compute $\lim\limits_{n\to\infty} \left(\dfrac{\pi^4}{48} - a_n b_n\right) n$.

We received 20 *solutions, all correct. We present two different solutions.*

Solution 1, based on similar arguments by Theo Koupelis and UCLan Cyprus Problem Solving Group (done independently).

Let $S_n = \left(\dfrac{\pi^4}{48} - a_n b_n\right) n$.

It is well known that $\sum_{k=1}^{\infty} \dfrac{1}{k^2} = \dfrac{\pi^2}{6}$ and $\sum_{k=1}^{\infty} \dfrac{1}{(2k-1)^2} = \dfrac{\pi^2}{8}$, so

$$a_n = \frac{\pi^2}{6} - \sum_{k=n+1}^{\infty} \frac{1}{k^2}, \quad \text{and} \quad b_n = \frac{\pi^2}{8} - \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2}.$$

Substituting we get

$$\begin{aligned} S_n &= \frac{n\pi^2}{48} - n\left(\frac{\pi^2}{6} - \sum_{k=n+1}^{\infty} \frac{1}{k^2}\right)\left(\frac{\pi^2}{8} - \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2}\right) \\ &= \frac{\pi^2}{6}\left(n \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2} + \frac{3n}{4} \sum_{k=n+1}^{\infty} \frac{1}{k^2} - \frac{6n}{\pi^2} \sum_{k=n+1}^{\infty} \frac{1}{(2k-1)^2} \cdot \sum_{k=n+1}^{\infty} \frac{1}{k^2}\right). \end{aligned}$$

Next by straightforward computations we have

$$\frac{1}{2}\left(\frac{1}{2k+2} - \frac{1}{2k+4}\right) < \frac{1}{(2k+1)^2} < \frac{1}{2}\left(\frac{1}{2k} - \frac{1}{2k+2}\right),$$

and

$$\frac{1}{k+1}-\frac{1}{k+2}=\frac{1}{(k+1)(k+2)}<\frac{1}{(k+1)^2}<\frac{1}{k}-\frac{1}{k+1}.$$

Summing from $k=n$ to infinity in the two inequalities above, we the obtain:

$$\frac{1}{4(n+1)}<\sum_{k=n+1}^{\infty}\frac{1}{(2k-1)^2}<\frac{1}{4n}\quad\text{and}\quad\frac{1}{n+1}<\sum_{k=n+1}^{\infty}\frac{1}{k^2}<\frac{1}{n}.$$

Therefore

$$\frac{\pi^2}{6}\left(\frac{n}{4(n+1)}+\frac{3n}{4(n+1)}-\frac{6n}{4n^2\pi^2}\right)<S_n<\frac{\pi^2}{6}\left(\frac{n}{4n}+\frac{3n}{4n}-\frac{6n}{4(n+1)^2\pi^2}\right),$$

or

$$\frac{\pi^2}{6}\left(1-\frac{1}{n+1}-\frac{6}{4n\pi^2}\right)<S_n<\frac{\pi^2}{6}\left(1-\frac{3n}{2(n+1)^2\pi^2}\right).$$

Therefore, by the Squeeze Theorem, it follows that $\lim\limits_{n\to\infty}S_n=\dfrac{\pi^2}{6}$.

Solution 2, by Corneliu Manescu-Avram, slightly enhanced by the editor.

It is well known that $\lim_{n\to\infty}a_n=\dfrac{\pi^2}{6}$ and

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}(a_{2n}-\frac{1}{4}a_n)=\frac{\pi^2}{6}-\frac{\pi^2}{24}=\frac{\pi^2}{8}.$$

Let $x_n=\dfrac{\pi^2}{48}-a_nb_n$ and $y_n=\dfrac{1}{n}$. Then $x_{n+1}-x_n=a_nb_n-a_{n+1}b_{n+1}$ and $y_{n+1}-y_n=\dfrac{1}{n+1}-\dfrac{1}{n}=\dfrac{-1}{n(n+1)}$. Hence

