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EDITORIAL

What a summer. Every May I think about all the projects that I will catch up to over the summer, that I will work on a few things I've put on the back-burner, that I will get a (huge) head start on my fall term. Most summers, I manage to get some/most planned things done and squeeze in some vacation time too. But it seems nothing can derail one's plans more effectively than a global pandemic. All my summer plans were cancelled to make room for new plans of getting ready for remote teaching, and vacations were cancelled altogether. I've attended dozens of webinars (this is not an exaggeration), organized Canada-wide bi-weekly online Teaching Meet Ups, served as a lead organizer of CMS COVID-19 Research and Education Meeting, co-wrote two pieces for CMS Education Notes on remote teaching, all while prepping 4 fall courses, 2 of which are new to me, all in a remote environment, which is new to everyone. The numerous dialogues and discussions provided spaces to build a rich community, voice fears and complaints, share useful hacks and small victories.

Then I started working on ***Crux***. The combination of open access and the pandemic surely has had an effect on our submissions! I'm proud to say that this issue alone received 325 solution submissions. Many new solvers have joined our ranks and we will be adding new editors to the Board to help support this ever-growing well-established community. I'm grateful to this community for continuing support and inspiration, for personal emails from regular contributors just checking in on the health of my extended ***Crux*** family, for the steady stream of problem proposals and solutions, for the rich collection of materials that we will feature in issue 8 dedicated to the memory of Richard Guy.

As the summer comes to the end, I'm sending out good thoughts to all of our readers out there: be safe, be healthy, be kind.

P. S. The pandemic has certainly put me behind in my emails as every possible conversation has now turned into an email or a conference call. Gone are the days of "I'll just stop by your office for a quick chat". So I must apologize to Bill Sands for his wonderful timbit of math in media spotted on January 10th that he sent me on January 11th that I only dug up in July and immediately had to include in this issue.

Kseniya Garaschuk

MathemAttic

No. 17

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2020**.

MA81. Find the sum of all positive integers smaller than 1260 which are not divisible by 2 and not divisible by 3.

MA82. Let $a_n = n^2 + 2n + 50$, $n = 1, 2, \ldots$. Let d_n be the largest positive integer that is a divisor of both a_n and a_{n+1}. Find the maximum value of d_n, $n = 1, 2, \ldots$

MA83. Prove that the numbers 26^n and $26^n + 2^n$ have the same number of digits, for any non-negative integer n.

MA84. The area of the trapezoid $ABCD$ with $AB \parallel CD$, $AD \perp AB$ and $AB = 3CD$ is equal to 4. A circle inside the trapezoid is tangent to all of its sides. Find the radius of the circle.

MA85. A collection of items weighing 3, 4 or 5 kg has a total weight of 120 kg. Prove that there is a subcollection of the items weighing exactly 60 kg.

. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA81. Déterminer la somme de tous les entiers positifs plus petits que 1260 qui sont ni divisibles par 2 ni divisibles par 3.

MA82. Soit $a_n = n^2 + 2n + 50$, $n = 1, 2, \ldots$. Aussi soit d_n le plus grand entier positif diviseur de a_n et a_{n+1}. Déterminer la valeur maximale de d_n, $n = 1, 2, \ldots$

MA83. Démontrer que 26^n et $26^n + 2^n$ ont le même nombre de chiffres, pour tout entier non négatif n.

MA84. Un trapèze $ABCD$ a une surface égale à 4 ; aussi, $AB \parallel CD$, $AD \perp AB$ et $AB = 3CD$. Or, un cercle inscrit à l'intérieur du trapèze est tangent à tous les côtés. Déterminer le rayon de ce cercle.

MA85. Une collection d'items de poids 3, 4 ou 5 kg a un poids total de 120 kg. Démontrer qu'il existe une sous collection de poids total égal à 60 kg.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(2), p. 49–52.

MA56. For a given arithmetic series the sum of the first 50 terms is 200, and the sum of the next 50 terms is 2700. What is the first term of the series?

Originally from MAA Problem Book II (1961–1965), Question #26, 1961 exam.

We received 17 submissions of which 16 were correct and complete. We present the solution by Maria Hessami Pilehrood, Lester B. Pearson Collegiate Institute (grade 10), Scarborough, ON, Canada.

The first 100 terms of the arithmetic sequence are

$$a, a+d, a+2d, a+3d, \ldots, a+49d, a+50d, a+51d, \ldots, a+99d.$$

The first 50 terms

$$a, a+d, a+2d, a+3d, \ldots, a+49d$$

sum to $50a + 1225d$, and the next 50 terms

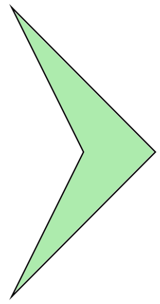
$$a+50d, a+51d, \ldots, a+99d$$

sum to $50a + 3725d$. So we get two equations

$$\begin{cases} 50a + 1225d = 200 \\ 50a + 3725d = 2700. \end{cases}$$

Subtracting the first equation from the second, we get $2500d = 2500$ and therefore, $d = 1$. Then from the first equation we get $50a + 1225 = 200$ or $50a = -1025$ and therefore, $a = -41/2 = -20.5$.

MA57. Define a *boomerang* as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees (see the accompanying figure). Let C be a convex polygon having s sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + \dfrac{s-2}{2}$.



Originally Problem 3 from the 1995 Canadian Math Olympiad.

We received one submission. We present a solution based on the submission by Arya Kondur, completed and corrected by the editor.

The sum over all interior angles of the q quadrilaterals is equal to $q \cdot 360°$.

Consider a boomerang. Let A be the vertex with the reflex angle, i.e. the angle that is greater than $180°$. Then A has to lie in the interior of the polygon. All the other quadrilaterals incident with A have an angle of less than $180°$ at A. All the angles around A are therefore interior angles of quadrilaterals (i.e. don't lie on a side of a quadrilateral). Since there cannot be two reflex angles around the same vertex, we have at least b points in the interior of the polygon that have the property that all angles around it are interior angles of quadrilaterals. Furthermore we have the s vertices of the polygon, whose angles sum to $(s-2)\cdot 180°$. Therefore we get for the sum of the interior angles of the quadrilaterals that

$$q \cdot 360° \geq b \cdot 360° + (s-2) \cdot 180°.$$

Dividing by $360°$ finishes the proof.

MA58. *Proposed by John McLoughlin.*

If the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are randomly ordered to form a nine-digit number, what is the probability that the number is divisible by 99?

We received 11 submissions of which 10 were correct and complete. We present the solution by Maria Hessami Pilehrood, modified by the editor.

In order for a number to be divisible by 99, it must be divisible by 9 and 11. Any 9-digit number with the digits 1-9 in some order is divisible by 9 since the sum of digits $1+2+\cdots+9 = 45$ is divisible by 9. Now we must find the number of such 9-digit numbers that are also divisible by 11.

Let $abcdefghi$ be such a 9-digit number. In order for it to be divisible by 11, we must have

$$(a+c+e+g+i) - (b+d+f+h) \equiv 0 \pmod{11}.$$

We consider the following cases:

Case 1. $(a+c+e+g+i)-(b+d+f+h)=0$.

Then $a+c+e+g+i=b+d+f+h=45/2=22.5$. This is impossible.

Case 2. $(a+c+e+g+i)-(b+d+f+h)=11$.

As $(a+c+e+g+i)+(b+d+f+h)=45$, we conclude that $2(a+c+e+g+i)=56$. Therefore, $a+c+e+g+i=28$ and $b+d+f+h=17$. All sets of five distinct integers from 1-9 that sum to 28 are

$$\begin{array}{lll} \{9,8,7,3,1\} & \{9,7,6,5,1\} & \{8,7,6,5,2\} \\ \{9,8,6,4,1\} & \{9,7,6,4,2\} & \{8,7,6,4,3\} \\ \{9,8,6,3,2\} & \{9,7,5,4,3\} & \\ \{9,8,5,4,2\} & & \end{array}$$

There are 9 sets in total. There are 5! ways to arrange a,c,e,g,i, 4! ways to arrange b,d,f,h and 9 different sets of values they can take, for a total of $(5!\cdot 4!)\cdot 9$ such 9-digit numbers.

Case 3. $(a+c+e+g+i)-(b+d+f+h)=-11$.

As $(a+c+e+g+i)+(b+d+f+h)=45$, we conclude that $a+c+e+g+i=17$ and $b+d+f+h=28$. All sets of five distinct integers from 1-9 that sum to 17 are $\{7,4,3,2,1\}$ and $\{6,5,3,2,1\}$. Thus there are $(5!\cdot 4!)\cdot 2$ such 9-digit numbers.

There are no more cases since we cannot have $(a+c+e+g+i)-(b+d+f+h)=\pm 22$ because 22 is even and $a+c+e+g+i+b+d+f+h=45$ is odd, which would give that $2(a+c+e+g+i)$ is odd and this is impossible. We cannot have $(a+c+e+g+i)-(b+d+f+h)\geq 33$ because then $a+c+e+g+i\geq 39$ and $b+d+f+h\leq 6$, but the sum of 4 digits is at least 10 (since 1+2+3+4=10). We also cannot have $(a+c+e+g+i)-(b+d+f+h)\leq -33$ because then $a+c+e+g+i\leq 6$, which is impossible.

Hence the probability that the number is divisible by 99 is

$$\frac{(5!\cdot 4!)\cdot 9+(5!\cdot 4!)\cdot 2}{9!}=\frac{5!\cdot 4!\cdot 11}{9!}=\frac{11}{126}.$$

MA59. Find positive integer solutions of

$$x^{x^{x^x}}=(19-y^x)y^{x^y}-74.$$

Originally problem 1 from the 1974 Grade 9 competition at the Leningrad Math Olympiad.

We received 8 submissions of which 7 were correct and complete. We present the solution by Aditya Gupta, modified by the editor.

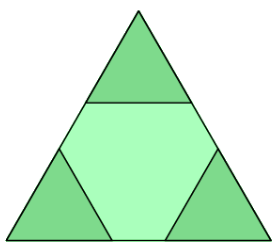
Examining both sides of the equality we see that $y^x<19$, else the LHS is positive and the RHS is negative.

For $y = 1$, the RHS is $(19-1)1-74 = -56 < 0$. As the LHS is positive, we conclude that $y \geq 2$ and $x \leq 4$ (recall $y^x < 19$).

For $x = 1$, we have the equality $1 = (19-y)y - 74 \Rightarrow y^2 - 19y + 75 = 0$ which has no integer solutions.

For $x \geq 2$, by requiring that $y^x < 19$, we see that $(x, y) = (2, 2), (3, 2), (4, 2), (2, 3)$ are the only cases to now consider. Checking these cases computationally yield $x = 2, y = 3$ as the unique solution to this equality.

MA60. Three equilateral triangles with sides of length 1 are shown shaded in a larger equilateral triangle. The total area of the three small triangles is half the area of the large triangle. What is the side-length of the larger equilateral triangle?



Originally Problem 20 from the 2019 UK Intermediate Mathematical Challenge.

We received 13 submissions, all correct. We present the solution with generalization provided by G. C. Greubel.

Consider a slightly more general problem with side length a of the smaller equilateral triangles, and side length b for the larger triangle. If the sum of the three smaller triangles is $\frac{1}{\phi}$ times the larger then:

With side length x the area of an equilateral triangle is $4A_x = \sqrt{3}\,x^2$. With this, and $3A_a = A_b/\phi$, then

$$3\left(\frac{\sqrt{3}\,a^2}{4}\right) = \frac{\sqrt{3}\,b^2}{4\,\phi},$$

or $b^2 = 3\,\phi\,a^2$. The value of the larger triangle's side length, b, is then given by $b = \sqrt{3\phi}\,a$.

For the problem given the side length of the larger triangle is $\sqrt{6}$.

PROBLEM SOLVING VIGNETTES

No. 12

Shawn Godin

Fun with Factorials

When I was in high school I really enjoyed participating in mathematics contests. In grades 9, 10 and 11 I wrote the Junior Mathematics Contest (University of Waterloo), and in grade 12 I wrote the Descartes contest (University of Waterloo) and the American High School Mathematics Examination. I used to look forward to the one day a year when I would be faced with some non-routine problems to challenge myself. I even enjoyed being faced with problems that I couldn't solve.

One problem from grade 10 sticks in my head all these years later.

> If $90! = 90 \times 89 \times 88 \times \cdots \times 2 \times 1$,
> then the exponent of the highest power of 2 which will divide 90! is
> (A) 86 (B) 45 (C) 90 (D) 75 (E) 85

That was problem 24 from the 1981 Junior Mathematics Contest. I don't remember if I solved the problem correctly or not, but something about it intrigued me so that I remember it to this day.

The problem involves something called a *factorial*, which, for positive integers, is defined as

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1.$$

These expressions arise quite often in counting problems. For example, if we have n distinct objects, the number of ways we can order them is $n!$. Thus, the number of ways to arrange the letters A, B, and C is $3! = 3 \times 2 \times 1 = 6$, since we have 3 letters to choose for the first position, then 2 are left for the middle position and 1 for the last.

Factorials show up in math competitions from time to time, so let's look at a few problems involving them. The following was problem 6 from the 2020 AMC12B.

