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IN THIS ISSUE / DANS CE NUMÉRO

This issue is dedicated to the memory of Canadian mathematical legend Richard Guy, who passed away on March 9th, 2020 at the age of 103. We are proud to present the materials that have been contributed to this issue by Richard's friends and colleagues as well as many people that he has inspired mathematically. We are also featuring a previously unpublished article by Richard Guy and Elwyn Berlekamp, in memory of both authors.

We celebrate Richard's legacy with this 103-page issue.

Richard celebrated his 100th birthday with his colleagues at the December Luncheon of Emeritus Association of University of Calgary in 2016. This picture was taken by Tom Swaddle. Here are some quotes from his colleagues.

"I had several discussions with Richard and his late wife, about what was the secret of their good health throughout the years, and was informed that *consistent exercise, including dance and love of music were important aspects*." – Dr. Carole-Lynne Le Navenec, Associate Professor Emerita of Nursing, University of Calgary.

"I, too, once asked Richard about the secret of his longevity. He said, with a deadpan expression, "I always put my socks on standing up!" He may not have been joking (hard to tell with Richard). He and I shared a passion for mountaineering; in fact, it was I who introduced him to the Alpine Club of Canada, in which he became very active, and we were very conscious of the need to preserve a sense of balance for climbing." – Tom Swaddle, Professor Emeritus of Chemistry, University of Calgary.

Fibonacci Plays Billiards

Elwyn Berlekamp and Richard Guy

Abstract

A *chain* is an ordering of the integers 1 to n such that adjacent pairs have sums of a particular form, such as squares, cubes, triangular numbers, pentagonal numbers, or Fibonacci numbers. For example 4 1 2 3 5 form a Fibonacci chain while 1 2 8 7 3 12 9 6 4 11 10 5 form a triangular chain. Since 1 + 5 is also triangular, this latter forms a triangular *necklace*. A search for such chains and necklaces can be facilitated by the use of paths of billiard balls on a rectangular or other polygonal billiard table.

Foreword

This manuscript dating back to 2003 lay dormant in Richard Guy's files for many years. Richard resurrected it in the summer of 2017 when, at age 100, he recruited his last student, Ethan White. Under Richard's and my joint supervision, Ethan, then an undergraduate student at the University of Calgary and an NSERC USRA recipient, conducted research on sum and difference necklaces.

The inaugural *Richard and Louise Guy lecture*, of the same title as this manuscript, was delivered in 2006 at the University of Calgary by Elwyn Berlekamp. Following Berlekamp's death in April 2019, Richard felt that he owed it to his long-time friend and collaborator to make this work more widely known. An arXiv version (`https://arxiv.org/abs/2002.03705`), augmented with an up-to-date appendix on square necklaces, was posted less than five weeks before Richard passed away on March 9.

Richard would be pleased to finally see this work formally published. Given his life-long delight in playful and recreational mathematics, *Crux Mathematicorum* is an ideal home for *Fibonacci Plays Billiards*.

Renate Scheidler, University of Calgary
September 2020

. . . .

At the 2002 Combinatorial Games Conference in Edmonton we found Yoshiyuki Kotani looking for values of n which would enable him to arrange the numbers 1 to n in a chain so that adjacent links summed to a perfect cube. Part of such a chain might be $\ldots 61 \quad 3 \quad 5 \quad 22 \quad 42 \ldots$. He had seen the corresponding problem asking for squares. Later Ed Pegg informed us that this problem, with squares and with $n = 15$, was proposed by Bernardo Recaman Santos, of Colombia, at the 2000 World Puzzle Championship. More recently it appeared as Puzzle 30 in [6].

$$(16\rightarrow)9\rightarrow 7\rightarrow 2\leftarrow 14\rightarrow 11\rightarrow 5\rightarrow 4\leftarrow 12\leftarrow 13\rightarrow 3\leftarrow 6\leftarrow 10\leftarrow 15\rightarrow 1\leftarrow 8(\leftarrow 17)$$

Figure 1: Solution(s) to Recaman's problem for $n = 15$, 16, 17.

This inspired Joe Kisenwether to ask for the numbers 1 to 32 to be arranged as a necklace whose neighboring beads add to squares (Figure 2).

4	21	28	8	1	15	10	26	23
32								2
17								14
19								22
30								27
6								9
3								16
13								20
12	24	25	11	5	31	18	7	29

Figure 2: A necklace with adjacent pairs of beads adding to squares.

The extension to cubes was suggested by Nob Yoshigahara. The least n for such a chain or necklace may be greater than 300. But it seems certain that such chains and necklaces can be found for all sufficiently large n, and for any other powers or polynomials, e.g., figurate numbers of various kinds; see Figure 3.

		3		
	7		12	
8				9
2				6
1	5	10	11	4

Figure 3: A necklace with adjacent pairs of beads adding to triangular numbers.

So we asked about more rapidly divergent sequences. For powers of 2, it is not possible to connect chains of odd numbers to chains of even numbers, and there are similar difficulties with powers of larger numbers.

However, the corresponding problem with neighbors summing to Fibonacci numbers, $F_0 = 0$, $F_1 = 1$, $F_{k+1} = F_k + F_{k-1}$, has a better balanced solution.

We can draw a graph with the numbers 1 to n as vertices and edges joining pairs whose sum is a Fibonacci number: for $n = 11$, this is Figure 4.

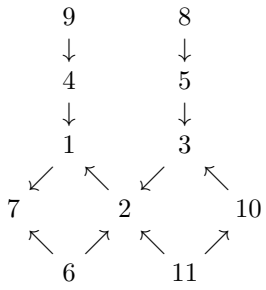

Figure 4: Graph whose adjacencies are Fibonacci sums

The arrows are drawn from the larger to the smaller number to emphasize that the larger number is not part of the graph unless the smaller is already present. From the graph we can read off 1 2; 1 2 3; 4 1 2 3; 4 1 2 3 5; 4 1 7 6 2 3 5; 4 1 7 6 2 3 5 8; 9 4 1 7 6 2 3 5 8 and 9 4 1 7 6 2 11 10 3 5 8. We can also verify that 6 and 10 can't be included in a chain unless some larger number is also present (in the former case 4, 5 and 6 are monovalent vertices and all three can't be ends of the chain; in the latter case, 8, 9 and 10). Evidently the Law of Small Numbers is at work. Six and ten are the only numbers which are not powers of primes. Is there some connexion with projective planes? No, but the Law of Small Numbers is indeed at work, but the villains are 9 and 11.

Theorem 1. *There is a chain formed with the numbers 1 to n with each adjacent pair adding to a Fibonacci number, just if* $n = 9$*, 11, or* F_k *or* $F_k - 1$*, where* F_k *is a Fibonacci number with* $k \geq 4$*. The chain is essentially unique.*

Proof. For $n \leq 11$ ($k = 4, 5, 6$) this follows from Figure 2. If $k = 7$, then $12 = F_7 - 1$ can be appended to the 11-chain, forming a 4-circuit; also, $F_7 = 13$ can be appended at the other end, as shown in Figure 5.

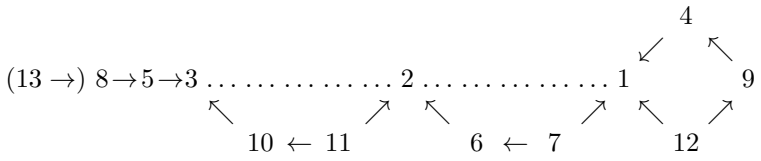

Figure 5: Ball and chain for 12 or 13.

Although 2 is adjacent to 1 and 3, the chain for 12 or 13 is essentially unique, except that the right tail may be 12 or 4 for either chain. None of the Fibonacci chains that we have seen will form a necklace; nor will any others.

The rest of the proof is by induction, but the comparatively simple pattern is made more difficult to describe because only every third Fibonacci number is even.

Balls and chains occur just for $F_{3m+1} - 1$ and F_{3m+1} with $m \geq 1$; other cases are simple chains. The chain 1—2—3 can be thought of as the "zeroth ball" (Fig. 6).

There are no chains for $n = 14$, 15, 16, 17, 18 or 19, since, when we successively append these numbers to the graph, the first three are monovalent vertices, as also is 17 ($= \frac{1}{2}F_9$), though this last can be accommodated by breaking the ball and allowing 17 to become an end of the chain. When we adjoin 18 & 19 they repectively allow 16 & 15 to become bivalent, but a chain is not reachieved until we append $F_8 - 1 = 20$ at 1 & 14.

Note that all the partitions (5&3, 2&6, 7&1) of $F_6 = 8$ into two distinct parts have been bypassed by the partitions of $F_9 = 34$ into parts of size less than $F_8 = 21$, which itself can then be appended to form a new tail to the chain. Because F_9 is even, as is every third Fibonacci number, $\frac{1}{2}F_9 = 17$ can only be appended to 4 ($= \frac{1}{2}F_6$).

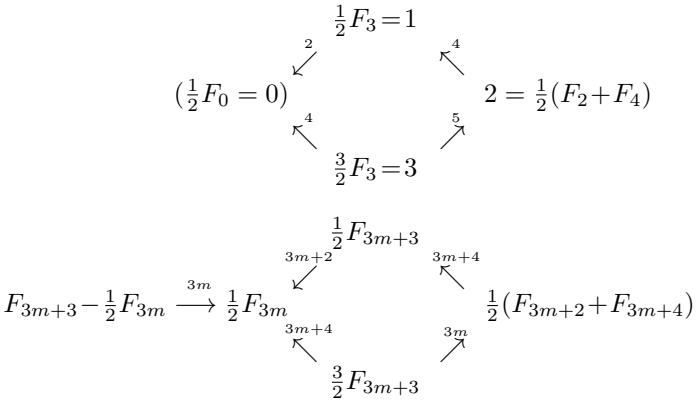

Figure 6: Zeroth ball and general ball. Small numbers above the arrows are ranks of Fibonacci numbers to which pairs of linked numbers sum.

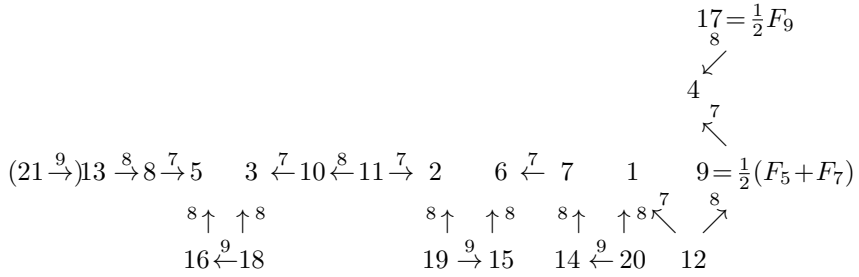

Figure 7: Fibonacci chains for $F_8 - 1 = 20$ and $F_8 = 21$.

If we continue, we find that a chain cannot again be achieved until we have replaced the six partitions of $F_7 = 13$ by links of partitions of $F_{10} = 55$ into two parts of size at most $F_9 - 1 = 33$ (Figure 8).