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=\lim_{n\to\infty}n(n+1)(a_{n+1}b_{n+1}-a_nb_n). \tag{1}$$

Since $a_{n+1}=a_n+\dfrac{1}{(n+1)^2}$ and $b_{n+1}=b_n+\dfrac{1}{(2n+1)^2}$, we have

$$a_{n+1}b_{n+1}-a_nb_n=\frac{a_n}{(2n+1)^2}+\frac{b_n}{(n+1)^2}+\frac{1}{(n+1)^2(2n+1)^2}. \tag{2}$$

From (1) and (2), we obtain

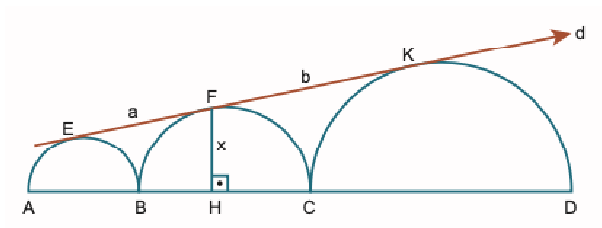
$$\begin{aligned}\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}&=\lim_{n\to\infty}\frac{n(n+1)}{(2n+1)^2}a_n+\lim_{n\to\infty}\frac{n}{n+1}b_n+\lim_{n\to\infty}\frac{n}{(n+1)(2n+1)^2}\\&=\frac{\pi^2}{24}+\frac{\pi^2}{8}=\frac{\pi^2}{6}.\end{aligned}$$

Hence by the Stolz-Cèsaro Theorem for the $(\frac{0}{0})$ case, we conclude that

$$\lim_{n\to\infty}\left(\frac{\pi^2}{24}-a_nb_n\right)n=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=\frac{\pi^2}{6}.$$

4620. *Proposed by Alpaslan Ceran.*

Consider three semicircles in the configuration below:

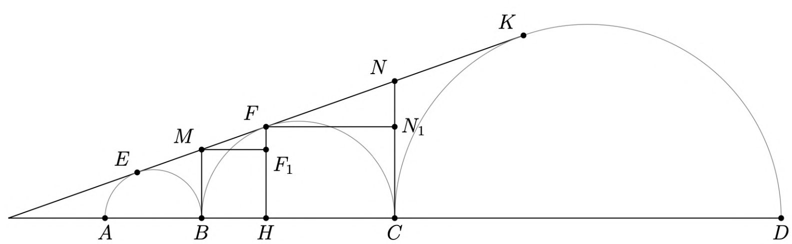


Prove that $\dfrac{1}{x}=\dfrac{1}{a}+\dfrac{1}{b}$.

We received 28 solutions, of which 20 were correct. We present the solution by Theo Koupelis.

Let BM be the common internal tangent of the circles with diameters AB and BC, where M is on the line EK. Similarly, let CN be the common internal tangent of the circles with diameters BC and CD, where N is on the line EK. By the fact that different tangent segments from the same point to the same circle have the same length, we find that

$$ME=MB=MF=\frac{a}{2},\qquad NF=NC=NK=\frac{b}{2}.$$



Moreover, $BM\parallel HF\parallel CN$. Let F_1 be the foot of the perpendicular from M to HF, and let N_1 be the foot of the perpendicular from F to CN. Then the right triangles MF_1F and FN_1N are similar because $\angle MFF_1=\angle FNN_1$. Also, $HF_1=BM$ and $CN_1=HF=x$ because BMF_1H and HFN_1C are rectangles. Therefore,

$$\frac{MF}{FF_1}=\frac{FN}{NN_1}\implies\frac{\frac{a}{2}}{x-\frac{a}{2}}=\frac{\frac{b}{2}}{\frac{b}{2}-x}\implies x(a+b)=ab\implies\frac{1}{x}=\frac{1}{a}+\frac{1}{b}.$$