> For all integers $n \geq 9$, the value of
>
> $$\frac{(n+2)! - (n+1)!}{n!}$$
>
> is always which of the following?
>
> (A) a multiple of 4 (B) a multiple of 10 (C) a prime number
> (D) a perfect square (E) a perfect cube

Using the definition of a factorial, we can write

$$\begin{aligned} n! &= n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1 \\ &= n \times [(n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1] = n \times (n-1)! \\ &= n \times (n-1) \times [(n-2) \times (n-3) \times \cdots \times 2 \times 1] = n \times (n-1) \times (n-2)! \\ &\vdots \end{aligned}$$

Hence we can simplify the expression in the problem as

$$\begin{aligned} \frac{(n+2)! - (n+1)!}{n!} &= \frac{(n+2) \times (n+1)! - (n+1)!}{n!} \\ &= \frac{(n+1)!((n+2)-1)}{n!} \\ &= \frac{(n+1)!(n+1)}{n!} \\ &= \frac{(n+1)^2 n!}{n!} \\ &= (n+1)^2 \end{aligned}$$

which is a perfect square.

Let's return to our original problem. Since we are interested in the largest power of 2 that divides 90!, we can imagine separating the even and odd numbers to get

$$\begin{aligned} 90! &= 90 \times 89 \times \cdots \times 2 \times 1 \\ &= (90 \times 88 \times \cdots \times 4 \times 2) \times (89 \times 87 \times \cdots \times 3 \times 1) \\ &= 2^{45} \times (45 \times 44 \times \cdots \times 2 \times 1) \times (89 \times 87 \times \cdots \times 3 \times 1) \\ &= 2^{45} \times (44 \times 42 \times \cdots \times 4 \times 2) \times (45 \times 43 \times \cdots \times 3 \times 1) \times (89 \times 87 \times \cdots \times 1) \\ &= 2^{45} \times 2^{22} \times (22 \times 21 \times \cdots \times 1) \times (45 \times 43 \times \cdots \times 1) \times (89 \times 87 \times \cdots \times 1) \end{aligned}$$

We can repeat this process over and over until we get

$$90! = 2^{45} \times 2^{22} \times 2^{11} \times 2^5 \times 2^2 \times 2^1 \times O$$

where O is the product of a number of odd numbers. From this we can see that our answer is $45 + 22 + 11 + 5 + 2 + 1 = 86$. Note, we can also interpret this as: there are 45 multiples of 2, 22 multiples of 4, 11 multiples of 8, 5 multiples of 16, 2 multiples of 32 and 1 multiple of 64 in the numbers from 1 to 90. As such, we can also evaluate our result as

$$\sum_{i=1}^{\infty} \left\lfloor \frac{90}{2^i} \right\rfloor = \left\lfloor \frac{90}{2} \right\rfloor + \left\lfloor \frac{90}{2^2} \right\rfloor + \left\lfloor \frac{90}{2^3} \right\rfloor + \cdots = 45+22+11+5+2+1+0+0+\cdots = 86,$$

where $\left\lfloor \frac{90}{2^3} \right\rfloor = \left\lfloor \frac{90}{8} \right\rfloor = 11$, for example, means that there are 11 multiples of 8 from 1 to 90. See if you can use this technique to find the largest power of 3 that divides 90!.

Let's move on to something a little more interesting. The problem below is question B2 from the 2019 Canadian Open Mathematics Challenge.

What is the largest integer n such that the quantity

$$\frac{50!}{(5!)^n}$$

is an integer?

We can use our line of thought from the last problem. Since

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 2^3 \times 3 \times 5,$$

then $(5!)^n = 2^{3n} \times 3^n \times 5^n$, which must divide evenly into $50!$. We can determine the largest exponents of powers of 2, 3, and 5 that divide $50!$ as

$$\left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{4} \right\rfloor + \left\lfloor \frac{50}{8} \right\rfloor + \left\lfloor \frac{50}{16} \right\rfloor + \left\lfloor \frac{50}{32} \right\rfloor = 25 + 12 + 6 + 3 + 1 = 47$$

$$\left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{9} \right\rfloor + \left\lfloor \frac{50}{27} \right\rfloor = 16 + 5 + 1 = 22$$

$$\left\lfloor \frac{50}{5} \right\rfloor + \left\lfloor \frac{50}{25} \right\rfloor = 10 + 2 = 12$$

Thus n must satisfy $3n \le 47,\ n \le 22,\ n \le 12$, which simplifies to $n \le 12$, so the largest value of n is 12.

We will return to factorials and investigate their role in counting problems at a later date. Enjoy the following problems involving factorials:

1. What is the units digit of the sum $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$? (2015 Cayley Contest #15)

2. Explain why there is no positive integer n for which $n!$ is divisible by 7^7 but is not divisible by 7^8. (2019 Galois Contest #3c)

3. If x and y are integers and $\frac{30!}{36^x 25^y}$ is equal to an integer, what is the maximum possible value of $x + y$? (2018 Cayley Contest #22)

4. How many positive integers n are there such that n is a multiple of 5, and the least common multiple of $5!$ and n equals 5 times the greatest common divisor of $10!$ and n? (2020 AMC12A #21)

5. A positive integer has k trailing zeros if its last k digits are all zero and it has a non-zero digit immediately to the left of these k zeros. For example, the number 1 030 000 has 4 trailing zeros. Define $Z(m)$ to be the number of trailing zeros of the positive integer m. Lloyd is bored one day, so makes a list of the value of $n - Z(n!)$ for each integer n from 100 to 10 000, inclusive. How many integers appear in his list at least three times? (2013 Cayley Contest #25)

OLYMPIAD CORNER

No. 385

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **November 15, 2020**.

OC491. Let ABC be a triangle such that $AB \neq AC$. Prove that there exists a point $D \neq A$ on its circumcircle satisfying the following property: For any points M, N outside the circumcircle on the rays AB and AC, respectively, satisfying $BM = CN$, the circumcircle of AMN passes through D.

OC492. Let ABC be a triangle with $AB = AC$ and let I be its incenter. Let Γ be the circumcircle of ABC. Lines BI and CI intersect Γ in two new points M and N respectively. Let D be another point on Γ lying on arc BC not containing A, and let E, F be the intersections of AD with BI and CI, respectively. Let P, Q be the intersections of DM with CI and of DN with BI respectively.

(i) Prove that D, I, P, Q lie on the same circle Ω.

(ii) Prove that lines CE and BF intersect on Ω.

OC493. Let a, b be real numbers such that $a < b$ and let $f : (a, b) \to \mathbb{R}$ be a function such that the functions $g : (a, b) \to \mathbb{R}$, $g(x) = (x - a)f(x)$ and $h : (a, b) \to \mathbb{R}$, $h(x) = (x - b)f(x)$ are increasing. Prove that the function f is continuous on (a, b).

OC494. Let n and q be two natural numbers, $n \geq 2$, $q \geq 2$ and $q \not\equiv 1 \pmod 4$ and let K be a finite field having exactly q elements. Prove that for every $a \in K$ there exist $x, y \in K$ such that $a = x^{2^n} + y^{2^n}$.

OC495. A box contains 2017 balls. On each ball is written exactly one integer. We randomly select two balls with replacement from the box and add the numbers written on them. Prove that the probability of getting an even sum is greater than 1/2.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC491. Soit ABC un triangle tel que $AB \neq AC$. Démontrer qu'il existe un point $D \neq A$ sur le cercle circonscrit de ABC et ayant la propriété suivante : pour tous points M et N à l'extérieur du cercle circonscrit et faisant partie des rayons AB et AC respectivement, puis tels que $BM = CM$, le cercle circonscrit de AMN passe par D.

OC492. Soit ABC un triangle tel que $AB = AC$ et soit I le centre de son cercle inscrit. Aussi, soit Γ le cercle circonscrit de ABC. Les lignes BI et CI intersectent Γ de nouveau en M et N respectivement. Aussi, soit D un autre point sur Γ, situé sur l'arc BC ne contenant pas A et soient E et F les intersections de AD avec BI et CI respectivement. Enfin, soient P et Q les intersections de DM avec CI puis DN avec BI, respectivement.

(i) Démontrer que D, I, P, Q se situent sur un même cercle Ω.

(ii) Démontrer que les lignes CE et BF intersectent en un point situé sur ce Ω.

OC493. Soient a et b des nombres réels tels que $a < b$ et soit $f : (a, b) \to \mathbb{R}$ une fonction telle que les fonctions $g : (a, b) \to \mathbb{R}$, $g(x) = (x-a)f(x)$ et $h : (a, b) \to \mathbb{R}$, $h(x) = (x-b)f(x)$ sont croissantes. Démontrer que la fonction f est continue sur (a, b).

OC494. Soient n et q deux nombres naturels tels que $n \geq 2$, $q \geq 2$, puis $q \not\equiv 1 \pmod 4$. Aussi, soit K un corps fini ayant exactement q membres. Démontrer que pour tout $a \in K$ il existe $x, y \in K$ tels que $a = x^{2^n} + y^{2^n}$.

OC495. Une boîte contient 2017 jetons. Sur chaque jeton est inscrit un nombre entier. On choisit deux jetons, avec remplacement, et on fait la somme des deux nombres. Démontrer que la probabilité que cette somme est paire est plus grande que 1/2.

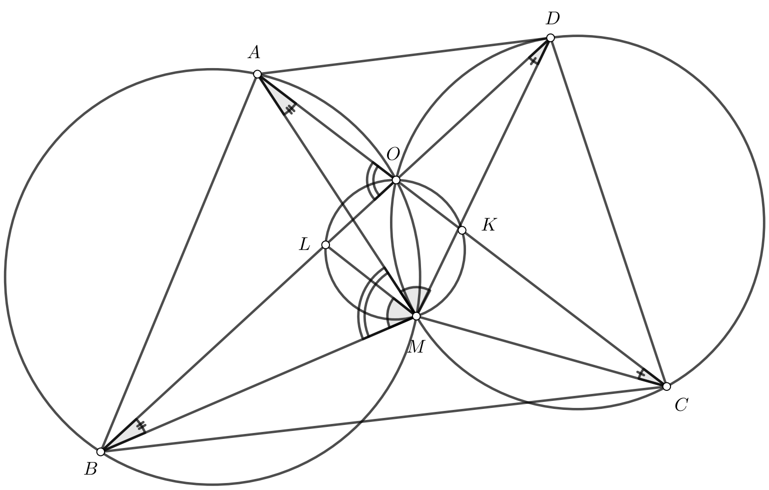
OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(2), p. 61–62.

OC466. In a convex quadrilateral $ABCD$ the diagonals AC and BD intersect at point O. The points A_1, B_1, C_1, D_1 are respectively on the segments AO, BO, CO, DO such that $AA_1 = CC_1$ and $BB_1 = DD_1$. Let M and N be, respectively, the second intersections of the circumcircles of $\triangle AOB$ and $\triangle COD$ and the circumcircles of $\triangle AOD$ and $\triangle BOC$, and let P and Q be, respectively, the second intersections of the circumcircles of $\triangle A_1OB_1$ and $\triangle C_1OD_1$ and the circumcircles of $\triangle A_1OD_1$ and $\triangle B_1OC_1$. Prove that the points M, N, P, Q lie on a circle.

Originally Bulgaria Math Olympiad, 1st Problem, Grade 9-12, Final Round 2017.

We received 2 correct solutions and one solution using Mathematica. We present the solution by UCLan Cyprus Problem Solving Group.



Let K be the midpoint of AC and L the midpoint of BD. We show that M, N, P, Q lie on the circumcircle of $\triangle OKL$. We provide complete details as to why O, K, L, M are concyclic. Since K is the midpoint of A_1C_1 and L is the midpoint of B_1D_1, similar arguments can be used for N, P, and Q. At the end we mention later what happens if $\triangle OKL$ is degenerate, equivalently $K = O$ or $L = O$.

To avoid configuration issues we use directed angles. We have

$$\angle ACM = \angle OCM = \angle ODM = \angle BDM,$$

$$\angle CAM = \angle OAM = \angle OBM = \angle DBM\,.$$

So $\triangle ACM$ and $\triangle BDM$ are directly similar. Since K and L are the midpoints of AC and BD respectively, it follows that $\angle KMA = \angle LMB$. Thus

$$\angle KML = \angle KMA + \angle AML = \angle LMB + \angle AML = \angle AMB = \angle AOB,$$

and O, K, L, M are concyclic as required.

If exactly one of K or L is equal to O, for example $K = O \neq L$, then the same proof as before shows that $\angle KML = \angle AOB$. Similar equalities holds for N, P, Q in place of M, so again M, N, P, Q are concyclic.

If $K = O = L$ then $ABCD$ is a parallelogram and $M = N = P = Q = O$. To establish this, let O_1 be the circumcentre of $\triangle AOB$ and O_2 be the circumcentre of $\triangle COD$. We have $\angle AOO_1 = 90^\circ - \angle ABO = 90^\circ - \angle ODC = \angle COO_2$. So O_1, O, O_2 are collinear, therefore the circumcircles of $\triangle AOB$ and $\triangle COD$ are externally tangent and $M = O$.