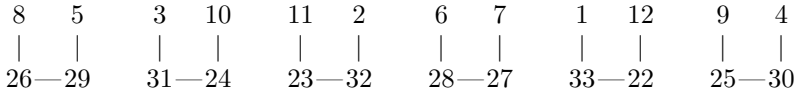

Figure 8: Links extending the chain to $F_9 - 1 = 33$.

$F_9 = 34$ can then be appended to $21 = F_8$ to make a new tail to the chain.

The next chain is for $F_{10} - 1 = 54$, obtained by appending links of partitions of $F_{11} = 89$ into parts of size at most 54:

$$54 - - - 35, 53 - - - 36, \ldots, 45 - - - 44$$

to the ten partitions 1—20, 2—19, …, 10—11, of $F_8 = 21$. The chain for $F_{10} = 55$ can be formed by appended it at the end $F_9 = 34$.

Note, that the link —51—38— need not immediately replace the end link —4—17 of the chain, but the latter can remain as part of a new ball, the case $m = 2$ of Figure 6, until we wish to append $\frac{1}{2}F_{12} = 72$, which we will do when forming the 88- and 89-chains.

We have seen several stages of the induction. In Figure 5 the numbers between $F_5 = 5$ and $F_6 = 8$ and F_6 itself are appended, as also are the numbers between $F_6 = 8$ and $F_7 = 13$ and 13 itself. In Figures 7 and 8, the numbers between F_k and F_{k+1} are appended for $k = 7$ and 8 respectively. Note that in the former $\frac{1}{2}F_{k+2} = 17$ is appended to $\frac{1}{2}F_{k-1} = 4$.

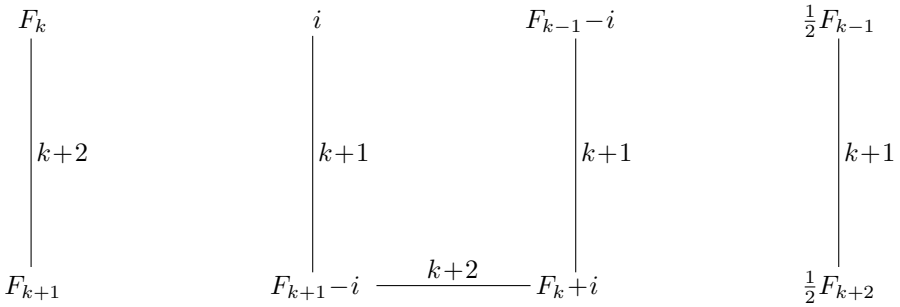

Figure 9: Extending $F_k - 1$ and F_k chains to those for $F_{k+1} - 1$ and F_{k+1} The appendage on the right is required only when $k = 3m + 1$.

Generally, as in Figure 9, we append the pairs of numbers $F_k + i$, $F_{k+1} - i$ for $1 \le i \le \frac{1}{2}(F_{k-1} - 1)$, except that, when $k = 3m + 1$, $\frac{1}{2}(F_{k-1} - 1)$ is not an integer and we have a new tail, $\frac{1}{2}F_{k+2}$, which **is** an integer, appended to $\frac{1}{2}F_{k-1}$. □

These last numbers are denominators of the convergents to the continued fraction for $\sqrt{5}$, sequence A001076 in Neil Sloane's Online Encyclopedia of Integer Sequences [5].

The proof can be made much more perspicuous with billiards diagrams, which will also throw light on the other kinds of chain in which we are interested.

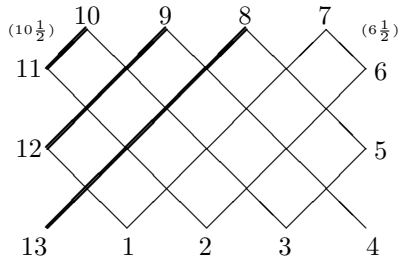

Figure 10: Fibonacci plays billiards. The thick upward paths connect 21-sums. The other upward paths connect 8-sums. The down paths connect 13-sums.

Figure 10 is equivalent to Figure 5. The 'ball' may be achieved by connecting the Fibonacci sum 1—+—4 = 5.

This billiard table viewpoint is useful for depicting long chains whose adjacent pair-sums all lie in a set of only three or four elements. If successive corners are at a, b, c, d, where $a < b < c < d$, then the semi-perimeter must be $c - a = d - b$, and the perimeter is $P = 2(c - a) = 2(d - b)$. One side must be $b - a = d - c$, and the other must be $c - b = a - d \pmod P$. Viewed along the 45 degree path taken by the billiard ball, each integer along the side of the table has valence 2, and each integer in a corner has valence 1. Hence, if the corners include 2 integers (called pockets) and 2 non-integers, then the path beginning at either pocket must eventually terminate in the other pocket.

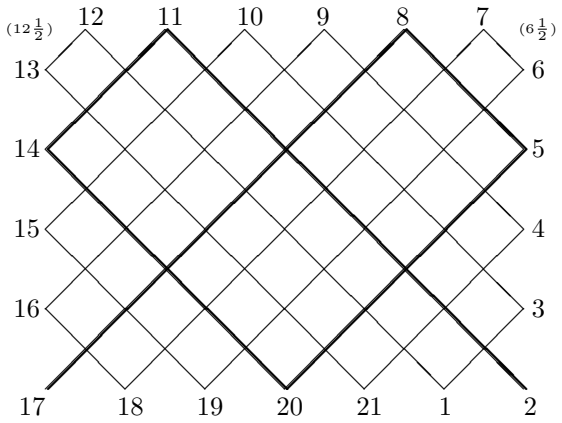

Figure 11: A billiard table with $A = 4$, $B = 13$, $C = 25$, $D = 34$ and perimeter $P = 21$. The double-sides $B - A = 9$ and $C - B = 12$ are not relatively prime.

Figure 11 shows a rectangle of perimeter 21, whose corners are at $a = 2$, $b = 6.5$, $c = 12.5$, $d = 17$. The sequence between pockets (thick lines) is 2, 11, 14, 20, 5, 8, 17. This sequence fails to reach many of the other integers along the perimeter, which lie in the following cycle: 1, 3, 10, 15, 19, 6, 7, 18, 16, 9, 4, 21, 13, 12, 1. The question of which rectangular billiard tables yield a single covering path and which yield a degeneracy of this sort is answered by the following lemma.

Lemma. Let A, B, C, D, be positive integers such that $A < B < C < D$ and $C - A = D - B$. Let $a = A/2$; $b = B/2$; $c = C/2$, and $d = D/2$. Further suppose that exactly two of a, b, c, d are integers, so that the corresponding billiard table has two corner pockets. Then the 45 degree path between the pockets touches **all** of the integers along the perimeter just if the rectangle's double-sides, $B - A$ and $C - B$, are relatively prime.

Note. In Figure 11 the sides are $6.5 - 2 = 4.5$ and $12.5 - 6.5 = 6$, so the double-sides are 9 and 12. They have a common factor of 3. So we could color each integer of shape $3k + 2$ and both pockets would be colored. Every integer along

the ball's path would then also be colored. In general, this argument shows that a degeneracy occurs whenever the double-sides are not relatively prime.

Proof of non-degeneracy. If the double-sides are prime to each other, and hence to the perimeter $P = C - A = D - B$, so that, mod P, $A \equiv C$ and $D \equiv B$, then consider any two integers separated by exactly one bounce along the ball's path. If the bounce is at x, these integers, mod P, are at $A - x$ and $B - x$, and the distance between them is $B - A \equiv D - C$ if measured in one direction mod P, or $A - B \equiv C - D \equiv A - D \equiv C - B$ if measured in the other direction. But since $B - A$ is a double-side, which is relatively prime to P, it follows that the sequence, obtained by looking at **alternate** bounce-points along the ball's path, cannot cycle back to itself, mod P, without first reaching a pocket. Since this is true for all values of x, the ball-path from one pocket to the other must go through every integer point on the rectangle's perimeter.

We can take three corners of a rectangle as the halves of any three consecutive Fibonacci numbers (recall that the corners are allowed to be half-integers). The perimeter of this rectangle will be the middle of these three Fibonacci numbers. Since any pair of adjacent Fibonacci numbers is relatively prime, the path from pocket to pocket is complete.

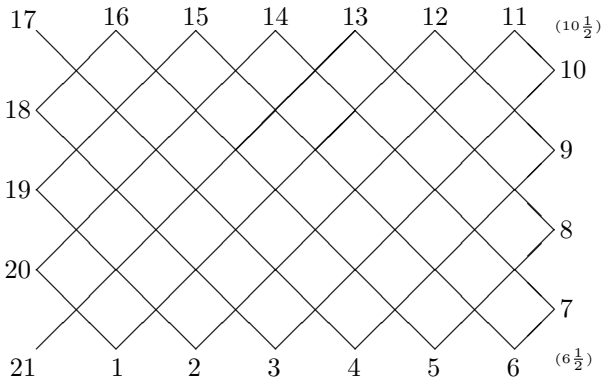

Figure 12: A billiard table giving a Fibonacci chain of length $P = 21$.

Square chains. For the 'square' chains and necklaces which we mentioned at the outset, Ed Pegg and Edwin Clark have verified that there are chains for $n = 15$, 16, 17, 23, 25 to 31 and necklaces (and hence chains) for $n = 32$ upwards. The existence problem was solved quite recently; more in the appendix at the end.

The billiards technique allows us to construct arbitrarily large specimens. Figure 13 shows how our billiard table technique can be used to find a 'square' chain of length 16.

We may delete 16, or append 17, giving the 15-, 16- and 17-chains of Figure 1.

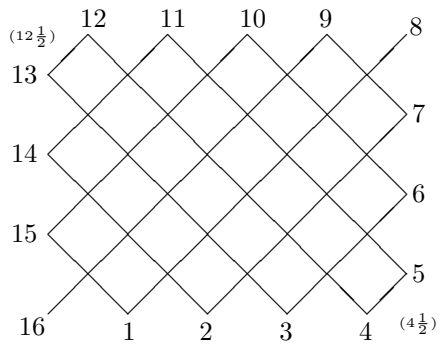

Figure 13: A billiard table with $A = 9$, $B = 16$, $C = 25$ (all squares), and $D = 32$ (half a square) and perimeter $P = 16$.

It is possible to accommodate other numbers by using billiard tables with more than four corners! Figure 14 shows such a table with corners at 4.5, 8.5, 9, 12.5, 24.5, and 32. The corner at 8.5 is reflex; the others are right. The perimeter is 39. There are two pockets: a conventional corner pocket at 32, and a side pocket at 9. The path between these two pockets is complete.