OC467. Let $p > 2$ be a prime number and let $x, y \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$. Prove that if $x\,(p-x)\,y\,(p-y)$ is a perfect square, then $x = y$.

Originally Poland Math Olympiad, 6th Problem, Second Round 2017.

We received 6 correct submissions. We present 2 solutions.

Solution 1, by Corneliu Manescu-Avram.

Let a, b, and c be positive integers such that $xy = a^2c$ and $(p-x)(p-y) = b^2c$. Then p divides $(p-x)(p-y) - xy = c(b-a)(b+a)$. Since p does not divide $c(b-a)$ it implies that p divides $b+a$ and $p \leq a+b$.

Indeed, if p divides c then p divides x and y; which is a contradiction. It is obvious that $1 \leq a < b \leq p$. Then $1 \leq b-a \leq p-1$ and p does not divide $b-a$.

Suppose that $x \neq y$. Then AM-GM inequality implies $a^2 \leq a^2c = xy < (x+y)^2/4$, and $a < (x+y)/2$. Similarly, $b < (p-x+p-y)/2$. Hence,

$$p \leq a+b < \frac{x+y}{2} + \frac{(p-x)+(p-y)}{2}.$$

This is a contradiction. It must be that $x = y$.

Solution 2, by Aditya Gupta.

Assume $xy(p-x)(p-y)$ is a perfect square. Hence there exists an integer $k > 0$ such that $xy(p-x)(p-y) = k^2$. Then $pxy(p-(x+y)) = (k-xy)(k+xy)$, and p divides $(k-xy)(k+xy)$. Note that $x+y < p$, and $k > xy$.

First, suppose that p is a divisor of $k-xy$, and $k-xy = ap$ for some integer $a > 1$. It follows that $pxy(p-(x+y)) = ap(ap+2xy)$, equivalently $p(xy-a^2) = xy(x+y+2a)$. As p is prime and p does not divide x or y, it implies that p divides $x+y+2a$. Then, $x+y+2a \geq p$. The equality $p(xy-a^2) = xy(x+y+2a)$ cannot hold, as the left side $p(xy-a^2) < pxy$ and the right side $xy(x+y+2a) \geq pxy$.

Second, suppose that p is a divisor of $k + xy$, and $k + xy = ap$ for some integer $a > 0$. Similarly, it follows that $p(xy-a^2) = xy(x+y-2a)$, and p divides $x+y-2a$. Note that $x + y - 2a \geq 0$, as AM-GM inequality implies:

$$a = \frac{k + xy}{p} = \frac{\sqrt{x(p-y)y(p-x)} + xy}{p} \leq \frac{1}{p}\left(\frac{x(p-y) + y(p-x)}{2} + xy\right) = \frac{x+y}{2}.$$

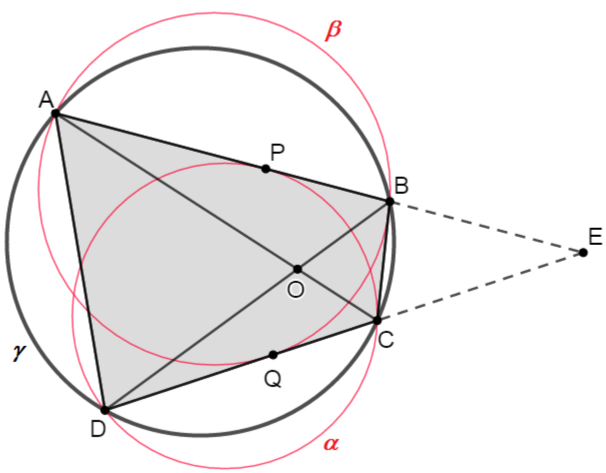
As before, if $x + y - 2a > 0$ a contradiction is obtained. Since p divides $x + y - 2a$ it follows that $x + y - 2a \geq p$. Then $p(xy - a^2) = xy(x + y - 2a)$ cannot hold. Therefore, $x + y - 2a = 0$ and $xy = a^2$, implying that $x = y$.

OC468. Let $ABCD$ be a cyclic quadrilateral. The point P is chosen on the line AB such that the circle passing through C, D and P touches the line AB. Similarly, the point Q is chosen on the line CD such that the circle passing through A, B and Q touches the line CD. Prove that the distance between P and the line CD equals the distance between Q and the line AB.

Originally Germany Math Olympiad, 4th Problem, Final Round 2017.

We received 9 submissions of which 7 were correct and complete. We present a solution based on the submissions of Ivko Dimitrić for the case of $AB \parallel CD$, Ronald Martins and C. R. Pranesachar for the case of $AB \nparallel CD$.

If $AB \parallel CD$, then the distance from P to the line DC is equal to the distance from Q to the line AB, both being equal to the distance between these two parallel lines.



If AB and CD intersect, let E be their intersection point as shown in the figure. Let α be the circle passing through C, D, and P, β the circle passing through A, B, and Q, and γ the circle that circumscribes $ABCD$.

If $P_{E,\alpha}$, $P_{E,\beta}$, and $P_{E,\gamma}$ are powers of point E in relation to circles α, β, and γ, then $P_{E,\alpha} = EP^2 = EC \cdot ED$, $P_{E,\beta} = EQ^2 = EB \cdot EA$.

But, since $P_{E,\gamma} = EB \cdot EA = EC \cdot ED$, we conclude that $EP = EQ$, implying that $\triangle EPQ$ is isosceles. Therefore, the heights relatives to the sides EP and EQ

are congruent. In other words, the distance between P and the line CD equals the distance between Q and the line AB.

It is interesting to note the observation of C. R. Pranesachar. The circle α is not unique. There exist two circles passing through C and D and being tangent to AB. If P_1 and P_2 are the tangency points on AB then P_1 and P_2 are equidistant from E by the tangent-secant theorem $EP_1^2 = EP_2^2 = EC \cdot ED$. Moreover P_1 and P_2 are equidistant from CD. Similarly, there exist two circles passing through A, B and being tangent to CD. The tangency points Q_1 and Q_2 are equidistant from E and AB. The above proof holds for both versions of the circle α and β.

OC469. Prove that for all nonnegative real numbers x, y, z satisfying $x+y+z=1$ it holds

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

Originally Germany Math Olympiad, 5th Problem, Final Round 2017.

We received 17 submissions of which 16 were correct and complete submissions. We present two solutions.

Solution 1, by Subhankar Gayen.

As $x, y, z \geq 0$ and $x+y+z=1$ it follows that $0 < 1-yz \leq 1$, $0 < 1-zx \leq 1$, $0 < 1-xy \leq 1$, and

$$1 = \frac{x}{1} + \frac{y}{1} + \frac{z}{1} \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy}.$$

Due to AM-GM inequality it follows that $yz \leq (y+z)^2/4$. Moreover $y+z = 1-x$, hence $yz \leq (1-x)^2/4 \leq 1/4$ and

$$\frac{x}{1-yz} \leq \frac{x}{1-(1-x)^2/4} = \frac{4x}{3+2x-x^2} \leq \frac{63x+3}{64}. \tag{1}$$

The last inequality is due to $(7x-9)(3x-1)^2 \leq 0$ for $0 \leq x \leq 1$. Inequalities similar to (1) can be obtained if permuting x, y, and z. In conclusion,

$$\frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{63(x+y+z)+9}{64} = \frac{9}{8},$$

as required.

Solution 2, by Antonopoulos Panagiotis.

By setting $x+y+z=p$, $xy+yz+zx=q$, and $xyz=r$ we get $p=1$, $x^2+y^2+z^2 = 1-2q$, and $x^2(y+z)+y^2(z+x)+z^2(x+y) = q-3r$. We conclude

$$\frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} = \frac{x^2}{x-r} + \frac{y^2}{y-r} + \frac{z^2}{z-r} = \frac{(4-2q)r-q+1}{-r^2+r-q+1}.$$

Denote by A the expression on the line above. We have to prove that $1 \leq A \leq 9/8$, or equivalently

$$0 \leq r - 2q + 3 \quad \text{and} \quad 9r^2 + 23r - 1 + q(1 - 16r) \leq 0. \qquad (2)$$

Due to linearity in q of the algebraic expressions in (2) we can apply the Q-lemma of the PQR-method. Hence the validity of $1 \leq A \leq 9/8$ in general follows from its validity for $(x, y, z) = (x, 1 - 2x, x)$ and $0 < x < 1/2$. However, for $(x, y, z) = (x, 1 - 2x, x)$ we have that $1 \leq A \leq 9/8$ is equivalent

$$1 \leq \frac{2x}{1 - x(1 - 2x)} + \frac{1 - 2x}{1 - x^2} \leq \frac{9}{8},$$

and with $(2x - 1)(x - 3) \geq 0$ and $(3x - 1)^2(2x^2 - 5x - 1) \leq 0$. The last two inequalities are valid for $0 < x < 1/2$.

Editor's Comments. Šefket Arslanagić provided the values of x, y, z for which the inequalities are in fact equalities. For the left inequality the values are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$, and $(0, 1/2, 1/2)$ and for the right side the values are $(1/3, 1/3, 1/3)$.

Oliver Geupel mentioned that the inequality was previously published as problem 273 in *Inequalities: Theorems, Techniques and Selected Problems*, Zdravko Cvetkovski, 2012, Springer, 400-402.

OC470. Prove that there is a natural number n having more than 2017 divisors d such that

$$\sqrt{n} \leq d < 1.01\sqrt{n}.$$

Originally Poland Math Olympiad, 5th Problem, First Round 2017.

We received 5 correct submissions. We present 2 solutions.

Solution 1, by Richard Hess.

Let $n = (k!)^2$, for some $k \geq 2118$. Then $k!(m + 1)/m$ is a divisor of n for any m such that $101 \leq m \leq 2117$. In conclusion, we constructed 2018 distinct divisors of n, and each of these divisors has the property:

$$\sqrt{n} = k! < k!\frac{m + 1}{m} < k!\frac{101}{100} = 1.01\sqrt{n}.$$

Solution 2, by UCLan Cyprus Problem Solving Group.

Let $n = 3^{2017} \cdot 100^m \cdot 101^m$ where m is chosen such that $3^{2017} < (101/100)^m$. For each $0 \leqslant i \leqslant 2017$ and each $1 \leqslant k \leqslant m$ define $a_{i,k} = 3^i \cdot 100^{m-k} \cdot 101^k$. Clearly the $a_{i,k}$'s are distinct divisors of n.

For each $0 \leqslant i \leqslant 2017$ let k_i be minimal such that $a_{i,k_i} \geqslant \sqrt{n}$ and define $d_i = a_{i,k_i}$. Note that there is such a k_i as $a_{i,m} = 3^i \cdot 101^m \geqslant 101^m > \sqrt{n}$.

So we constructed 2018 distinct divisors of n, d_i, such that $\sqrt{n} < d_i$ for each i. Furthermore,

$$d_i = a_{i,k_i} = \frac{101}{100} \cdot 3^i \cdot 100^{m-(k_i-1)} \cdot 101^{k_i-1} = a_{i,k_i-1} < 1.01\sqrt{n}\,.$$

Note that the above solutions can be easily modified to prove the obvious generalization: For every $\delta > 0$ and every $N \in \mathbb{N}$ there is a positive integer n such that n has more than N divisors d satisfying $\sqrt{n} < d < (1+\delta)\sqrt{n}$.

Timbits, Fruit Loops and math in the media

In a January 10 editorial of the Vancouver Island Free Daily (https://www.vancouverislandfreedaily.com/) titled "Timbit cereal seems like a mistake" by Max Winkelman, the following paragraph appears:

"For those wondering, the chocolate-glazed Timbit cereal version has 17 grams of sugar for every 36 grams of cereal and sugar is the first item listed on the ingredients, according to the Post Consumer Brands website. To compare that to another well-known sugary cereal, Froot Loops has 12 grams of sugar for every 39 grams of cereal. Once you even out portion sizes that means there is 16.4 per cent more sugar content than in (regular) Froot Loops."

The number 16.4 sounded wrong to me, so I investigated. Here are three questions:

1. Is the number 16.4 correct?
2. If not, what is the correct percentage? That is, in equal amounts of cereal, how much more sugar in percentage terms does Timbits have than Froot Loops has?
3. Where (likely) did the number 16.4 come from?

Find the answers on page 308.
By Bill Sands.

FOCUS ON...

No. 42

Michel Bataille

Three Inversions

Introduction

In Focus On... No 8 [2013 : 307], which was also devoted to inversions, the stress was put on those of possibly negative power. In the present number, we consider three particular inversions and their interventions in problems about the triangle.

I am aware that inversion is not often taught nowadays (in France, it disappeared off the high-school syllabus some fifty years ago), but only the more elementary properties of this transformation are needed here; the beginner can find them for example in [**1**].

Throughout what follows, we adopt the following notations: ABC is a triangle with circumcircle Γ centered at O and orthocenter H. The feet of the altitudes from A, B, C are A', B', C', respectively, and M is the midpoint of BC. To avoid particular cases, it is always assumed that ABC is scalene and not right-angled.