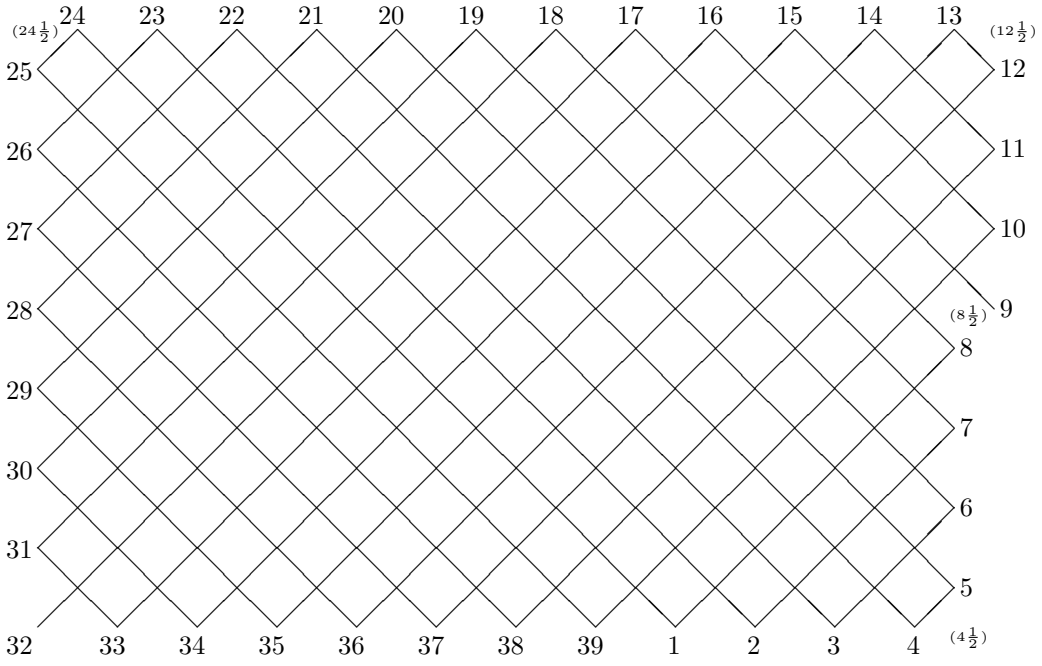

Figure 14: A billiard table with six corners.

Square necklaces. In order to connect the two pockets and make a necklace, we must be sure that they sum to a square. Two half-squares summing to a square are provided by the parametric equation

$$\left((r+s)^2-2r^2\right)^2+\left((r+s)^2-2s^2\right)^2=2\left(r^2+s^s\right)^2$$

For example, $1^2+7^2=2\cdot 5^2$. We multiply the solution by 6 to get the parity right and to avoid the sides having a common factor of 3. $42^2-6^2=2^6\cdot 3^3$ can be arranged as the difference of two odd squares, which are not multiples of 3, in just two different ways, 43^2-11^2 and 433^2-431^2. Billiard tables with half these squares as corners have perimeters 1728 and 185725. Their double-sides, $(5\cdot 17, 31\cdot 53)$ and $(2^6\cdot 3^3, 11\cdot 43\cdot 389)$ are coprime, so the chains contain every integer on the perimeter. Moreover, the ends of the chains are $\frac{1}{2}6^2$ and $\frac{1}{2}42^2$ which sum to 30^2 so that they may be joined to form necklaces.

Here are some small square necklaces. The bold numbers are $6x$, $6y$.

r,s	$x^2+y^2=2z^2$ x,y	corners are half the squares of:	double sides are coprime	perimeter P
2,1	1,7	**6**,11,**42**,43	85,1643	1728
3,2	7,17	**42**,**102**,119,151	3757,8640	12397
4,3	17,31	**102**,**186**,197,251	4213,24192	28405
7,3	1,41	**6**,23,**246**,247	493,59987	60480
7,5	23,47	109,**138**,269,**282**	7163,53317	60480
5,4	31,49	**186**.**294**,373,437	51840,52693	104533
7,3	1,41	**6**,**246**,397,467	60480,97093	157573
2,1	1,7	**6**,**42**,431,433	1728,183997	185725
5,3	7,23	**42**,**138**,859,869	17280,718837	736117
7,3	1,41	**6**,**246**,2153,2167	60480,4574893	4635373
3,2	7,17	**42**,**102**,2159,2161	8640,4650877	4659517
5,4	31,49	**186**,**294**,2587,2597	51840,6606133	6657973
5,3	7,23	**42**,**138**,4319,4321	17280,18634717	18651997
4,3	17,31	**102**,**186**,6047,6049	24192,36531613	36555805
7,5	23,47	**138**,**282**,15119,15121	60480,228504637	228565117

Of course, if one looked for square chains by putting halves of odd squares at the corners of a billiard table, then, by Theorem 0 of number theory, namely that odd squares are congruent to 1 mod 8, we would find that our tour broke up into four separate loops, those containing 0 and 1, -1 and 2, -2 and 3, and those containing -3 and 4 modulo 8. However, we are able to make a single necklace, by breaking the loops at places which sum to a square on other loops. For example, the billiard table with corners at 4.5, 24.5, 40.5 and 60.5 yields four 18-loops which may be connected to form a 72-necklace as follows

$$\ldots 1 — 3 \ldots 6 — 10 \ldots 71 — 29 \ldots 52 —48 \ldots$$

where the dots represent the other 16 members of each of the four loops.

More generally, if the odd squares are $(s-2r)^2$, $(s+2r)^2$, $(2s-r)^2$ and $(2s+r)^2$, we will have $n = 3(s^2 - r^2)$. In order that the point 1 is on an edge adjacent to the smallest square, we must have

$$s \geq r + \sqrt{(9r^2-1)/2}.$$

Cubic chains. The billiard table with corners at {62.5, 171.5, 256, 365} has perimeter 387. The sides are relatively prime, so the path between the pockets is complete. The adjacent pair-sums are 125, 343, 512 and 730. In pursuit of a chain all of whose pair-sums are cubes, we move the corner from 365 to 364.5, and insert a new reflex corner at 386.5 and a side pocket at 387. A detailed calculation reveals that the path between the pockets at 387 and 256 is complete, so we then have a cubic chain among the numbers from 1 through 387. This chain uses only the cubes 125, 343, 512 and 729.

By deleting the endpoint at 387 we obtain a cubic chain among the numbers from 1 through 386. Since each of our Fibonacci chains also has a pocket at its highest number, we can similarly delete that maximum number and obtain a Fibonacci chain among the numbers from 1 to $F_k - 1$, for any $k > 3$. We leave the reader to design billiard tables with extra corners to accommodate such numbers.

No doubt, in answer to Nob Yoshigara's question, cubic chains and necklaces exist for all sufficiently large n, but not for $n < 295$. When $n = 295$ the graph has just two monovalent vertices, at 216 and 256, which have to form the tails of a chain, but it cannot be completed. We can construct a **cubic necklace** if we can find a number which is the sum of two odd cubes in two different ways. If the cubes are $a^3 + d^3 = b^3 + c^3$, then we also need that $a^3 < c^3 - b^3$ (to make sure the necklace includes all the numbers from 1 on) and that $\gcd(c^3 - b^3, b^3 - a^3) = 2$ (else the necklace will split up into smaller necklaces). The smallest try is $23^3 + 163^3 = 121^3 + 137^3$, but the relevant gcd is 14 and we have 7 small necklaces each of length 114256 instead of a single necklace of length 799792. Fortunately, Andrew Bremner observes that $21^3 + 257^3 = 167^3 + 231^3$ where $167^3 - 21^3 = 2 \cdot 13 \cdot 31 \cdot 73 \cdot 79$ and $231^3 - 167^2 = 2^6 \cdot 119827$ have gcd 2, so that if we put halves of these four odd cubes at the corners of a billiard table, we will have a cubic necklace of length the latter number, 7668928. Surely there are smaller ones.

Triangular chains exist for $n = 2$ and probably for all $n \geq 9$. Necklaces appear to exist for $n \geq 12$, except for $n = 14$. We would like to see proofs of these statements, which we have verified to $n = 70$. It is easy to find arbitrarily large triangular chains, by taking numbers which are the sum of two triangular numbers in two different ways. If the triangular numbers $A < B < C < D$ are odd and not all multiples of three (in fact two will have to be multiples of 3 and two of them congruent to 1 mod 3), then, by placing their halves at the corners of a billiard table, we will have a **triangular necklace** of length $C - A$, provided that the sides of the table are coprime, and that $A < C - B$ (else we will lose some of the beads from the beginning of the necklace).

The table on the next page shows some triangular necklaces.

The existence of 'triangular triples', such as

$$---29---91---62--,$$

$$---44---92---61--,$$

$$---27---93---78--$$

where each pair sums to a triangular number, enables us to expand the 90-necklace at the head of the last list, to 91-, 92- and 93-necklaces, as in Figure 15.

In the same way, we can insert —101—199—152— and —100—200—53— into the 198-necklace which is the second in the list.

corners are half the triangular numbers:	sides are coprime	perimeter; # of beads
1, 15, 91, 105	7, 38	90
55, 153, 253, 351	49, 50	198
91, 231, 325, 465	47, 70	234
15, 253, 465, 703	106, 119	450
21, 55, 561, 595	17, 253	540
45, 153, 595, 703	54, 221	550
91, 253, 741, 903	81, 244	650
253, 703, 1035, 1485	166, 225	782
3, 325, 903, 1225	161, 289	900
325. 703, 1275, 1653	189, 286	950
45, 91, 1035, 1081	23, 472	990
465, 703, 1653, 1891	119, 475	1188
171, 1225, 1431, 2485	103, 527	1260
45, 325, 1431, 1711	140, 553	1386
1, 55, 1431, 1485	27, 688	1430
45, 1035, 1711, 2701	338, 495	1666
1, 435, 1711, 2145	217, 638	1710
171, 703, 1953, 2485	266, 625	1782
91, 153, 1891, 1953	31, 869	1800
55, 1485, 1891, 3321	203, 715	1836
105, 595, 2211, 2701	245, 808	2106
15, 231, 2485, 2701	108, 1127	2470
91, 1485, 2701, 4095	608, 697	2610
55, 435, 2701, 3081	190, 1133	2646
21, 595, 3081, 3655	287, 1243	3060
3, 325, 3081, 3403	161, 1378	3078
171, 253, 3321, 3403	41, 1584	3250
1, 91, 4005, 4095	45, 1957	4004

Triangular necklaces

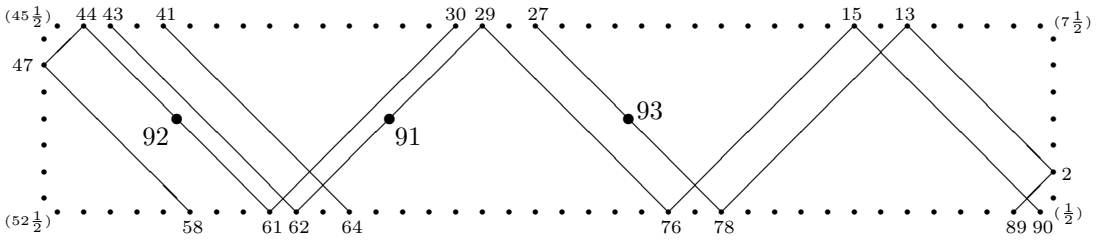

Figure 15: Expanding a 'triangular' 90-necklace by one, two or three beads.