Exchanging the orthocenter and a foot of altitude

In this paragraph, we consider the inversion $\mathbf{I}$ with centre A such that $\mathbf{I}(H) = A'$. To become familiar with it, here are some simple, readily proved properties of $\mathbf{I}$:

(1) The line BC inverts into the circle Γ_h with diameter AH, that is, the circumcircle of the triangle $AB'C'$. Note that its centre is the midpoint U of AH and recall that $\overrightarrow{AU} = \overrightarrow{UH} = \overrightarrow{OM}$.

(2) We deduce that $\mathbf{I}(B) = C'$ and $\mathbf{I}(C) = B'$. It follows that $\mathbf{I}(\Gamma)$ is the line $B'C'$.

(3) The line BB' (respectively CC') inverts into the circle with diameter AC (respectively AB).

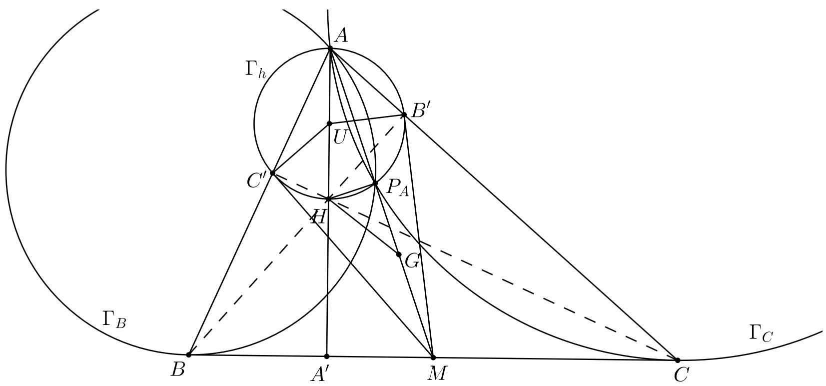
Our first application offers a surprising, perhaps new property of isogonal lines in $\angle BAC$:

> Let D, E be points on the line BC, with $D, E \neq B, C$, such that AD and AE are symmetrical in the internal bisector of $\angle BAC$. Then the circumcircle of ΔADE is tangent to Γ.

The points $D' = \mathbf{I}(D)$ and $E' = \mathbf{I}(E)$ lie on the circle Γ_h (by (1)). The reflection in the diameter of Γ_h perpendicular to $B'C'$ exchanges B' and C' and leaves Γ_h invariant, hence exchanges D' and E' (since $\angle(AC', AD') = \angle(AE', AB')$). Therefore $D'E'$ and $B'C'$ are parallel lines and their inverses, namely the circumcircles of ADE and ABC, are tangent at A.

Our next example is a variant of solution to problem **12073** of *The American Mathematical Monthly*, proposed in November 2018 and solved in March 2020:

> Let G be the centroid of ABC and P_A be the second point of intersection of the two circles through A that are tangent to BC at B and C. Similarly define P_B and P_C. Prove that G, H, P_A, P_B, and P_C are concyclic.



Let Γ_B (resp. Γ_C) be the circle through A and tangent to BC at B (resp. C). Since $MB^2 = MC^2$, M has the same power with respect to Γ_B and Γ_C, hence lies on their radical axis AP_A. In other words, P_A is on the median AG.

It suffices to show that HP_A is perpendicular to this median, since then P_A is on the circle with diameter HG; in a similar way, the same is true of P_B, P_C and the concyclicity of G, H, P_A, P_B, P_C follows.

We observe that U, A', B', C', M lie on the nine-point circle of which UM is a diameter (since $UA' \perp A'M$), hence $UC' \perp C'M$ and $UB' \perp B'M$.

The inverse of Γ_B is a line tangent at C' to $\mathbf{I}(BC) = \Gamma_h$, hence is the line $C'M$. Similarly, $\mathbf{I}(\Gamma_C) = MB'$ and therefore $\mathbf{I}(P_A) = M$. We deduce that H, A', P_A, M are concyclic. Because $HA' \perp A'M$, the circle containing these points is the circle with diameter HM. As a result, $HP_A \perp P_AM$, as desired.

With the help of the inversion $\mathbf{I}$, we can prove the following interesting supplement:

> The point P_A satisfies $\dfrac{P_AB}{P_AC} = \dfrac{AB}{AC}$.

Indeed, since $\mathbf{I}(P_A) = M$, we have

$$P_AB = \frac{|p| \cdot MC'}{AM \cdot AC'}, \qquad P_AC = \frac{|p| \cdot MB'}{AM \cdot AB'}$$

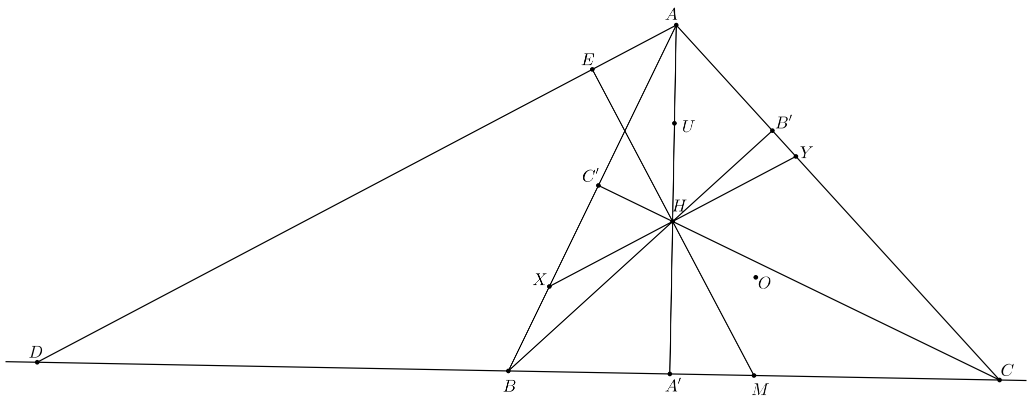
where $p = \overrightarrow{AH} \cdot \overrightarrow{AA'}$ and so

$$\frac{P_AB}{P_AC} = \frac{MC'}{MB'} \cdot \frac{AB'}{AC'}.$$

But M is the centre of the circle through B, B', C, C', hence $MB' = MC'$; also $\frac{AB'}{AC'} = \frac{AB}{AC}$ (since $AB \cdot AC' = AB' \cdot AC (= |p|)$) and the result follows.

As a new example, we consider a problem inspired by, and actually equivalent to, problem **4355** [2018 : 258 ; 2019 : 277], of which an analytic solution was featured. Thanks to the inversion **I**, we can propose a geometric solution.

> If the perpendicular to HM through H meets AB at X and AC at Y, prove that H is the midpoint of XY.



Let the perpendicular to HM through A intersect the line BC at D and the line HM at E. The key idea is to prove that D is the harmonic conjugate of A' with respect to B, C. Then, the line AD, which is parallel to XY, is the harmonic conjugate of AA' with respect to AB and AC and so H is the midpoint of XY (see Focus On... No 32 if necessary).

First, we prove that E is on the circumcircle Γ of ABC. To this end, we observe that OU is parallel to HM (since $\overrightarrow{OM} = \overrightarrow{UH}$), hence is perpendicular to DE. In addition, since $\angle AEH = 90°$, E is on Γ_h and therefore $UA = UE$. It follows that the line OU is the perpendicular bisector of AE. Thus $OA = OE$ and E lies on Γ.

Now, because E is on Γ_h, we have $\mathbf{I}(E) = D$. Since E is also on Γ, we obtain that the point D is on the line $B'C'$ (see property (2) above). The tranversals $DB'C'$ and DBC show that H is on the polar of D with respect to the circle with centre M passing through B, C, B', C'. Finally, this polar is the altitude AH and A' is the harmonic conjugate of D with respect to B, C.

For more examples involving the inversion **I**, we refer the reader to my solutions to problems **4244** [2017 : 215 ; 2018 : 221] and **4440** [2019 : 198 ; 2019 : 533], as well as exercise 1 at the end.

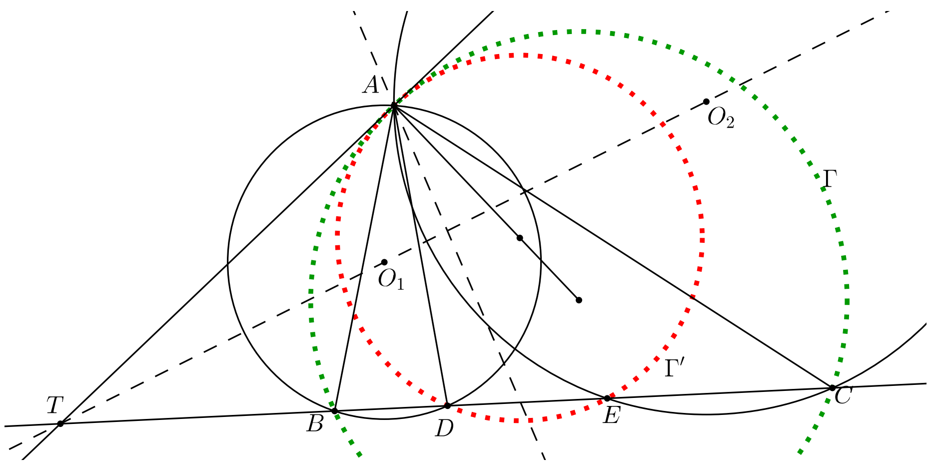
We will mention two more inversions connected to the triangle, less frequently met than **I**, but of interest nevertheless.

An inversion in a circle orthogonal to Γ

Denoting by T the point of intersection of BC and the tangent to the circumcircle Γ at A, we consider the inversion $\mathbf{J}$ in the circle with centre T and radius TA.

We use this inversion to give an alternative solution to problem **2052**, proposed in the October 2018 issue of *The Mathematics Magazine*.

> If D is a point on line BC such that $D \neq B$ and $D \neq C$, let E lie on BC so AE is the reflection of AD across the bisector of $\angle BAC$. Let O_1, O_2 be the circumcentres of triangles ΔABD and ΔACE, respectively. Prove that there exists a point P, independent of the choice of D, such that the line O_1O_2 passes through P.

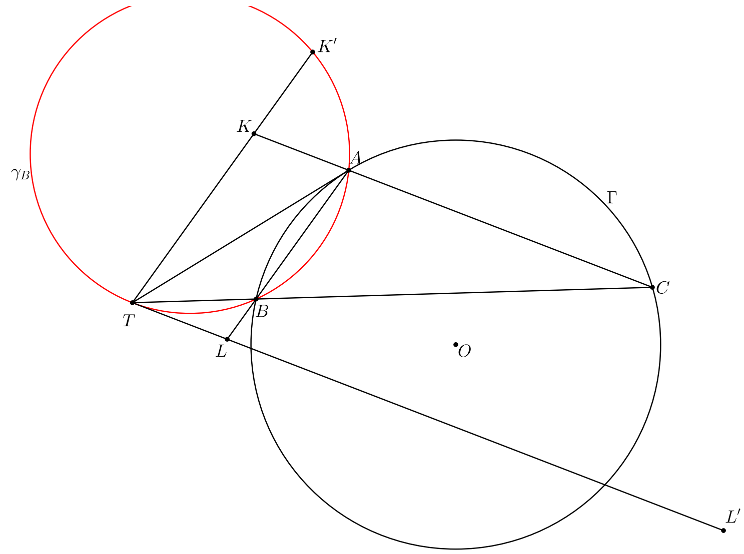


As we have seen in the previous paragraph, the circumcircle Γ' of ΔADE is tangent to Γ at A, hence TA^2 is the power of T with respect to Γ as well as to Γ'. Therefore $\overrightarrow{TB} \cdot \overrightarrow{TC} = \overrightarrow{TD} \cdot \overrightarrow{TE}$ and the inversion $\mathbf{J}$ satisfies $\mathbf{J}(B) = C$, $\mathbf{J}(D) = E$ and $\mathbf{J}(A) = A$. Note that $T \neq D, E$ since otherwise AE or AD is parallel to BC. Since the circumcircles ΔABD and ΔACE are inverse of each other, their centres are on a line through T. Thus $P = T$ answers the problem.

Another example of application of $\mathbf{J}$ is provided by problem **3439** [2009 : 395, 397 ; 2010 : 401], here adapted to agree with our notations:

> The parallel to AB through T intersects the line AC in K and the parallel to AC through T intersects the line AB in L. Prove that the lines KL and OT are perpendicular.

The inversion $\mathbf{J}$ transforms the lines AC into the circumcircle γ_B of ΔTAB. In consequence, the point $K' = \mathbf{J}(K)$ is the point of intersection other than T of γ_B and the line TK.



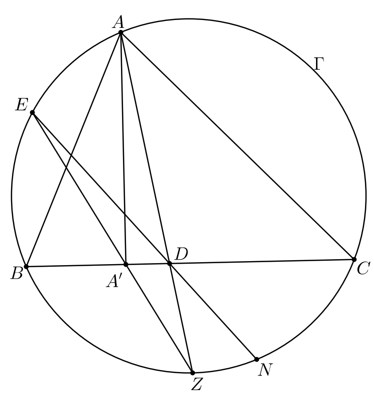
Observing that the parallel chords TK' and AB of γ_B have the same perpendicular bisector, we deduce that $OK' = OT$. Similarly, the point $L' = \mathbf{J}(L)$ satisfies $OL' = OT$. Thus, O is the centre of the circumcircle of $\Delta TK'L'$. But this circle inverts into the line KL, hence this line is perpendicular to OT.

An inversion centered on Γ

Finally, let Z be the midpoint of the arc BC of the circumcircle Γ not containing A and consider the inversion $\mathbf{K}$ in the circle with centre Z and radius $ZB = ZC$. Obviously, B and C are invariant under $\mathbf{K}$ so that $\mathbf{K}(\Gamma) = BC$.

This inversion provides a very simple solution to slightly modified problem **4216** [2017 : 68,70; 2018 : 78]:

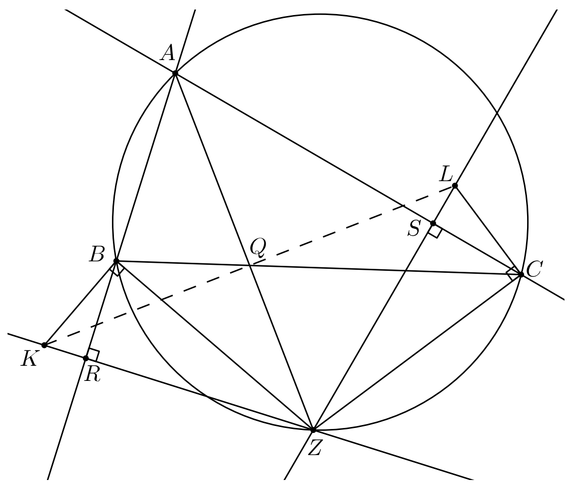
> Let $AZ \cap BC = \{D\}$, $ZA' \cap \Gamma = \{Z, E\}$, $ED \cap \Gamma = \{E, N\}$. Prove that AN is a diameter of Γ.



Since $\mathbf{K}(BC) = \Gamma$, we have $\mathbf{K}(A') = E$ and $\mathbf{K}(D) = A$, hence E, A', A, D are concyclic. It follows that E is on the circle with diameter AD and so $AE \perp EN$. Now, the circle with diameter AN passes through the points A, N, E, hence coincides with Γ.

Our second example is problem **3756** [2012 : 242,244 ; 2013 : 282]:

> The perpendiculars to AB through Z and to ZB through B intersect at K and the perpendiculars to AC through Z and to ZC through C intersect at L. Prove that the lines BC, AZ intersect at the midpoint of KL.



We introduce the orthogonal projections R and S of Z onto the lines AB and AC, respectively. Note that BR (resp. CS) is the altitude from B (resp. C) in the right-angled triangle ZBK (resp. ZCL). It follows that

$$ZR \cdot ZK = ZB^2 = ZC^2 = ZS \cdot ZL,$$

so that $\mathbf{K}(R) = K$, $\mathbf{K}(S) = L$.

Now, let $Q = \mathbf{K}(A)$. Clearly, Q is on the line AZ and on BC (since A is on Γ). We have to show that Q is also the midpoint of KL. To this aim, note that the circle with diameter AZ passes through R and S, hence its inverse, namely the line KL, is orthogonal to the line AZ.

As a result, $Q = \mathbf{K}(A)$ is on the line KL and observing that $ZK = ZL$ (note that $ZR = ZS$, Z being on the internal bisector of $\angle BAC$), AZ is the perpendicular bisector of KL. Thus, Q is the midpoint of KL (and $AK = AL$ as a bonus).

As usual we conclude with exercises. Use $\mathbf{I}$ in exercise 1 and $\mathbf{J}$ in exercise 2. We keep the notations introduced above.

Exercises

1. Let the line parallel to BC through H intersect AB at P and AC at Q and let the perpendicular to AC through A intersect the line PQ at Y. Let X be the point of intersection other than A of the circle with diameter AC and the circumcircle of ΔAPQ. Prove that C, X, Y are collinear.

2. Let A_1 be the reflection of A in the line BC. The circle passing through A_1 and tangent to BC at B intersects Γ at D $(D \neq B)$. Prove that $TD = TA$.

Reference

[**1**] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 2007.

Timbits, Fruit Loops and math in the media (answers)

1. No.
2. 53.47%, arrived at by the calculation $(17/36)/(12/39) = 221/144 = 1.5347\ldots$. Note that this figure is much worse than the percentage quoted!
3. Just a guess of course, but I bet the writer (or the person he asked) used subtraction instead of division, getting $17/36 - 12/39 = .164\ldots$. The correct interpretation of this is that there are 16.4 more grams of sugar in 100 gm of Timbits than in 100 gm of Froot Loops.

By Bill Sands.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **November 15, 2020**.

4561. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$ and let $w_1, w_2, \ldots, w_n$ be distinct complex numbers such that $w_1 + w_2 + \cdots + w_n = 1$. For $k = 1, 2, , \ldots, n$, let $P_k(x) = \prod_{j=1, j\neq k}^{n} (x - w_j)$. If z is a complex number, evaluate

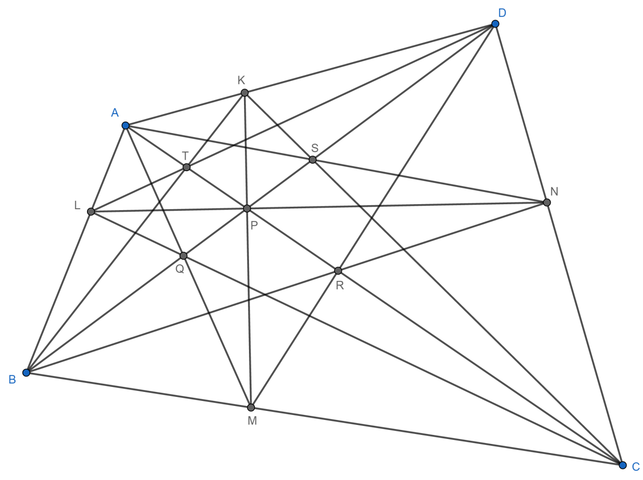
$$\sum_{k=1}^{n} \frac{w_k P_k(zw_k)}{P_k(w_k)}.$$

4562. *Proposed by Pericles Papadopoulos.*

Let P be the intersection point of the diagonals AC and BD of a convex quadrilateral $ABCD$. The angle bisector of the opposite angles $\angle APD$ and $\angle BPC$ intersects AD and BC at points K and M respectively, and the angle bisector of the opposite angles $\angle APB$ and $\angle CPD$ intersects AB and DC at points L and N respectively. Show that:

(a) $(DK)(AL)(BM)(CN) = (KA)(LB)(MC)(ND)$.

(b) Cevians AM, BP, CL concur at point Q, cevians BN, CP, DM concur at point R, cevians AN, DP, CK concur at point S, and cevians DL, BK, PA concur at point T.



4563. *Proposed by George Stoica.*

Find all perfect squares in the sequence $x_0 = 1, x_1 = 2, x_{n+1} = 4x_n - x_{n-1}, n \geq 1$.

4564. *Proposed by Alijadallah Belabess.*

Let a, b, c and d be non-negative real numbers with $ab + bc + cd + da = 4$. Prove that:

$$a^3 + b^3 + c^3 + d^3 + 4abcd \geq 8$$

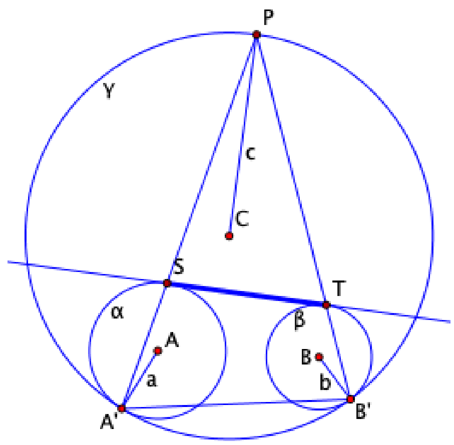
4565. *Proposed by Daniel Sitaru.*

Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC. Prove that

$$4(am_bm_c + bm_cm_a + cm_am_b) \geq 9abc.$$

4566. *Proposed by J. Chris Fisher.*

Given three circles, α, β, and γ with centers A, B, C and radii a, b, c, respectively, where γ is tangent to α at A' and to β at B', either both circles externally or both internally. One exterior common tangent line is tangent to α at S and to β at T.



(a) Prove that the lines $A'S$ and $B'T$ intersect at a point of γ.
(b) Show that

$$(A'B')^2 = \frac{c^2 \cdot ST^2}{(c \pm a)(c \pm b)},$$

where the plus signs are used when α and β are externally tangent to γ, and the negative signs when internally tangent to γ.

Comment. Part (b) is problem 1.2.8 on page 5 of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems, San Gaku* (The Charles Babbage Research Centre, 1989). Instead of a proof, the authors provide (on page 82) a reference to a 19th century Japanese geometry text together with the comment, “Called ‘Three Circles and Tangent Problem’, or ‘Sanen Bousha’, and applied in the solution to many problems.”

4567. *Proposed by Paul Bracken.*

Prove that for any $n \in \{0, 1, 2, 3, \ldots\}$, the following holds

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1)^{2n+1} = (-1)^n 2^{2n} (2n+1)!$$

4568. *Proposed by Song Qing, Leonard Giugiuc and Michael Rozenberg.*

Let k be a fixed positive real number. Consider positive real numbers x, y and z such that

$$xy + yz + zx = 1 \quad \text{and} \quad (1+y^2)(1+z^2) = k^2(1+x^2).$$

Express the maximum value of the product xyz as a function of k.

4569. *Proposed by Nguyen Viet Hung.*

Solve the following equation in the set of real numbers

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125.$$

4570. *Proposed by Lorian Saceanu.*

If ABC is an acute angled triangle, then

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{9}{\sqrt{11 + \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}}} \leq \frac{3\sqrt{3}}{2}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 novembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4561. *Proposé par Michel Bataille.*

Soit n un entier tel que $n \geq 2$ et soient $w_1, w_2, \ldots, w_n$ des nombres complexes distincts tels que $w_1 + w_2 + \cdots + w_n = 1$. Pour $k = 1, 2, \ldots, n$, soit $P_k(x) = \prod_{j=1, j\neq k}^{n} (x - w_j)$. Si z est un nombre complexe, évaluer

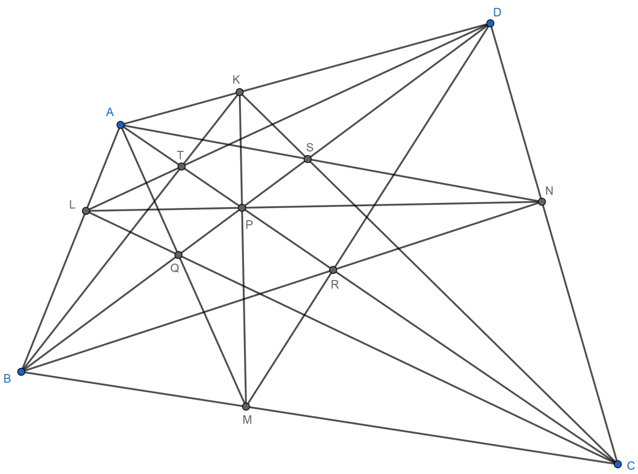
$$\sum_{k=1}^{n} \frac{w_k P_k(z w_k)}{P_k(w_k)}.$$

4562. *Proposé par Pericles Papadopoulos.*

Soit P le point d'intersection des diagonales AC et BD d'un quadrilatère convexe $ABCD$. Les bissectrices des angles opposés $\angle APD$ et $\angle BPC$ intersectent AD et BC en K et M respectivement, tandis que les bissectrices des angles opposés $\angle APB$ et $\angle CPD$ intersectent AB et DC en L et N respectivement. Démontrer:

(a) $(DK)(AL)(BM)(CN) = (KA)(LB)(MC)(ND)$.

(b) Les céviannes AM, BP and CL sont concourantes en un point Q, les céviannes BN, CP et DM sont concourantes en un point S et les céviannes DL, BK and PA sont concourantes en un point T.



4563. *Proposé par George Stoica.*

Déterminer tous les carrés parfaits dans la suite $x_0 = 1, x_1 = 2, x_{n+1} = 4x_n - x_{n-1}, n \geq 1$.

4564. *Proposé par Alijadallah Belabess.*

Soient a, b, c et d des nombres réels non négatifs tels que $ab + bc + cd + da = 4$. Démontrer:

$$a^3 + b^3 + c^3 + d^3 + 4abcd \geq 8.$$

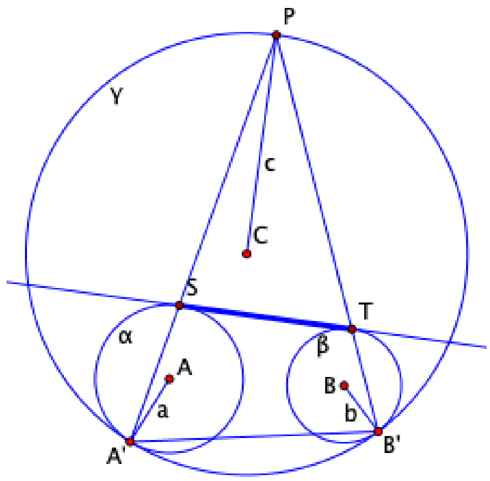
4565. *Proposé par Daniel Sitaru.*

Soient m_a, m_b et m_c les longueurs des médianes du triangle ABC. Démontrer que

$$4(am_bm_c + bm_cm_a + cm_am_b) \geq 9abc.$$

4566. *Proposé par J. Chris Fisher.*

Soient α, β et γ trois cercles de centres A, B et C, puis de rayons, a, b et c respectivement ; de plus, γ est tangent à α en A' et à β en B', les deux de façon externe ou les deux de façon interne. Une ligne externe est tangente à α en S et à beta en T.