Pentagonal chains are those in which adjacent links sum to the pentagonal numbers, 1, 2, 5, 7, 12, 15, ..., $\frac{1}{2}n(3n \pm 1)$. They appear to exist for all $n \geq 4$ (e.g., 1—4—3—2) and necklaces for all $n \geq 9$, e.g., –6–1–4–8–7–5–2–3–9–6– or

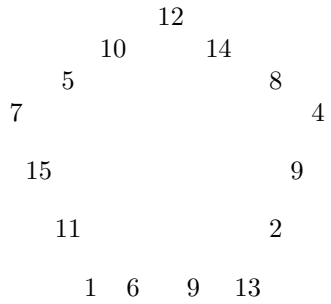

Figure 16: A necklace with adjacent pairs of beads adding to pentagonal numbers.

This has been checked to $n = 49$. Here are some other necklaces.

corners are half the pentagonal numbers:	sides are coprime	perimeter; # of beads
15, 35, 57, 77	10, 11	42
1, 7, 51, 57	3, 22	50
7, 35, 117, 145	14, 41	110
35, 77, 145, 187	21, 34	110
15, 117, 145, 247	51, 14	130
57, 155, 247, 345	49, 46	190
7, 145, 287, 425	69, 71	280
1, 15, 287, 301	7, 136	286
7, 51, 301, 345	22, 125	294
7. 77, 425, 495	35, 174	418

Prime chains have been considered from time to time [3, 4], but as in all cases except the Fibonacci numbers and the Lucas numbers, existence proofs for all large enough n are elusive.

Theorem 2. *There is a chain formed with the numbers 1 to n with each adjacent pair adding to a Lucas number, just if $n = 5$, or L_k or $L_k - 1$, where L_k is a Lucas number with $k \geq 2$ ($L_2 = 3$, $L_3 = 4$, $L_{n+1} = L_n + L_{n-1}$). The chain is essentially unique.*

The proof can follow either of the methods used for Theorem 1.

There are corresponding theorems for sequences satisfying the same recurrence relation. For example, the chains that can be formed using the numbers 4, 5, 9, 14, 23, 37, ... have length one of those numbers, or one less than one of them.

Appendix on square necklaces (February 2020)

In the more than seventeen years since this paper was written, one author has collected square necklaces for $32 \leq n \leq 252$. They are not unique. Figure 17 shows a pair of necklaces for $n = 40$.

1	3	6	19	30	34	15	10	39	25	24
8										40
17										9
32										16
4										33
21										31
28										18
36										7
13										2
12										23
37	27	22	14	35	29	20	5	11	38	26

3	6	19	30	34	15	10	39	25	24	40
22										9
27										16
37										33
12										31
13										18
36										7
28										29
21										20
4										5
32	17	8	1	35	14	2	23	26	38	11

Figure 17: A pair of square necklaces for $n = 40$.

At a recent MathFest presentation by the other author, a member of the audience claimed to have used a computer to find square necklaces for $32 \leq n \leq 1000$.

We were delighted to learn that the problem was recently solved by Robert Gerbicz; see the Mersenne Forum blog thread [1]. Square necklaces exist for any length of the form $n = (71 * 25^k - 1)/2$ with $k \geq 0$. A generalization of this construction proves the existence of square necklaces of any length $n \geq 32$ and square chains of any length $n \geq 25$. Gerbicz's C code for generating square necklaces is available for download [2].

Acknowledgment. Thanks to Alex Fink for finding one of the necklaces in Figure 17 and to Ethan White for discovering Robert Gerbicz's blog post.

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[3] Guy, R. K., Prime pyramids, *Crux Mathematicorum*, **19** (1993) 97–99.

[4] Kenney, M. J., Student Math. Notes, *NCTM News Bull.*, Nov. 1986.

[5] Sloane, N. J. A. (editor), *Online Encyclopedia of Integer Sequences*. Published electronically at
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TRIBUTE

Richard K. Guy's passing at age 103 invokes awe as much as sadness. It is truly the end of an era.

I first learned of Richard Guy in the late 1960's, when as an undergraduate at the University of Manitoba I got interested in the "no-three-in-line" problem, which is: *In an n by n grid of points, can you always choose $2n$ of these n^2 points so that no three are collinear (in any direction)*? Richard had co-authored a paper on this problem, conjecturing that the answer is **no** for large enough n (still unsettled to this day, by the way). I got nowhere beyond drawing some examples, but had a lot of fun in the process.

As a graduate student in the 1970's I visited the University of Calgary, first meeting Richard then, and joined the U of C Math Department in 1979. In the mid-80's, for some reason Richard put my name forward to replace Léo Sauvé as Editor of *Crux*; an odd choice, as I had some experience at the time with local math contests and other school enrichment activities in the Department, but had never even heard of *Crux*, much less subscribed or contributed to it. However, it was a move on his part that I am forever grateful for, as it led to my enjoying ten years of intense but ever-fascinating involvement with this publication. During this time, Richard's formidable knowledge and library on topics such as geometry, number theory and inequalities were always available and invaluable to me. In 1991, Richard became a member of *Crux*s very first Editorial Board, and he remained as a Board member through 2003. Beyond that, he was always around for advice.

Richard's international prominence in certain areas of mathematics was due in part to his impressive list of publications, which include some classic books such as *Unsolved Problems in Geometry* (written with Croft and Falconer), *Unsolved Problems in Number Theory*, and the two-volume *Winning Ways for your Mathematical Plays* (written with Berlekamp and Conway). His contributions to mathematics even extended to terminology, in particular the phrase "n choose k" for the binomial symbol $\binom{n}{k}$. Closer to home, his decades-long support of mathematical enrichment for students in Calgary was an inspiration to his colleagues, as was his love of the outdoors and of life gently lived. He and his late wife and companion Louise were a delight to know and to be around. The mathematical community, and particularly the mathematics department at Calgary, was the richer for his presence, and is much the poorer without him.

Bill Sands

MathemAttic

No. 18

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **December 15, 2020**.

MA86. On a $2 \times n$ board, you start from the square at the bottom left corner. You are allowed to move from square to adjacent square, with no diagonal moves, and each square must be visited at most once. Moreover, two squares visited on the path may not share a common edge unless you move directly from one of them to the other. We consider two types of paths, those ending on the square at the top right corner and those ending on the square at the bottom right corner. The diagram below shows that there are 4 paths of each type when $n = 4$. Prove that the numbers of these two types of paths are the same for $n = 2014$.

MA87. One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Let a_1 denote the number of clothespins holding up the first piece of clothing, a_2 the number of clothespins holding up the second piece of clothing, and so forth. You want to remove all the clothing from the line, obeying the following rules:

(i) you must remove the clothing in the order that they are hanging on the line;

(ii) you must remove either 2, 3 or 4 clothespins at a time, no more, no less;

(iii) all the pins holding up a piece of clothing must be removed at the same time.

Find all sequences $a_1, a_2, \ldots, a_n$ of any length for which all the clothing can be removed from the line.

MA88. *Proposed by Konstantin Knop.*

a) Sort the numbers from 1 to 100 in increasing order of their digit-sums; in case of a tie, sort in increasing order of the numbers themselves. Consider the resulting sequence $a(n)$: $a(1) = 1, a(2) = 10, a(3) = 100, \ldots$ Find at least one number $n > 1$ such that $a(n) = n$.

b) Consider the same problem but for numbers from 1 to 100 000 000.

MA89. *Proposed by Bill Sands.*

Two robots R2 and D2 are at the origin O on the x, y plane. R2 can move twice as fast as D2. There are two treasures located on the plane, and whichever robot gets to each treasure first gets to keep it (in case of a tie, neither robot gets the treasure). One treasure is located at the point $P = (-3, 0)$, and the other treasure is located at a point $X = (x, y)$. Find all $X \neq O$ so that D2 can prevent R2 from getting both treasures, no matter what R2 does. Which such X has the largest value of y?

Note: D2 does not care if R2 gets one of the treasures, only that R2 shouldn't get both treasures. D2 also doesn't care if it gets either treasure itself, it only wants to prevent R2 from getting both treasures.

MA90. *Proposed by Michel Bataille.*

Two positive integers are called co-prime if they share no common divisors other than 1. Find all pairs of co-prime x, y such that $\dfrac{y(x+y)}{x-y}$ is a positive integer.

. . . . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA86. Sur un damier de taille $2 \times n$, on commence au carré en bas et à gauche. Par la suite, il est permis de se déplacer d'un carré à un carré adjacent, sans déplacement en diagonale; on peut visiter chaque carré au plus une seule fois. Enfin, deux carrés faisant partie du même parcours ne peuvent partager un côté que si on se déplace de l'un immédiatement vers l'autre. On considère deux sortes de parcours, ceux terminant au sommet à droite en haut puis ceux terminant à droite en bas. Le schéma montre qu'il y a bien 4 parcours de chaque sorte lorsque $n = 4$. Démontrer qu'il y a le même nombre de parcours de chaque sorte lorsque $n = 2014$.

MA87. Un nombre d'items de linge se trouvent suspendus sur une corde à linge. Chaque item y est retenu par 1, 2 ou 3 épingles à linge. Soit a_1 le nombre d'épingles retenant le premier item de linge, a_2 le nombre d'épingles retenant le deuxième item de linge, et ainsi de suite. On voudrait enlever tous les items de linge de la corde, en respectant les directives suivantes:

(i) on doit enlever les items de la corde dans le même ordre qu'ils sont suspendus;

(ii) on doit enlever 2, 3 ou 4 épingles à la fois, ni plus ni moins;

(iii) toutes les épingles retenant un item doivent être enlevées en même temps.

Pour n donné, déterminer toutes les suites $a_1, a_2, \ldots, a_n$ permettant d'enlever tous les items de linge de la corde.

MA88. *Proposé par Konstantin Knop.*

a) Trier les nombres de 1 à 100 en ordre croissant de leurs sommes de chiffres, où les ex æquo sont départagés selon l'ordre des nombres eux-mêmes. Ceci donne lieu à la suite $a(n)$: $a(1) = 1, a(2) = 10, a(3) = 100, \ldots$ Déterminer au moins un nombre $n > 1$ tel que $a(n) = n$.

b) Considérer le même problème pour les nombres de 1 à 1 000 000 000.

MA89. *Proposé par Bill Sands.*

Au départ, deux robots R2 et D2 se trouvent à l'origine O dans le plan x, y. Deux trésors se situent dans le plan et le premier robot à y arriver reçoit le trésor en question (en cas d'égalité, ni l'un ni l'autre reçoit le trésor). Un trésor se trouve au point $P = (-3, 0)$ et le second à un point $X = (x, y)$. Or, R2 se déplace à deux fois la vitesse de D2. Déterminer tout $X \neq O$ tel que D2 peut empêcher R2 de recevoir les deux trésors, quoi que R2 fasse. Lequel tel X a la plus grande valeur de y?

Note: D2 se balance si R2 reçoit un des trésors, pourvu qu'il ne reçoive pas les deux; aussi, D2 se balance s'il reçoit un quelque trésor ou non, il ne veut qu'empêcher R2 de recevoir les deux trésors.