(a) Démontrer que les lignes $A'S$ et $B'T$ intersectent en un point sur γ.
(b) Démontrer que

$$(A'B')^2 = \frac{c^2 \cdot ST^2}{(c \pm a)(c \pm b)},$$

où les signes plus sont utilisés lorsque α et β sont tangents à l'externe de γ, et les signes moins lorsqu'ils sont tangents à l'interne de γ.

Commentaire. La partie (b) est le problème 1.2.8 en page 5 de H. Fukugawa et D. Pedoe, *Japanese Temple Geometry Problems, San Gaku* (Charles Babbage Research Centre, 1989). Au lieu de fournir une preuve, les auteurs fournissent, en page 82, une référence à une texte japonais de géométrie du 19ième siècle, puis le commentaire "Dit 'Problème des trois cercles et tangentes' ou 'Sanen Bousha' et appliqué à la solution de bien des problèmes."

4567. *Proposé par Paul Bracken.*

Démontrer que pour tout $n \in \{0, 1, 2, 3, \ldots\}$, la suivante tient

$$\sum_{k=0}^{n} (-1)^k \binom{2n+1}{n-k} (2k+1)^{2n+1} = (-1)^n 2^{2n}(2n+1)!$$

4568. *Proposé par Song Qing, Leonard Giugiuc et Michael Rozenberg.*

Soit k un nombre réel positif. Soient x, y et z des nombres réels positifs tels que

$$xy + yz + zx = 1 \quad \text{et} \quad (1 + y^2)(1 + z^2) = k^2(1 + x^2).$$

Exprimer la valeur maximale du produit xyz en termes de k.

4569. *Proposé par Nguyen Viet Hung.*

Résoudre l'équation suivante dans l'ensemble des nombres réels

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125.$$

4570. *Proposé par Lorian Saceanu.*

Si ABC est un triangle acutangle, alors

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{9}{\sqrt{11 + \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}}} \leq \frac{3\sqrt{3}}{2}.$$

BONUS PROBLEMS

These problems appear as a bonus. Their solutions will **not** *be considered for publication.*

B26. *Proposed by Leonard Giugiuc.*

Let ABC be a triangle and let M be a point in its interior. Prove that if

$$PA^3 + PB^3 + PC^3 \geq MA^3 + MB^3 + MC^3$$

for any point P in the triangle's plane, then

$$\frac{MA}{[MBC]} = \frac{MB}{[MCA]} = \frac{MC}{[MAB]}.$$

B27. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

Let D be the foot of the altitude to the hypotenuse BC of a triangle ABC having a right angle at A, and let Γ be the circle with centre A and radius AD. If S is an arbitrary point on the leg AB exterior to Γ whose tangents to that circle intersect the hypotenuse in a pair of points P and Q, prove that $\dfrac{AS}{SB}$ is the arithmetic mean of $\dfrac{CP}{PB}$ and $\dfrac{CQ}{QB}$.

B28. *Proposed by Francisco Javier Garcia Capitan, Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be a scalene triangle with incenter I. Let M and N be the midpoints of the sides AB and AC, respectively. The lines MI and AC intersect each other at Q. Similarly, the lines NI and AB intersect each other at P. Prove that if $[ABC] = 2[APQ]$, then $\angle A = \frac{\pi}{2}$.

B29. *Proposed by Leonard Giugiuc and Michael Rozenberg.*

Find real numbers a, b and c such that

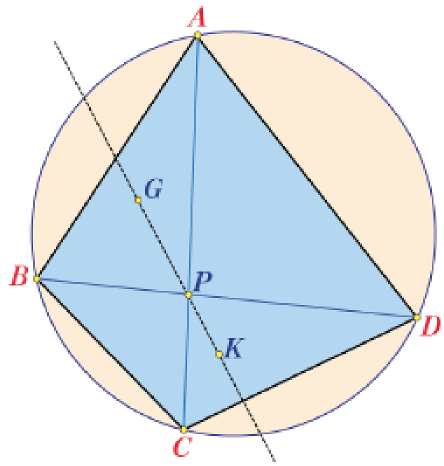
$$\left|\frac{a+b}{a-b}\right| + \left|\frac{b+c}{b-c}\right| + \left|\frac{c+a}{c-a}\right| = 2.$$

B30. *Proposed by Leonard Giugiuc.*

Let ABC be a scalene acute angled triangle such that $AC > BC > AB$. Prove that the Euler line of the triangle does not intersect the side BC.

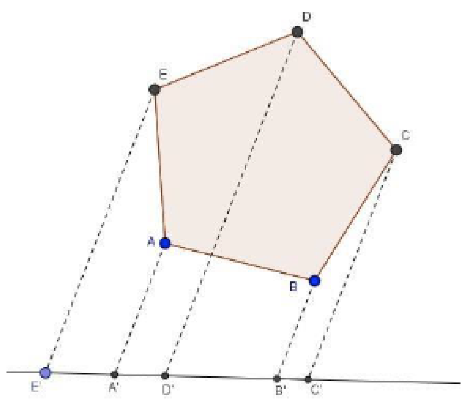
B31. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Consider a quadrangle $ABCD$ such that $AB \cdot CD = AD \cdot BC$. Let P be the point of intersection of its diagonals, G be the centroid of APB and K a symmedian point of PCD. Prove that G, P and K are collinear.



B32. *Proposed by Leonard Giugiuc, Borislav Mirchev and Kadir Altintas.*

Let $ABCDE$ be a regular pentagon. Through its vertices A, B, C, D and E we draw five parallel lines that intersect a line outside of the pentagon in points A', B', C', D' and E', respectively. Show that $A'D' \cdot A'E' = B'C' \cdot B'D'$.



B33. *Proposed by Leonard Giugiuc.*

Determine all possible values of angle A in a triangle ABC such that

$$(\cot B + \cot C)^2(\cot B \cot C + 1) = (\cot^2 B + 1)(\cot^2 C + 1).$$

B34. *Proposed by Tran Quan Hung, Leonard Giugiuc and Kadir Altintas.*

Let ABC be a triangle with an acute angle A. Suppose the point P lies inside ABC on the bisector of the angle A such that $\angle BPC = \pi - \angle A$. Let PB and PC intersect the sides AC and AB at E and F, respectively. Let K be the projection of P on EF and suppose that the line PK intersects the side BC at L. Prove that $\frac{PK}{PL} = \cos A$.

B35. *Proposed by Leonard Giugiuc.*

Let $(a_n)_{n\geq 1}$ be a sequence of positive rational numbers defined as follows:

$$a_1 = 2, \quad a_{n+1} = a_n + \frac{1}{p_n q_n} \quad (n \geq 1),$$

where p_n and q_n are mutually prime positive integers such that $a_n = \frac{p_n}{q_n}$. Find $a_{n+1} + \frac{1}{a_n}$ for $n \geq 1$.

B36. *Proposed by Leonard Giugiuc, Kadir Altintas and Marian Dinca.*

Let ABC be a triangle with centroid G and incircle ω. If r and R are respectively inradius and the circumradius of ABC, find the range of values $\frac{r}{R}$ given that G lies on ω.

B37. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be a scalene acute-angled triangle with centroid G, orthocenter H, incenter I and circumcenter O. Let M be the midpoint of $[BC]$. Define ω as the circle with diameter AO and, similarly, θ as the circle with diameter HM. Prove that $GI||BC$, given that circles ω and θ are externally tangent.

B38. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Miguel Ochoa Sanchez.*

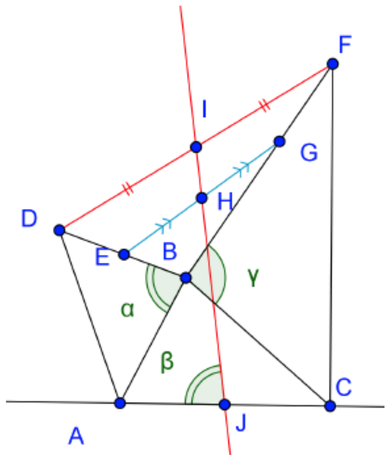
Let $\mathcal{L}$ be a fixed line with point A not on $\mathcal{L}$. Suppose points B and C move along $\mathcal{L}$ so that $\angle BAC = 90°$. For an arbitrary fixed pair of points B and C, define the square $MNPQ$ with M on side AB, N on AC, P and Q on side BC. If $T = BN \cap CM$, find the locus of T.

B39. *Proposed by Thanos Kalogerakis and Leonard Giugiuc.*

Let I be the incenter of triangle ABC, a, b, c be its sides, and D be the foot (on BC) of the bisector of the angle at A. Denote the midpoint of BC by M, and let a tangent from M touch the incircle at K while a tangent from M touches at L the circle whose diameter is AD. Prove that $MK = ML = |b - c|$.

B40. *Proposed by Leonard Giugiuc and Dao Thanh Oai.*

In the configuration below, show that $\beta = \alpha$ and $\dfrac{AC}{2IH} = \dfrac{AB}{DE}$.



B41. *Proposed by Leonard Giugiuc, Alexander Bogomolny and Marian Dinca.*

Let a, b and c be real numbers that satisfy the following two properties

$$a + b + c = 6 \quad \text{and} \quad a^2 + b^2 + c^2 \in \left[12, \frac{68}{3}\right].$$

If $a = \max\{a, b, c\}$, prove that $2(b^2 + c^2) - a^2 \leq 12$.

B42. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers such that $ab + bc + ca = 3$. Determine all $x > 0$ such that

$$(abc)^x(a + b + c) \leq 3.$$

B43. *Proposed by Leonard Giugiuc and Tran Hoang Nam.*

Prove that if a, b and c are real numbers, then

$$(a^2 + b^2 + c^2)^3 \geq 2((a - b)(a - c)(b - c))^2 + \frac{(a + b + c)^6}{27}.$$

When does equality hold?

B44. *Proposed by Leonard Giugiuc and Michael Rozenberg.*

Find the smallest positive k so that for all $a, b, c \geq 0$, we have

$$((a - b)(a - c)(b - c))^2 \leq k(a^3 + b^3 + c^3 - 3abc)^2.$$

B45. *Proposed by Leonard Giugiuc and Kadir Altintas.*

Let ABC be a triangle with centroid G, incircle ω, circumradius R and inradius r. If G lies on ω, prove that

$$(a^2+b^2+c^2)^2 \geq 972r^3(8R-25r).$$

When does the equality hold?

B46. *Proposed by Leonard Giugiuc and Sladjan Stankovik.*

Let a, b and c be real numbers greater than or equal to 1 such that $a+b+c = 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Find the minimum value of the expression $a+b+c$.

B47. *Proposed by Ardak Mirzakhmedov, Leonard Giugiuc and Michael Rozenberg.*

Prove that for any real numbers a, b and c, we have

$$\sqrt{a^2+3b^2}+\sqrt{b^2+3c^2}+\sqrt{c^2+3a^2} \geq \sqrt{7(a^2+b^2+c^2)+5(ab+bc+ca)}.$$

B48. *Proposed by Leonard Giugiuc.*

Let A, B and C be angles of an acute angled triangle. Prove that

$$\sqrt{6(1+\cos A\cos B\cos C)} \geq 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

B49$\star$. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a cyclic quadrilateral inscribed in a circle with radius r and center O. Let a, b, c and d be the distances from O to the sides AB, BC, CD and DA of the quadrilateral, respectively. If O is in the interior of $ABCD$, prove or disprove that

$$\frac{AB}{a}+\frac{BC}{b}+\frac{CD}{c}+\frac{DA}{d} \geq r\sqrt{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

B50. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a regular tetrahedron with edge length a. Let O be an interior point of $ABCD$. Suppose the lines AO, BO, CO and DO intersect the faces BCD, ACD, ABD and ABC at points A_1, B_1, C_1 and D_1, respectively. Prove that

$$OA_1+OB_1+OC_1+OD_1 < a.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(2), p. 77–82.

4511. *Proposed by Robert Frontczak.*

Evaluate the following sum in closed form:

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-1}\right).$$

We received 32 submissions, out of which 28 were correct and complete. We present the one by Theo Koupelis, modified by the editor.