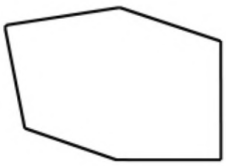
MA90. *Proposé par Michel Bataille.*

Deux entiers positifs sont dits relativement premiers s'ils ne partagent aucun diviseur autre que 1. Déterminer tout couple d'entiers relativement premiers x, y tels que $\dfrac{y(x+y)}{x-y}$ est un entier positif.

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(3), p. 98–101.

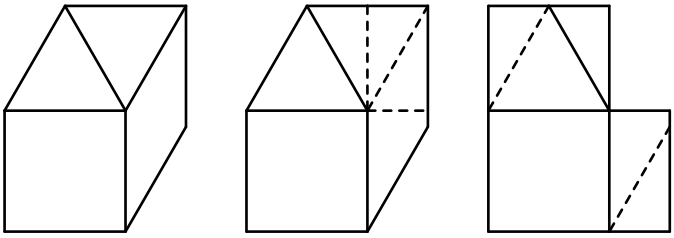
MA61. A hexagon has consecutive angle measures of 90°, 120°, 150°, 90°, 120° and 150°. If all of its sides are 4 units in length, what is the area of the hexagon?



Originally problem 14 of the 2014 Mathworks Math Contest.

We received 15 solutions, 14 of which were correct. We present the solution by Joel Schlosberg.

The hexagon can be dissected into a square, two equilateral triangles, and a parallelogram; those can in turn be cut and rearranged into two rectangles whose dimensions are 4×6 and $4 \times 2\sqrt{3}$. Therefore, the area of the hexagon is $24 + 8\sqrt{3}$.



MA62. A positive integer n is called "savage" if the integers $\{1, 2, \ldots, n\}$ can be partitioned into three sets A, B and C such that

i) the sum of the elements in each of A, B and C is the same,

ii) A contains only odd numbers,

iii) B contains only even numbers, and

iv) C contains every multiple of 3 (and possibly other numbers).

Now consider the following:

(a) Show that 8 is a savage integer.

(b) Prove that if n is an even savage integer, then $\frac{n+4}{12}$ is an integer.

Originally problem 10 of the 2003 Euclid Contest.

We received 8 solutions, all of which were correct. We present the solution by Corneliu Manescu-Avram.

(a) From $A = \{5, 7\}$, $B = \{4, 8\}$, $C = \{1, 2, 3, 6\}$, it follows that 8 is a savage number.

(b) Let $S(M)$ be the sum of elements of the set M. From

$$S(A) = S(B) = S(C) = \frac{n(n+1)}{6},$$

we deduce that 6 divides $n(n+1)$. The set B contains only even numbers, so that $S(B)$ is even, whence it follows that 12 divides $n(n+1)$. If $n = 12k+r$, with $r \in \{0, 2, 4, 6, 8, 10\}$, this is possible only for $r \in \{0, 8\}$. But for $n = 12k$, we have $S(C) = 2k(12k + 1)$ and

$$3 + 6 + \cdots + 12k = 6k(4k + 1) \leq S(C),$$

a contradiction.

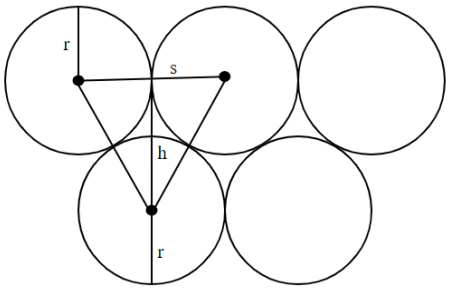
The only remaining possibility is $r = 8$, so that $n+4 = 12k+8+4 = 12(k+1)$.

MA63. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

Originally problem 25 of the 1998 Cayley Contest.

We received 2 submissions, both of which were correct and complete. We present the solution by Arya Kondur.

In order to pack the maximum number of circles, we need to create rows that alternate between 100 and 99 circles. A small portion of two rows is depicted in Figure 1, as well as some denotations regarding the radius of each circle and the side length of the equilateral triangle.



We see that $s = 2r$, where s represents the side length of the equilateral triangle and r represents the radius of a circle. The height of the equilateral triangle can

be calculated as $h = s \cdot \sin 60°$ since the angle opposite h is equivalent to $60°$. This yields $h = \frac{s}{2}\sqrt{3} = r\sqrt{3}$.

Notice that if we simply have one row of circles, the height of the row would be $2r$. However, referring to the figure, we see that with two rows of circles, the height is $r + h + r$. That is, the height of two rows is $2r + h = 2r + r\sqrt{3}$. With three rows, we find that the height is $r + 2h + r$. This simplifies to $2r + 2r\sqrt{3}$. We see a pattern starting to form. The height of n rows is $2r + (n-1)(r\sqrt{3})$.

We are given that the diameter of a circle is 1, meaning the radius is $1/2$. We substitute this value into our formula for the height of n rows. Thus, the height is $1 + (n-1)(\sqrt{3}/2)$. Since the maximum height is 100, we set this equal to 100 and solve for n. It follows that the maximum number of rows is $n = 115.3$. However, since we want an integer number of rows, we round down to $n = 115$.

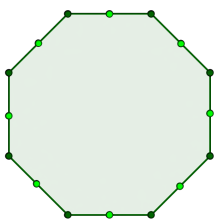
Since we have 115 rows, we have 58 rows with 100 circles and 57 rows with 99 circles. We do not make 58 rows of 99 circles and 57 rows of 100 circles since this would result in fewer total circles. With our current arrangement, the total number of circles we can fit in the 100 by 100 square is $58 \cdot 100 + 57 \cdot 99 = 11443$ circles. Therefore, we conclude that a maximum of $11443 - 10000 = 1443$ additional circles can be packed.

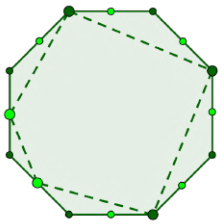
MA64. A regular octagon is shown in the first diagram below, with the vertices and midpoints of the sides marked.

An "inner polygon" is a polygon formed by traversing the octagon in a clockwise manner, selecting some of the marked points as you go, ensuring that each side of the original octagon contains exactly one selected point. Then each selected point is connected to the next with a line segment, and the last is connected to the first to complete the inner polygon.

An example of an inner polygon is shown in the second diagram.

How many inner polygons does the regular octagon have?





Originally problem 13 of the 2017 Mathworks Math Contest (with modified wording).

We received three submissions, out of which two were correct and complete. Both are presented below.

Solution 1, by Aditya Gupta, slightly edited.

We observe that an inner polygon is uniquely determined by the vertices it has in common with the octagon.

Since an inner polygon cannot contain adjacent vertices of the octagon, it can contain at most 4 of those vertices. Thus we can divide the problem into five cases:

(a) The chosen points of the inner polygon do not contain a vertex of the octagon and hence consist entirely of midpoints of sides. There is 1 such inner polygon.

(b) One vertex of the octagon is chosen. We have 8 such inner polygons.

(c) Two vertices are chosen. Since the vertices cannot be adjacent, we then have $\frac{8\cdot 5}{2} = 20$ such inner polygons.

(d) Three vertices are chosen. There are $\frac{8\cdot\binom{4}{2}}{3} = 16$ such inner polygons.

(e) Four vertices are chosen. There are 2 such inner polygons.

Hence the total number of inner polygons is $1 + 8 + 20 + 16 + 2 = 47$.

Solution 2, by the Missouri State University Problem Solving Group, slightly edited.

We claim that for any regular n-gon with $n \geq 5$, the number of inner polygons is $F_{n+1} + F_{n-1}$, where F_k is the k-th Fibonacci number. In particular, when $n = 8$, we have $F_9 + F_7 = 34 + 13 = 47$ inner polygons. [*Ed: The formula also holds for $n < 5$ if we allow degenerate polygons*]. Note that $F_{n+1} + F_{n-1}$ is also known as the n-th Lucas number L_n.

We first show that the number of sequences of 0's and 1's of length n having no two consecutive 1's is F_{n+2}. We call such a sequence *nice*. Let A_n denote the number of nice sequences of length n. Then $A_1 = 2 = F_3$ and $A_2 = 3 = F_4$. There is a bijection between nice sequences of length n starting with 0 and nice sequences of length $n - 1$ obtained by deleting the leading 0. There is also a bijection between nice sequences of length n starting with 1 and nice sequences of length $n-2$, obtained by deleting the leading two terms, which have to be 1 and 0 by definition. Therefore $A_n = A_{n-1} + A_{n-2}$. Since the Fibonacci sequence obeys the same recursion with shifted initial terms, the result follows.

We next note that there is a bijection between inner polygons and nice sequences that do not both begin and end with a 1. Fix a vertex of the original polygon. Beginning at that vertex and circling through the other vertices of the original polygon, write a 1 if the vertex is a vertex of the inner polygon and a 0 if it is not. By definition, no two 1's can be consecutive, nor can both the first and the last terms be 1's. Conversely, given a sequence, we construct an inner polygon by taking as vertices all the vertices of the original polygon that correspond to a 1 in the sequence and all the midpoints of sides of the original polygon that do not contain one of those vertices.

Finally, we show that the number of such sequences is as claimed. If a sequence begins with a 0, we can append a nice sequence of length $n-1$, of which there are F_{n+1}. If the sequence begins with a 1, then the second and last term must both be 0. This leaves a nice sequence of length $n-3$, of which there are F_{n-1}. Therefore the total number of inner polygons is $F_{n+1} + F_{n-1}$.

MA65. There are four unequal, positive integers a, b, c, and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$?

Originally problem 23 of the 1998 Cayley Contest.

We received 17 submissions of which 16 were correct and complete. We present the solution by Doddy Kastanya.

We have two equations:

$$N = 5a + 3b + 5c, \tag{1}$$

$$N = 4a + 5b + 4c. \tag{2}$$

If we multiply equation (1) by 4, multiply equation (2) by 5 and then take the difference, we get:

$$N = 13b \tag{3}$$

Also, from (1) and (2), we have:

$$5a + 3b + 5c = 4a + 5b + 4c$$

or $a + c = 2b$. Since b is a positive integer and N is between 131 and 150, the only value that satisfies both equation (3) and the condition on the value of N is $b = 11$. Since $a + c = 2b$, then $a + b + c = 3b = 33$.

PROBLEM SOLVING VIGNETTES

No. 13

Shawn Godin

A Problem from Richard Guy

I am not sure when I first heard about Richard Guy, I suspect it was from one of Martin Gardner's articles. Over the years I have accumulated a few of his books. Through my involvement with *Crux*, I have corresponded with Professor Guy on a number of occasions: soliciting material, asking for feedback or opinions on issues as well as receiving feedback or opinions on issues where I had not asked for one. I always felt that the journal, and I, benefited from his wisdom. I have also been fortunate enough to meet him on a couple of occasions at CMS meetings and I currently possess some of his old journals that he had donated to the University of Calgary. In his memory, we will explore a problem from one of his publications.

In [1], Professor Guy presents us with the first few terms of 35 patterns. The reader is then given the task to conjecture whether these patterns continue indefinitely, or not. In example 6, we are introduced to the numbers

$$31, 331, 3331, 33331, 333331, 3333331,$$

each of which is prime. This leads us to consider whether all numbers of the form $33\ldots331$ are prime.

We will refer to these numbers as $t_n = \overbrace{33\ldots33}^{n}1$. As the last digit of t_n is 1, 2 and 5 cannot be factors of t_n. Also, $t_n - 1$ and $t_n + 2$ are each clearly divisible by 3, so 3 cannot be a factor of any t_n. To aid our analysis, we will rewrite t_n in a different form.

We played with *rep-digit* numbers in a previous column [2019 : 45(6), p. 313-317]. It can be shown that the rep-digit number consisting of n copies of the digit k can be written as

$$\overbrace{kk\ldots kk}^{n} = \frac{k}{9}(10^n - 1).$$

Hence, $t_n + 2 = \overbrace{33\ldots333}^{n+1}$, so

$$t_n + 2 = \frac{3}{9}(10^{n+1} - 1)$$
$$t_n = \frac{10^{n+1} - 7}{3}.$$

Since we are looking for primes, $p > 5$, that divide t_n and we know 3 is not a factor of t_n, then we can check to see if p divides $3t_n = 10^{n+1} - 7$. If p divides

$3t_n$, it must also divide t_n and we will have $10^{n+1} \equiv 7 \pmod{p}$. In this form we can tell right away that 7 never divides t_n since $7 \nmid 10$.

Exploring powers of 10 for the next couple of primes we get the following.

n	$10^n \pmod{11}$	$10^n \pmod{13}$
1	10	10
2	1	9
3	10	12
4	1	3
5	10	4
6	1	1
7	10	10
8	1	9
9	10	12
10	1	3

Modulo 11, the residue classes of powers of 10 alternate back and forth between 10 and 1. This makes sense as we can write $10 \equiv -1 \pmod{11}$. When we look modulo 13 we get the residue classes $10, 9, 12, 3, 4, 1$ and then the pattern repeats. For all prime moduli the powers of 10 will form a periodic sequence of residue classes.

Fermat's little theorem states that if p is prime, a is an integer and $p \nmid a$, then

$$a^{p-1} \equiv 1 \pmod{p},$$

which was explored in an earlier issue [2012: 38(6) p.235 - 237]. For our problem this means that $10^{p-1} \equiv 1 \pmod{p}$, hence $10^p \equiv 10 \pmod{p}$ which means that the cycle has length $p-1$ or we have gone through several cycles by this time. We can conclude that the length of the cycle must divide $p-1$. This is verified by our examples: modulo 11 the period was 2 and $2 \mid 10$; and modulo 13 the period was 6 and $6 \mid 12$.

If we would have calculated the residues of 2 modulo 11 we would have had a cycle of length 10: $2, 4, 8, 5, 10, 9, 7, 3, 6, 1$. A number n is called a *primitive root* modulo p, for some prime p, if powers of n run through all the non-zero residue classes. Hence we would say 2 is a primitive root modulo 11. Notice that since $10 = 2 \times 5$, cycles modulo 11 can have period 1, 2, 5 or 10. Once we have a primitive root we can make some deductions. Since $4 = 2^2$ and $2^{10} \equiv 1 \pmod{11}$ which marks the end of the cycle, then $4^5 = (2^2)^5 \equiv 1 \pmod{11}$, hence the cycle for powers of 4 has length 5. Similarly we can see the cycle for 10 has length 2 since $10 \equiv 2^5 \pmod{11}$.

Primitive roots cycle through all non-zero residues modulo p, so if 10 is a primitive root modulo p, then $10^n \equiv 7 \pmod{p}$ for some n and then $10^k \equiv 7 \pmod{p}$ for *all* $k \equiv n \pmod{p-1}$. Artin conjectured in 1927 that all integers that are not perfect squares or -1 are primitive roots for infinitely many primes. If true, it means there are infinitely many primes that divide some terms of our sequence.

Even if it isn't true, if we can find one prime where 10 is a primitive root, we will be able to find infinitely many terms of our sequence that are not prime.

It turns out 10 is a primitive root modulo 17, and $10^9 \equiv 7 \pmod{17}$. Hence $17 \mid 10^9 - 7$ and therefore $17 \mid t_8$. Dividing yields $t_8 = 333\,333\,331 = 17 \times 19\,607\,843$ where the second factor is also prime. Since 10 is a primitive root modulo 17, powers of 10 cycle through the 16 non-zero residue classes modulo 17. Thus, since $17 \mid t_8$ we must also have $17 \mid t_{24}$, $17 \mid t_{40}, \ldots$ and in general $17 \mid t_{16k+8}$. So we get an infinite family of terms of the sequence that are divisible by 17. Similarly 10 is a primitive root modulo $19, 23, 29, 47, 59, 61, \ldots$ so each of these is tied to infinite families of terms in the sequence that are divisible by them.

Even if 10 is not a primitive root modulo p we might get a solution. Note that $10^2 = 100 \equiv 7 \pmod{31}$ which means that $31 \mid t_1 = 31$, but if we look at powers of 10 modulo 31 we get the cycle of residue classes

$$10 \to 7 \to 8 \to 18 \to 25 \to 2 \to 20 \to 14 \to 16 \to 5 \to 19 \to 4 \to 9 \to 28 \to 1 \to 10 \to 7 \to \cdots$$

which is periodic of period 15 (note $15 \mid 30$), which means that $31 \mid t_{15k+1}$.

We can generalize this idea: if $t_n = p$ we must have $10^{n+1} \equiv 7 \pmod{p}$ which means that 7 is in the cycle of powers of 10 modulo p, which is periodic with period that divides $p - 1$. This means that $p \mid t_{kd+n}$ for all positive integers k, where $d \mid p - 1$ is the period. Thus the list of numbers must contain infinitely many composites.

For example, the next prime on the list is $t_2 = 331$, which means that, at the very least, $331 \mid t_{330k+2}$. Using Excel, I investigated the residues of powers of 10 modulo 331 and found that the length of the cycle is actually $110 \mid 330$. Therefore, we have the more precise result $331 \mid t_{110k+2}$. Wolfram Alpha confirms that

$$\frac{3t_{112}}{331} = \frac{10^{113} - 7}{331}$$

is an integer, showing that $331 \mid t_{112}$.

There are some other phenomena related to ideas from our solution. Earlier, we saw that the periods of powers of 10 modulo 11, 13 and 17 were 2, 6 and 16 respectively. Notice that the fractions

$$\begin{aligned}
\frac{1}{11} &= 0.\overline{09} \\
\frac{1}{13} &= 0.\overline{076923} \\
\frac{1}{17} &= 0.\overline{0588235294117647}
\end{aligned}$$

are also periodic with periods 2, 6 and 16. There is a strong connection here. If you do the long division for $\frac{1}{13}$, for example, you will see, in the remainders, the cycle of residue classes of powers of 10 modulo 13. Think about why this must be true. In [2], Professor Guy and John H. Conway (another great mathematician

we lost this year) investigate decimal representations of fractions. They introduce what they call *long primes* as primes, p, whose decimal representation of $\frac{1}{p}$ has the maximum length $p-1$. As we have seen, these are the primes where 10 is a primitive root.

Along with [2], readers of *MathemAttic* may enjoy a few of Professor Guy's other books. In particular [3], a collection of over 200 problems with hints, solutions, and discussions; and [4] for any interested in seeing some unsolved problems and progress that has been made towards their solutions.

References

[1] Richard K. Guy, The Strong Law of Small Numbers, in Richard K. Guy and Robert Woodrow (Eds.) *The Lighter Side of Mathematics*, Spectrum Series, The Mathematical Association of America, 1994, pp. 265-280. Reprinted from *The American Mathematical Monthly* **95**(1988) pp. 697-712.

[2] John H. Conway and Richard K. Guy, *The Book of Numbers*, Springer-Verlag, New York, 1996.

[3] Paul Vaderlind, Richard K. Guy, and Loren Larson, *The Inquisitive Problem Solver*, The Mathematical Association of America, Washington, 2002.

[4] Richard K. Guy, *Unsolved Problems in Number Theory, Second Edition*, Springer-Verlag, New York, 1994.

The Eugene Strens Collection

John McLoughlin

Did you know that one of the finest collections of recreational mathematics is housed at University of Calgary? The Eugene Strens Recreational Mathematics Collection was acquired through the dedication and generosity of Professor Richard and Mrs. Louise Guy, the estate of Eugene Strens and many other individuals who established the original collection of books, periodicals, puzzles and manuscripts. Among these others are Charles W. Trigg, Wade Philpott, Martin Gardner, William Schaaf, and Leon Bankoff.

Two related links appear here. The first of these is a link to the book collection in the library catalogue:
`https://asc.ucalgary.ca/book-collection/eugene-strens/`

The second link is for The Recreational Mathematics Archives. Here will appear a collection of links to the finding aids for the various fonds that make up the Strens archival collection (the C.M. Jones fonds will be of special interest because they contain correspondence between Jones and Richard Guy)
`https://asc.ucalgary.ca/special-collection/recreational-mathematics/`

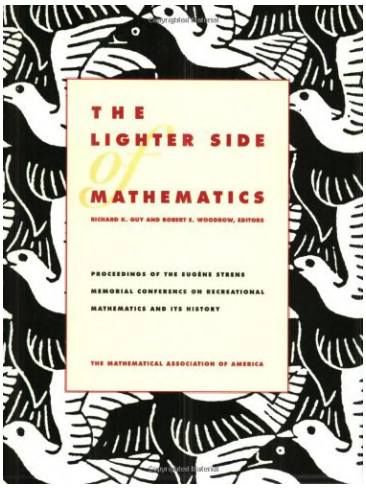
Personal Note

I had the opportunity to visit the collection in its pre-digital days back in the mid-90s. On two occasions I spent a couple of days there. The collection was under the supervision of Apollonia Steele at the time. I would go through the catalogue to select items for consideration. These would be retrieved and shared for use in the area. A few things struck me on my visit beyond the extraordinary wealth of the resources. First there were multiple copies of select books. Why? These books had been provided by the likes of the aforementioned contributors and hence, it was not as though one could simply pass them along. Rather it was not uncommon to find jottings or notes of the contributors in the works themselves. It was neat to be holding books from the collections of people who had authored the same books that had entertained me beforehand. Second, there was so much good material that it was challenging to make selections. I do recall learning some mathematics and leaving with photocopies of select articles, in particular, ideas around number.

One of those learnings concerned Kaprekar's number. Select any four different digits and arrange them to form the largest four-digit number and the smallest four-digit number before taking the positive difference between these numbers. Now repeat this using the four digits in your result. (It may be that the four digits are not all different any longer, but use them anyhow.) Continue this process. Lo and behold, the result will always be 6174. This is Kaprekar's number. Observe that $7641 - 1467 = 6174$, thus, returning the number itself.