First we note that the series

$$\frac{1}{1}-\frac{1}{7}+\frac{1}{9}-\frac{1}{15}\pm\dots \tag{1}$$

satisfies the alternating series test and thus converges (to say S). Thus we get that

$$S=\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-1}\right)=1+\sum_{n=1}^{\infty}\left(\frac{1}{8n+1}-\frac{1}{8n-1}\right),$$

since the partial sums of both series in the line above are partial sums of (1). Rewriting the right hand side, we obtain

$$S=1-\sum_{n=1}^{\infty}\frac{2}{64n^2-1}=1+\frac{1}{8}\sum_{n=1}^{\infty}\frac{2\cdot\frac{1}{8}}{\left(\frac{1}{8}\right)^2-n^2}.$$

Using the cotangent series

$$\pi\cot(\pi z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

for $z=\frac{1}{8}$, we calculate

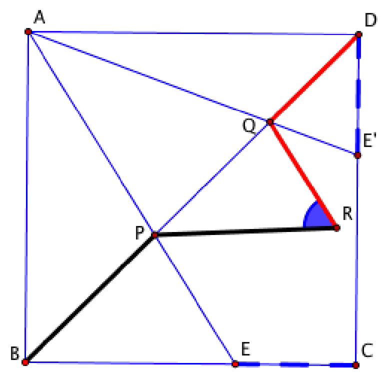
$$S=1+\frac{1}{8}\left(\pi\cot\frac{\pi}{8}-8\right)=\frac{\pi}{8}\cot\frac{\pi}{8}.$$

Taking into account that $\cot\dfrac{\pi}{8}=1+\sqrt{2}$, we finally have $S=\dfrac{\pi(1+\sqrt{2})}{8}$.

4512. *Proposed by J. Chris Fisher.*

For any point E on the side BC of the square $ABCD$ let E' be chosen on side CD so that $DE' = CE$, and let the lines AE and AE' intersect the diagonal BD in points P and Q, respectively. If R is either point whose distance from P equals PB and from Q equals QD, then prove that $\angle QRP = 60°$.

Comment from the proposer: this problem was shown to me 15 years ago; I do not know its source.



We received 27 submissions, of which 26 were correct and complete. We present the solution by Roy Barbara.

Without loss of generality let 1 be the length of the sides of the square. Let $CE = DE' = x$ for $0 < x < 1$, $RQ = DQ = y$ and $RP = BP = z$; then $BE = 1 - x$.

In any $\triangle XYZ$ the length of the angle bisector through X can be calculated via $\dfrac{2 \cdot XY \cdot XZ}{XY + XZ} \cos\left(\dfrac{\angle X}{2}\right)$. Apply this formula to the bisector from D in $\triangle ADE'$ and to the bisector from B in $\triangle ABE$ respectively (note that $\cos\left(\frac{\angle ADE'}{2}\right)$ and $\cos\left(\frac{\angle ABE}{2}\right)$ are both equal to $\frac{\sqrt{2}}{2}$) to get

$$y = DQ = \frac{\sqrt{2}x}{1+x} \text{ and } z = BP = \frac{\sqrt{2}(1-x)}{2-x}. \tag{1}$$

Since BD is a diagonal of the square, $PQ = BD - DQ - BP = \sqrt{2} - y - z$. Using the cosine law in $\triangle QRP$,

$$\begin{aligned} \cos(\angle QRP) &= \frac{PR^2 + QR^2 - PQ^2}{2 \cdot PR \cdot QR} \\ &= \frac{z^2 + y^2 - (\sqrt{2} - y - z)^2}{2yz} = \frac{\sqrt{2}(y+z) - 1}{yz} - 1. \end{aligned}$$

Substituting the formulas for y and z from (1) we get $\cos(\angle QRP) = \frac{1}{2}$. Thus $\angle QRP = 60°$, as desired.

4513. *Proposed by H. A. ShahAli.*

Let

$$a_n = \prod_{k=1}^{n} \frac{2k}{2k-1}$$

for each natural n. Prove that the number $A(m)$ of n's for which $\lfloor a_n \rfloor = m$ is non-zero and strictly increasing for any integer $m \geq 2$.

We received 7 submissions. Unfortunately, some of the purported solutions depended on reasonable-sounding but unsupported statements. The problem turns out to be incorrect. The sequence $\{A(m)\}$ is easily verified to not be strictly increasing, as pointed out by Walther Janous and Shrinivas Udpikar. In fact, the UCLan Cyprus Problem Solving Group provided computer evidence that it is not even increasing. The partial argument below is based on ideas drawn from the submissions of Ioannis Sfikas and the UCLan Cyprus Problem Solving Group.

Since

$$\frac{2(k+1)}{2k+1} < \frac{2k+1}{2k} < \frac{2k}{2k-1}$$

for every positive integer k,

$$2n+1 = \prod_{k=1}^{n} \frac{2k+1}{2k} \prod_{k=1}^{n} \frac{2k}{2k-1} < a_n^2 < 2^2 \prod_{k=1}^{n-1} \frac{2(k+1)}{2k+1} \prod_{k=1}^{n-1} \frac{2k+1}{2k} = 2(2n) = 4n,$$

Since

$$a_{n+1} = a_n \left(\frac{2n+2}{2n+1} \right) = a_n \left(1 + \frac{1}{2n+1} \right),$$

$$0 < a_{n+1} - a_n = \frac{a_n}{2n+1} < \frac{2\sqrt{n}}{2n+1} < \frac{1}{\sqrt{n}} \leq 1.$$

It follows that $\{a_n\}$ is a positive, increasing, divergent sequence, and that $\lfloor a_{n+1} \rfloor$ is equal to either $\lfloor a_n \rfloor$ or $\lfloor a_n \rfloor + 1$ for each $n \geq 2$. Therefore, since $a_1 = 2$, $\lfloor a_n \rfloor$ assumes each positive integer value exceeding 1.

Let $b_n = a_{n+1} - a_n$. Then

$$\begin{aligned} b_n - b_{n+1} &= \frac{a_n}{2n+1} - \frac{a_{n+1}}{2n+3} \\ &= a_n \left(\frac{1}{2n+1} - \frac{2n+2}{(2n+3)(2n+1)} \right) \\ &= \frac{a_n}{(2n+1)(2n+3)} > 0, \end{aligned}$$

so that $\{b_n\}$ is decreasing. Let $A(m) = k$ and $A(m+1) = h$ for some $m > 2$ and suppose r satisfies

$$\begin{aligned} m - 1 = \lfloor a_r \rfloor < m &= \lfloor a_{r+1} \rfloor = \cdots = \lfloor a_{r+k} \rfloor \\ &< m+1 = \lfloor a_{r+k+1} \rfloor = \cdots = \lfloor a_{r+k+h} \rfloor \\ &< m+2 = \lfloor a_{r+k+h+1} \rfloor. \end{aligned}$$

We have that

$$(k-1)b_{r+k-1} < b_{r+k-1} + \cdots + b_{r+1} = a_{r+k} - a_{r+1} < 1$$

and

$$\begin{aligned} a_{r+2k} - (m+1) < a_{r+2k} - a_{r+k} &= b_{r+2k-1} + \cdots + b_{r+k} \\ &< (k-1)b_{r+k} < (k-1)b_{r+k-1} < 1. \end{aligned}$$

Therefore $\lfloor a_{r+2k} \rfloor = m+1$ and so $k-1 \leq h$. It follows that $A(m+1) \geq A(m)-1$, so that $A(m)$ may decrease by 1. An extension of this argument shows that $A(m)$ cannot decrease by 1 twice in a row.

Comments from the editor. In fact, the result is not true. The values of $A(m)$ for $m = 2, 3, \ldots, 8$ are respectively 2, 2, 3, 4, 4, 5, 5. It might be imagined that $A(m)$, while not strictly increasing, still increases. However, UCLan Cyprus Problem Solving Group computed that $\lfloor a_{484} \rfloor = \lfloor a_{509} \rfloor = 39$, $\lfloor a_{510} \rfloor = 40$, $\lfloor a_{535} \rfloor = 41$ so that $A(39) \geq 26 > 25 \geq A(40)$.

The issue is basically that if you have a positive increasing sequence whose differences decrease, it is possible for the sequence to have one fewer term in the right of two adjacent unit intervals. Take for example $\{0.05, 0.53, 0.99, 1.43, 1.85, 2.25, \ldots\}$ whose successive differences are $\{0.48, 0.46, 0.44, 0.42, 0.40, \ldots\}$. Three terms lie in $[0, 1)$ but only two lie in $[1, 2)$.

Shrinivas Udpikar provided an interesting alternative argument that $\frac{a_n}{2n+1} < 1$:

$$a_n = \frac{2^n n!}{(2n)!/(2^n n!)} = \frac{2^{2n}(n!)^2}{(2n)!} = \frac{2^{2n}}{\binom{2n}{n}},$$

so that

$$b_n = \frac{a_n}{2n+1} = \frac{\sum_{i=0}^{2n} \binom{2n}{i}}{(2n+1)\binom{2n}{n}} < 1,$$

since $\binom{2n}{i} \leq \binom{2n}{n}$ with strict inequality when $i \neq n$.

4514. *Proposed by Leonard Giugiuc.*

Let a, b and c be real numbers all greater than or equal to $\frac{1}{2}$ such that $a+b+c = 3$. Prove that

$$\frac{a^2}{b+1} + \frac{b^2}{c+1} + \frac{c^2}{a+1} \geq \frac{a}{b+1} + \frac{b}{c+1} + \frac{c}{a+1}.$$

We received 18 submissions, all of which were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group, modified by the editor.

Equivalently, we need to prove that

$$\frac{a^2-a}{b+1} + \frac{b^2-b}{c+1} + \frac{c^2-c}{a+1} \geq 0\,.$$

Without loss of generality assume $a \geq b \geq c$. Note that the function $f(x) = x^2 - x$ is increasing in $[\frac{1}{2}, \infty)$ since given $x \geq y \geq \frac{1}{2}$ we have

$$f(x) - f(y) = (x - y)(x + y - 1) \geq 0\,.$$

Thus $a^2 - a \geq b^2 - b \geq c^2 - c$ and $\frac{1}{c+1} \geq \frac{1}{b+1} \geq \frac{1}{a+1}$. By the rearrangement inequality we have

$$\frac{a^2 - a}{b+1} + \frac{b^2 - b}{c+1} + \frac{c^2 - c}{a+1} \geq \frac{a^2 - a}{a+1} + \frac{b^2 - b}{b+1} + \frac{c^2 - c}{c+1}.$$

We now observe that for $x \geq 0$ we have

$$\frac{x^2 - x}{x+1} - \frac{x-1}{2} = \frac{2x^2 - 2x - (x^2 - 1)}{2(x+1)} = \frac{(x-1)^2}{2(x+1)} \geq 0\,.$$

So

$$\frac{a^2 - a}{a+1} + \frac{b^2 - b}{b+1} + \frac{c^2 - c}{c+1} \geq \frac{(a-1) + (b-1) + (c-1)}{2} = \frac{a+b+c-3}{2}\,.$$

As $a + b + c = 3$, we conclude that

$$\frac{a^2 - a}{a+1} + \frac{b^2 - b}{b+1} + \frac{c^2 - c}{c+1} \geq 0\,.$$

as desired.

This proof was selected due to its simplicity. As a point of interest, other submissions utilized the inequality of arithmetic and geometric mean and/or the Cauchy-Schwarz inequality. Additionally, many submissions noted that equality holds if and only if $a = b = c = 1$.

4515. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b)\sqrt{\dfrac{a}{b}} + (2b+c)\sqrt{\dfrac{b}{c}} + (2c+a)\sqrt{\dfrac{c}{a}}}{a+b+c} \geq 3.$$

We received 24 solutions, all correct. We present the solution by Richard B. Eden.

For any positive numbers x and y, by the AM-GM inequality we have:

$$(2x+y)\sqrt{\frac{x}{y}} = \sqrt{\frac{x^3}{y}} + \left(\sqrt{\frac{x^3}{y}} + \sqrt{xy}\right) \geq \sqrt{\frac{x^3}{y}} + 2\sqrt{\sqrt{\frac{x^3}{y}} \cdot \sqrt{xy}} = \sqrt{\frac{x^3}{y}} + 2x$$

with equality if and only if $\sqrt{\frac{x^3}{y}} = \sqrt{xy}$, which is true if and only if $x = y$.

It therefore suffices to prove that

$$\sqrt{\frac{a^3}{b}}+\sqrt{\frac{b^3}{c}}+\sqrt{\frac{c^3}{a}} \geq a+b+c. \tag{1}$$

By the Cauchy-Schwarz Inequality, we have

$$\sum_{\text{cyclic}} \sqrt{\frac{a^3}{b}} \cdot \sum_{\text{cyclic}} \sqrt{ab} \geq \left(\sum_{\text{cyclic}} \sqrt{\sqrt{\frac{a^3}{b}} \cdot \sqrt{ab}}\right)^2 = (a+b+c)^2 \tag{2}$$

with equality if and only if

$$\frac{a^3/b}{ab} = \frac{b^3/c}{bc} = \frac{c^3/a}{ca},$$

which is true if and only if $a = b = c$.

Since

$$a+b+c = \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2} \geq \sqrt{ab}+\sqrt{bc}+\sqrt{ca},$$

we have from (2) that

$$\sum_{\text{cyclic}} \sqrt{\frac{a^3}{b}} \cdot \sum_{\text{cyclic}} \sqrt{ab} \geq (a+b+c)(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}).$$

Dividing both sides by $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$, (1) follows and equality holds if and only if $a = b = c$. This completes the proof.