The Lighter Side of Mathematics

A recreational mathematics conference was held at University of Calgary in 1986 to celebrate the founding of the Strens Collection. A collection of articles by authors such as Doris Schattschneider, Elwyn Berlekamp, H.S.M. Coxeter, David Singmaster and many others were collated into a volume representing the proceedings. The resulting publication, *The Lighter Side of Mathematics*, was edited by Richard Guy and Robert Woodrow. The MAA published it as part of its Spectrum Series.



The articles are preceded by an introductory section offering insight into the origins of the collection. Richard Guy mentions in reference to the recreational mathematical manuscripts and items, how the children of Eugene Strens "were reluctant to see the careful collection of a lifetime broken up, but they did not have the facility to continue curating it themselves." He goes on to explain how Lee Sallows reached out to Martin Gardner who reached out to others concerning the possibilities for housing the collection. Ultimately it was Alan MacDonald, the director of libraries at University of Calgary who expressed willingness to house the special collection. An agreement by the Strens children to donate a significant chunk of the works along with an arrangement to purchase the balance made this possibility a reality. Alan MacDonald described the collection as "a pearl that will grow" and indeed it did with the contributions of many others to the volumes of material on hand.

A subsequent introductory chapter (prepared by Pascal Strens and translated by Lee Sallows) offers insight into the life story of Eugene Louis Charles Marie Strens. Amidst the biographical piece, Pascal writes "Not unnaturally, Eugene would communicate his enthusiasms to his children; a great treat for us would be those special occasions when the forbidden drawer was opened and each of us was invited to choose a game or puzzle to examine." He closes with an expression of gratitude for the arrangement to keep much of the collection intact as "a fitting memorial to the remarkable man who was our father."

Some additional comments

It has been challenging to give a sense of the importance of this collection and the effort made by Richard and Louise Guy to bring this to fruition. There was an emphasis on making the materials accessible to a broader audience. Perhaps some of the readers here have availed themselves of the collection, or may be inclined to learn more about it now. That is all good.

I want to acknowledge Regina Landwehr and Allison Wagner from the University

of Calgary Library for assistance with communications. They provided the web links including two additional ones shared here for the interest of readers. They can be reached via `speccoll@ucalgary.ca` with any further inquiries.

1. An article from March 2013 describing an exhibit of Strens materials curated by Regina Landwehr for an exhibition series:
   `https://wayback.archive-it.org/11669/20190427153210/https://www.ucalgary.ca/news/utoday/march21-2013/math`
2. And for general interest, though you may have seen it already, the memorial page the U of C created for Dr. Guy:
   `https://science.ucalgary.ca/mathematics-statistics/about/richard-guy`

Finally, concerning *The Lighter Side of Mathematics*, it is a wonderful collection. The *American Mathematical Monthly* review by Victor J. Katz in 1996 aptly pointed to its curiosities including the difficulty in distinguishing recreational and serious mathematics. He notes, "Despite its unevenness, however, the book contains many gems including prize-winning studies from other publications and summaries of important ideas by excellent expository writers."

The review outlines several of the papers that impressed Katz. Out of respect to Richard Guy, the bulk of the commentary on his contribution of "The Strong Law of Small Numbers" appears here verbatim.

There are many number patterns which seem to appear when we look at small values of n. But, as Guy notes, "there aren't enough small numbers to meet the demands made of them." Thus many of the patterns – and we need to discover which ones – do not hold in general. If you like numerical puzzles, the article will provide many hours of stimulation (provided you don't look at the answers provided).

I will add that *The Lighter Side of Mathematics* holds a place in my recreational mathematics collection. In closing, Richard Guy has made a valuable contribution to recreational mathematics through his writings as well as his initiative through the Strens Collection and other examples of making recreational mathematics more visible and accessible. Whether a collection, a conference, a book, or a personal connection... thank you Richard.

OLYMPIAD CORNER

No. 386

The problems in this section appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **December 15, 2020**.

OC496. The six digits 1, 2, 3, 4, 5, and 6 are used to construct a one-digit number, a two-digit number and a three-digit number. Each digit must be used only once and all six digits must be used. The sum of the one-digit number and the two-digit number is 47 and the sum of the two-digit number and the three-digit number is 358. Find the sum of all three numbers.

OC497. Does there exist a positive integer that is divisible by 2020 and has equal number of digits 0,1,2, ..., 9?

OC498. A collection of 8 cubes consists of one cube with edge-length k for each integer k, $1 \leq k \leq 8$. A tower is to be built using all 8 cubes according to the rules:

(a) Any cube may be the bottom cube in the tower.

(b) The cube immediately on top of a cube with edge-length k must have edge-length at most $k + 2$.

Let T be the number of different towers than can be constructed. What is the remainder when T is divided by 1000?

OC499. A self-avoiding rook walk on a chessboard (a rectangular grid of unit squares) is a path traced by a sequence of moves parallel to an edge of the board from one unit square to another, such that each begins where the previous move ended and such that no move ever crosses a square that has previously been crossed, i.e., the rookôs path is non-self-intersecting.

Let $R(m, n)$ be the number of self-avoiding rook walks on an $m \times n$ (m rows, n columns) chess board which begin at the lower-left corner and end at the upper-left corner. For example, $R(m, 1) = 1$ for all natural numbers m; $R(2, 2) = 2$; $R(3, 2) = 4$; $R(3, 3) = 11$. Find a formula for $R(3, n)$ for each natural number n.

OC500. An $n \times m$ matrix is *nice* if it contains every integer from 1 to mn exactly once and 1 is the only entry which is the smallest both in its row and in its column. Prove that the number of $n \times m$ nice matrices is $\dfrac{(nm)!n!m!}{(n + m - 1)!}$.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2020**. *La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problémes.*

OC496. Les six chiffres 1, 2, 3, 4, 5 et 6 servent à construire un nombre à 1 chiffre, un nombre à 2 chiffres, puis un nombre à 3 chiffres. Chaque chiffre doit servir une seule fois. La somme du nombre à 1 chiffre et celui à 2 chiffres est 47, tandis que la somme du nombre à 2 chiffres et celui à 3 chiffres est 358. Déterminer la somme des trois nombres.

OC497. Existe-t-il un entier positif, divisible par 2020 et utilisant le même nombre de chacun des chiffres 0,1,2, ..., 9?

OC498. Une collection de 8 cubes consiste d'un cube de côté k pour chaque entier k, $1 \leq k \leq 8$. Une tour doit être construite selon les règles qui suivent.

(a) N'importe quel cube peut être utilisé comme base.
(b) Le cube immédiatement au dessus du cube de côté k doit être de côté au plus $k+2$.

Soit T le nombre de différentes tours pouvant être construites. Quel est le reste lorsque T est divisé par 1000 ?

OC499. Une Marche autoévitante d'une tour sur un échiquier (une grille rectangulaire formée de carrés unitaires) est un chemin tracé par une suite de mouvements parallèles à un bord de l'échiquier partant d'un carré unitaire à un autre de sorte que chacun de ces mouvements commence où le mouvement précédent a terminé et qu'aucun mouvement ne croise un carré qui a été précédemment croisé, c'est-à-dire le chemin de la tour ne se croise pas.

Soit $R(m,n)$ le nombre de Marches autoévitantes d'une tour sur un échiquier $m \times n$ (m lignes, n colonnes) qui commencent au coin inférieur gauche et se terminent au coin supérieur gauche. Par exemple, $R(m,1) = 1$ pour tout entier naturel m; $R(2,2) = 2$; $R(3,2) = 4$; $R(3,3) = 11$. Trouver une formule pour $R(3,n)$ pour chaque entier naturel n.

OC500. Une matrice $n \times m$ est dite *spéciale* si elle contient tout entier de 1 à mn exactement une fois et si 1 est le seul entier à être le plus petit à la fois dans sa rangée et dans sa colonne. Démontrer que le nombre de matrices spéciales $n \times m$ est $\dfrac{(nm)!n!m!}{(n+m-1)!}$.

OLYMPIAD CORNER
SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(3), p. 111–112.

OC471. There are $n > 3$ distinct natural numbers less than $(n-1)!$ written on a blackboard. For each pair of these numbers, Sergei divided the bigger number by the smaller with the remainder and wrote on his notebook the resulting incomplete quotient. For example, if he divided 100 by 7, he got $100 = 14 \cdot 7 + 2$ and wrote 14 in the notebook. Prove that among the numbers in the notebook there are two that are equal.

Originally from Russia Math Olympiad, 5th Problem, Grade 9, Final Round 2017.

We received 9 *submissions. We present the solution by Corneliu Avram-Manescu.*

Let $x_1 < x_2 < \ldots < x_n$ be the given numbers and let $q_i = \left\lfloor \dfrac{x_{i+1}}{x_i} \right\rfloor$. If all the q_i are distinct, then

$$\prod_{i=1}^{n-1} \frac{x_{i+1}}{x_i} \geq \prod_{i=1}^{n-1} q_i \geq \prod_{i=1}^{n-1} i, \quad \text{i.e.} \quad \frac{x_n}{x_1} \geq (n-1)!,$$

which is a contradiction, because all the x_i are less than $(n-1)!$.

OC472. Let $P(x)$ be a polynomial of degree $n \geq 2$ with nonnegative coefficients and let a, b and c be the side lengths of an acute-angled triangle. Prove that the numbers $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$ and $\sqrt[n]{P(c)}$ are also the side lengths of a triangle.

Originally from Russia Math Olympiad, 6th Problem, Grade 10, Final Round 2017.

We received 7 *submissions. We present the solution by Ioannis D. Sfikas, slightly modified.*

First, we need a Lemma.

Lemma. For any real numbers a, b and for $0 \leq p \leq 1$, $|a+b|^p \leq |a|^p + |b|^p$.