4516. *Proposed by Hung Nguyen Viet.*

Find the values of:

(a) $(1-\cot 1^\circ)(1-\cot 2^\circ)(1-\cot 3^\circ)\cdots(1-\cot 44^\circ)$,

(b) $\dfrac{1}{1+\cot 1^\circ}+\dfrac{1}{1+\cot 2^\circ}+\dfrac{1}{1+\cot 3^\circ}+\cdots+\dfrac{1}{1+\cot 89^\circ}$.

We received 35 correct solutions to both parts. One additional submission gave the answer with no justification; two answered only part (a), one with no justification.

The solution of (a) depends on establishing that

$$(1-\cot\theta)(1-\cot(45^\circ-\theta)) = 2$$

and applying this to $\theta = k^\circ$ for $1 \leq k \leq 22$ to get the answer 2^{22}. Here are some approaches:

$$\begin{aligned}(1-\cot\theta)(1-\cot(45^\circ-\theta)) &= (\cot 45^\circ-\cot\theta)(\cot 45^\circ-\cot(45^\circ-\theta)) \\ &= \frac{\sin(\theta-45^\circ)}{\sin 45^\circ \sin\theta} \cdot \frac{\sin(-\theta)}{\sin 45^\circ \sin(45^\circ-\theta)} = \frac{1}{\sin^2 45^\circ} = 2,\end{aligned}$$

$$1 - \cot(45^\circ - \theta) = \frac{\sin(45^\circ - \theta) - \cos(45^\circ - \theta)}{\sin(45^\circ - \theta)} = \frac{\sin(45^\circ - \theta) - \sin(45^\circ + \theta)}{\sin(45^\circ - \theta)}$$
$$= \frac{-2\cos 45^\circ \sin\theta}{\sqrt{2}(\cos\theta - \sin\theta)} = \frac{2}{1 - \cot\theta},$$

$$(1 - \cot\theta)(1 - \cot(45^\circ - \theta)) = \left(1 - \frac{1}{\tan\theta}\right)\left(1 - \frac{1 + \tan\theta}{1 - \tan\theta}\right)$$
$$= \left(\frac{\tan\theta - 1}{\tan\theta}\right)\left(\frac{-2\tan\theta}{1 - \tan\theta}\right) = 2.$$

Multiply
$$1 - \cot\theta = \frac{\sin\theta - \cos\theta}{\sin\theta} = \frac{-\sqrt{2}\sin(45^\circ - \theta)}{\sin\theta}$$
by the analogous expression with θ replaced by $45^\circ - \theta$ to get the product 2.

For the solution of (b), note that
$$\frac{1}{1 + \cot\theta} + \frac{1}{1 + \cot(90^\circ - \theta)} = \frac{\tan\theta}{\tan\theta + 1} + \frac{1}{1 + \tan\theta} = 1.$$
Apply this to $\theta = k^\circ$ for $1 \le k \le 44$ and note that
$$\frac{1}{1 + \cot 45^\circ} = \frac{1}{2}$$
to get the answer $\dfrac{89}{2}$.

Alternatively, combining the fractions over a common denominator, we find that
$$\frac{1}{1 + \cot\theta} + \frac{1}{1 + \tan\theta} = \frac{2 + \tan\theta + \cot\theta}{1 + \cot\theta + \tan\theta + \cot\theta\tan\theta} = 1.$$
Rendering the cotangent and tangent functions in terms of sine and cosine gives an easy verification of the identity.

4517. *Proposed by Robert Frontczak.*

Let F_n denote the n-th Fibonacci number defined by $F_{n+1} = F_n + F_{n-1}$, $F_0 = 0$, $F_1 = 1$. Further, let T_n denote the n-th triangular number, that is $T_n = \dfrac{n(n+1)}{2}$. Show that
$$\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7.$$

We received 27 solutions, of which 18 were complete and correct. Most incomplete submissions failed to identify the interval of convergence of a power series before *evaluating it at a specific value. We present a solution which combines those of Michel Bataille and Marie-Nicole Gras.*

For $|x| < 2$, we have

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{2^{n+1}} = \frac{x}{2} \cdot \frac{1}{1-\frac{x}{2}} = \frac{x}{2-x}.$$

Differentiate to get

$$\sum_{n=0}^{\infty} (n+1)\frac{x^n}{2^{n+1}} = \frac{d}{dx}\left(\frac{x}{2-x}\right) = \frac{2}{(2-x)^2}$$

and again to get

$$\sum_{n=1}^{\infty} (n(n+1))\frac{x^{n-1}}{2^{n+1}} = \frac{d}{dx}\left(\frac{2}{(2-x)^2}\right) = \frac{4}{(2-x)^3}.$$

Let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$; note that $|\alpha| < 2$ and $|\beta| < 2$. Since $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$,

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} &= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \frac{n(n+1)(\alpha^n - \beta^n)}{2^{n+3}} \\
&= \frac{\alpha}{4\sqrt{5}} \sum_{n=1}^{\infty} n(n+1)\frac{\alpha^{n-1}}{2^{n+1}} - \frac{\beta}{4\sqrt{5}} \sum_{n=1}^{\infty} n(n+1)\frac{\beta^{n-1}}{2^{n+1}} \\
&= \frac{\alpha}{4\sqrt{5}} \cdot \frac{4}{(2-\alpha)^3} - \frac{\beta}{4\sqrt{5}} \cdot \frac{4}{(2-\beta)^3} \\
&= \frac{1}{\sqrt{5}}\left(\frac{\alpha}{(2-\alpha)^3} - \frac{\beta}{(2-\beta)^3}\right) \\
&= \frac{1}{\sqrt{5}}\left(\frac{\alpha^7}{(\alpha^2(2-\alpha))^3} - \frac{\beta^7}{(\beta^2(2-\beta))^3}\right) \\
&= \frac{\alpha^7 - \beta^7}{\sqrt{5}} = F_7,
\end{aligned}$$

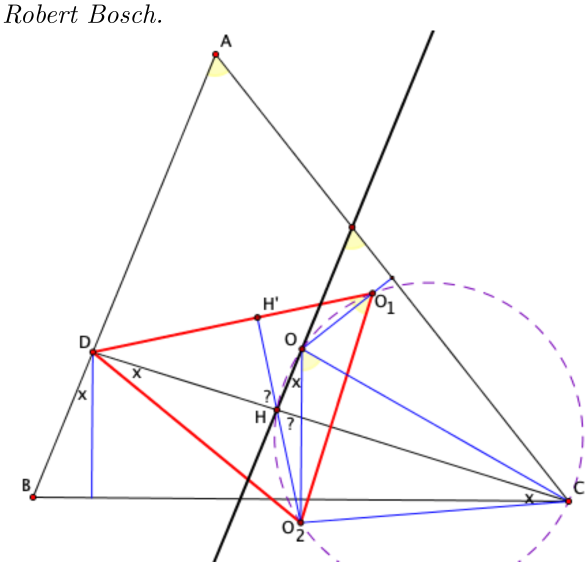
as required, where in the last line we used the equalities $\alpha^2 = \alpha + 1$, $\beta^2 = \beta + 1$ to calculate that $\alpha^2(2-\alpha) = 1$ and similarly $\beta^2(2-\beta) = 1$.

4518. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

If O is the circumcenter of a triangle ABC and D is any point on the line AB, let O_1 and O_2 be the respective circumcenters of triangles ADC and DBC. Prove that the orthocenter of triangle O_1DO_2 lies on the line through O that is parallel to AB.

We received 15 valid submissions, although two of the proofs were by computer. Should we classify a computer proof to be "correct and complete"? Aargh! Bringing in a computer for this problem seems to be somewhat like using the proverbial *cannon to kill a mosquito. We present two of the proofs that were managed without a computer.*

Solution 1, by Robert Bosch.



Let H be the intersection of CD with the line through O that is parallel to AB. It suffices to show that H is the orthocenter of ΔO_1DO_2. Because CD is the common chord of the circles centered at O_1 and O_2, it is the line through D that is perpendicular to O_1O_2, so it therefore contains the desired orthocenter. Our problem thus is reduced to showing that H is also on the altitude from O_2 to O_1D.

Let H' be the intersection of O_2H and O_1D; we will show that $\angle HH'D = 90°$. We have $\angle DO_1O_2 = A$ since O_1 is the circumcenter of triangle ADC, so then $\angle CDO_1 = 90° - A$. Thus, we need to prove that $\angle O_2HC = A$. But note that $\angle O_2OC = A$, since O is the circumcenter of triangle ABC. So it is enough to prove the quadrilateral OHO_2C is cyclic. Note that $\angle DCO_2 = \angle O_2DC = 90° - B$, because O_2 is the circumcenter of triangle BDC. We have $\angle HOO_2 = 90° - B$ (because $HO||DA$), proving the quadrilateral OHO_2C is cyclic. This completes the proof.

Solution 2, by Joel Schlosberg

Since

$$\angle O_2O_1D = \frac{1}{2}\angle CO_1D = \angle CAD = \angle CAB$$

and

$$\angle O_1O_2D = \frac{1}{2}\angle CO_2D = \angle CBD = \angle CBA,$$

$\triangle O_1O_2D \sim \triangle ABC$. Let $\alpha = \angle BAC$, $\beta = \angle CBA$, $\gamma = \angle ACB$, and $\theta = \angle CDA$. Since O_1O_2 is the perpendicular bisector of CD, by right angle trigonometry and the law of sines we have,

$$O_1O_2 = \frac{1}{2}CD(\cot\alpha + \cot\beta) = \frac{AC\cos\alpha + BC\cos\beta}{2\sin\theta} = \frac{AB}{2\sin\theta};$$

if H is the orthocenter of $\triangle O_1O_2D$,

$$DH = O_1O_2\cot\gamma = \frac{AB\cot\gamma}{2\sin\theta}.$$

The distance from H to AB is

$$DH\sin\theta = \frac{1}{2}AB\cot\gamma,$$

which is the same as the constant distance from O to AB. Therefore $HO \parallel AB$.

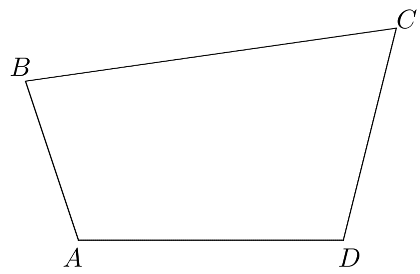
4519. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a non-degenerate convex quadrilateral. If

$$AB^2 + CD^2 + 2AD\cdot BC = AC^2 + BD^2,$$

prove that AD is parallel to BC.

We received 22 correct solutions. We present the solution by Marie-Nicole Gras.



We introduce cartesian coordinates with origin at A and define

$$A = (0,0),\quad D = (a,0),\quad B = (x,y),\quad C = (u,v),\ u \geq x.$$

We compute

$$\begin{aligned} AC^2 + BD^2 - AB^2 - CD^2 &= u^2 + v^2 + (x-a)^2 + y^2 - x^2 - y^2 - (u-a)^2 - v^2 \\ &= -2ax + 2au. \end{aligned}$$

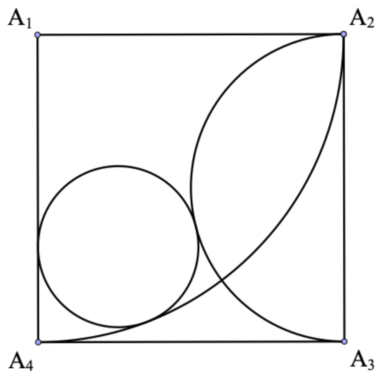
By assuption, we have $AC^2 + BD^2 - AB^2 - CD^2 = 2AD\cdot BC$; it follows that

$$2a(u-x) = 2a\sqrt{(u-x)^2 + (v-y)^2}.$$

This equality implies $y = v$; the points B and C have the same ordinate, and thus AD is parallel to BC.

4520. *Proposed by Arsalan Wares.*

The figure shows a square with a quarter circle, a semicircle and a circle inside it. The length of the square is the same as the length of the radius of the quarter circle which in turn is the same as the length of the diameter of the semicircle. The circle touches both the quarter circular arc (internally) and the semicircular arc (externally), and one of the sides of the square as shown. If the length of the square is 25, find the exact length of the radius of the circle.



We received 28 submissions, of which 27 were correct. We present the solution by I. J. L. Garces.

The desired radius of the circle is $9-\sqrt{6}$ units.

Let C and r be the center and radius of the circle, respectively, and O the center of the semicircle. Let the horizontal line (that is, parallel to A_1A_2) through C intersect A_1A_4 and A_2A_3 at A and B, respectively. Then A_1A_2BA is a rectangle.

Applying Pythagorean Theorem to the right triangle A_1BC, we get

$$A_1A = \sqrt{(25-r)^2 - r^2} = \sqrt{625-50r}, \qquad r \leq 12.5.$$

Again, applying Pythagorean Theorem to the right triangle OBC, we get

$$\left(\sqrt{625-50r} - \frac{25}{2}\right)^2 + (25-r)^2 = \left(r + \frac{25}{2}\right)^2.$$

We solve for r in the preceding equation.

$$\sqrt{625-50r} = -5r + 50, \quad r \leq 10$$
$$r^2 - 18r + 75 = 0$$
$$r = 9-\sqrt{6} \quad \text{or} \quad r = 9+\sqrt{6} > 10 \text{ (not accepted)}$$

Thus, the radius of the circle is $9-\sqrt{6}$ units.