Proof. If a and b have opposite signs it's obvious. Now, assume that $a \neq 0$ and let $t = b/a \geq 0$. We need to prove that for $0 \leq p \leq 1$,

$$(1+t)^p \leq 1 + t^p \qquad \forall t \geq 0.$$

If $p = 0$ or $p = 1$ the result is trivial. So, let $0 < p < 1$ and consider the function $f(t) = 1 + t^p - (1+t)^p$. Clearly, $f(0) = 0$ and

$$f'(t) = p\left(\frac{1}{t^{1-p}} - \frac{1}{(t+1)^{1-p}}\right).$$

Hence, $f'(t) > 0$ for all $t > 0$, i.e. $f(t)$ is increasing on $[0, \infty)$ and the conclusion follows. □

Now, let $P(x) = \sum_{k=1}^{n} a_k x^k$. By Minkowski Inequality, the Lemma and the fact that $a + b > c$, we have

$$\sqrt[n]{P(a)} + \sqrt[n]{P(b)} \geq \sqrt[n]{\sum_{k=0}^{n} a_k \left(a^{\frac{k}{n}} + b^{\frac{k}{n}}\right)^n} \geq \sqrt[n]{\sum_{k=0}^{n} a_k (a+b)^k} = \sqrt[n]{P(a+b)} > \sqrt[n]{P(c)}.$$

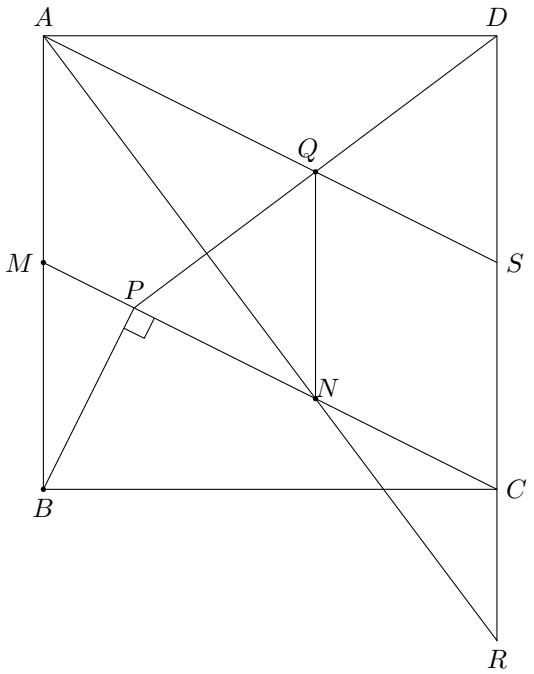
Similarly, $\sqrt[n]{P(b)} + \sqrt[n]{P(c)} > \sqrt[n]{P(a)}$ and $\sqrt[n]{P(c)} + \sqrt[n]{P(a)} > \sqrt[n]{P(b)}$. Therefore, $\sqrt[n]{P(a)}, \sqrt[n]{P(b)}, \sqrt[n]{P(c)}$ are also side lengths of a triangle.

OC473. In square $ABCD$, let M be the midpoint of AB, let P be the projection of point B onto line CM and let N be the midpoint of segment CP. The angle bisector of $\angle DAN$ intersects line DP at point Q. Prove that quadrilateral $BMQN$ is a parallelogram.

Originally from 2017 Romania Math Olympiad, 3rd Problem, Grade 7.

We received 18 *submissions. We present* 3 *solutions.*

Solution 1, by Miguel Amengual Covas.



We have $\frac{MB}{BC} = \frac{\frac{1}{2}AB}{AB} = \frac{1}{2}$. From similar triangles BPM, CPB and CBM, then

$$MP = \frac{1}{2}BP = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot PC\right) = \frac{1}{2} \cdot \left(\frac{1}{2} \cdot (2 \cdot NC)\right) = \frac{1}{2}NC,$$

implying

$$\frac{MN}{NC} = \frac{MP + PN}{NC} = \frac{MP + NC}{NC} = \frac{MP}{NC} + 1 = \frac{3}{2}.$$

Let AN (extended) intersect DC (extended) at R. From similar triangles AMN and RCN,

$$\frac{AM}{CR} = \frac{MN}{NC} = \frac{3}{2}.$$

That is to say, $AM = \frac{3}{2}CR$, and therefore

$$AB = 2 \cdot AM = 3 \cdot CR,$$

or

$$AD = 3 \cdot CR,$$

yielding

$$\frac{AD}{DR} = \frac{3}{4},$$

revealing that $\triangle ADR$ is a 3-4-5 right-angled triangle with

$$AD : DR : RA = 3 : 4 : 5.$$

Let AQ (extended) intersect DR at S. Applying the internal angle bisector theorem to $\triangle ADR$ at A, we obtain

$$\frac{DS}{AD} = \frac{DR}{AD + AR} = \frac{4}{3 + 5} = \frac{1}{2},$$

making $\triangle ADS$ congruent to $\triangle CBM$ with

$$\angle DAS = \angle BCM,$$

implying $AS \parallel CM$. Therefore, Q divides DP in the same ratio that S divides DC:

$$\frac{DQ}{DP} = \frac{DS}{DC} = \frac{DS}{AD} = \frac{1}{2},$$

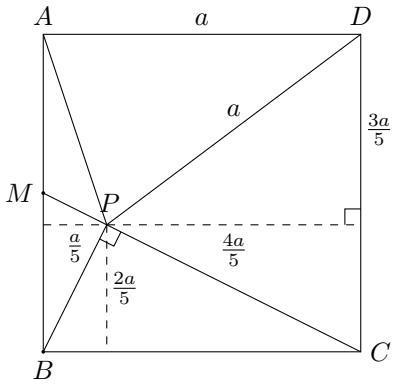
and so Q is the midpoint of PD.

Hence QN, connecting the midpoints of sides PD and PC of $\triangle PCD$, is parallel to CD and half as long or, equivalently,

$$QN \parallel MB \qquad \text{and} \qquad QN = MB,$$

making $MBNQ$ a parallelogram, as desired.

Note that triangles ADP and PDC are isosceles (see the following figure).



Solution 2, by Ivko Dimitrić

Let a be the side-length of the square, $s = PB$ and $L = \overleftrightarrow{AQ} \cap \overleftrightarrow{DC}$. Further, let

$$\alpha := \angle MCB = \angle PCB = \angle PBM \quad \text{and} \quad \beta := \angle CMB = \angle CBP,$$

so that $\beta = 90^\circ - \alpha$. Right triangles CMB, CBP and BMP are similar, so

$$\frac{MP}{PB} = \frac{PB}{PC} = \frac{BM}{BC} = \frac{1}{2},$$

from which

$$s = PB = \frac{1}{2}PC = PN = NC \quad \text{and} \quad MP = \frac{1}{2}PB = \frac{1}{2}s.$$

Then,

$$BC^2 = BP^2 + PC^2 \quad \Longrightarrow \quad a^2 = s^2 + (2s)^2 = 5s^2 \quad \Longrightarrow \quad s = \frac{a}{\sqrt{5}},$$

$$MN = MP + PN = \frac{1}{2}s + s = \frac{3}{2}s = \frac{3a}{2\sqrt{5}}, \qquad \sin\alpha = \frac{PB}{BC} = \frac{s}{a} = \frac{1}{\sqrt{5}}.$$

Since $\angle NMB = \angle CMB = \beta$, $\angle NMA = 180^\circ - \beta$ and

$$\cos(180^\circ - \beta) = \cos(90^\circ + \alpha) = -\sin\alpha = -\frac{1}{\sqrt{5}},$$

the Law of Cosines for $\triangle AMN$ gives

$$\begin{aligned} AN^2 &= AM^2 + MN^2 - 2AM \cdot MN \cdot \cos(180^\circ - \beta) \\ &= \frac{a^2}{4} + \frac{9a^2}{20} + 2 \cdot \frac{a}{2} \cdot \frac{3a}{2\sqrt{5}} \cdot \frac{1}{\sqrt{5}} = a^2. \end{aligned}$$

Hence, $AN = a = AB = AD$. Since $PN = PB$, the triangle BPN is right isosceles with $\angle NBP = \angle PNB = 45°$. Because $\triangle ABN$ is isosceles,

$$\angle ANB = \angle ABN = \angle ABP + \angle PBN = \alpha + 45°.$$

Then,

$$\angle BAN = 180° - 2\angle ABN = 180° - (2\alpha + 90°) = 90° - 2\alpha$$

and

$$\angle NAD = 90° - \angle BAN = 2\alpha.$$

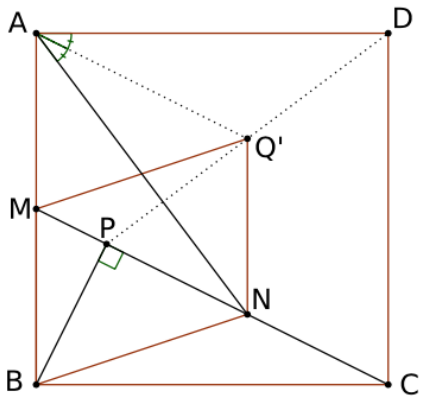
AQ is the angle bisector of $\angle NAD$, so $\angle NAQ = \angle QAD = \angle LAD = \alpha$, hence $\angle DLA = 90° - \alpha = \beta$. Therefore, right triangles ALD and CMB are congruent, L is the midpoint of CD and $\angle CMB = \beta = \angle DLA = \angle LAB$, showing that LA is parallel to CM. Since L is the midpoint of CD and LQ is parallel to CP, the segment LQ is the mid-line parallel to side CP in $\triangle DPC$, so Q is the midpoint of DP. Since N is the midpoint of PC, QN is the mid-line in the same triangle parallel to side DC and equal to half of that side in length. Therefore,

$$QN \parallel DC \parallel MB \quad \text{and} \quad QN = \frac{1}{2}DC = \frac{1}{2}AB = MB,$$

which proves that $BMQN$ is a parallelogram.

Remark. Another result that can be proved is the following: If K is the midpoint of BN and O is the center of the square, then $OPQN$ is also a square.

Solution 3, by Oliver Geupel.



Consider Cartesian coordinates with $A = (0, 10)$, $B = (0, 0)$, $C = (10, 0)$, and $D = (10, 10)$. Then, $M = (0, 5)$. Since BP is the altitude in the right $\triangle BCM$, we have $CP/PM = BC^2/BM^2 = 4$. Hence, $\vec{P} = \left(4\vec{M} + \vec{C}\right)/5 = (2, 4)$. For the midpoint N of CP, we have $\vec{N} = \left(\vec{C} + \vec{P}\right)/2 = (6, 2)$. Let Q' be the point that makes quadrilateral $BMQ'N$ a parallelogram, that is, $\vec{Q'} = \vec{M} + \vec{N} - \vec{B} = (6, 7)$. In order to prove $Q = Q'$, it is enough to show that Q' is the midpoint of DP and

$\cos\angle NAQ' = \cos\angle Q'AD$. In fact, $2\vec{Q'} = \vec{D} + \vec{P}$; whence Q' is the midpoint of DP. Moreover,

$$\cos\angle NAQ' = \frac{\overrightarrow{AN}\cdot\overrightarrow{AQ'}}{AN\cdot AQ'} = \frac{2}{\sqrt{5}} = \frac{\overrightarrow{AQ'}\cdot\overrightarrow{AD}}{AQ\cdot AD} = \cos\angle Q'AD.$$

Consequently, $Q = Q'$, that is, quadrilateral $MBQN$ is a parallelogram.

OC474. Given a right triangle ABC with hypotenuse AB, let D be the foot of the altitude drawn from point C, let M and N be the intersections of the angle bisectors of $\angle ADC$ and $\angle BDC$, respectively, with sides AC and BC. Prove that

$$2\cdot AM\cdot BN = MN^2.$$

Originally from 2017 Czech-Slovakia Math Olympiad, 5th Problem, First Round.

We received 22 *submissions. We present* 2 *solutions.*

Solution 1, by Panagiotis Antonopoulos.

Since $\widehat{MDN}+\widehat{ACB} = 180^0$ then $MDNC$ is cyclic, therefore $\widehat{CNM} = \widehat{CDM} = 45^\circ$ therefore $\widehat{CNM} = \widehat{CMN}(= 45^\circ)$ and we conclude

$$CM = CN \iff CM^2 + CN^2 = 2MC\cdot NC \iff MN^2 = 2MC\cdot NC. \quad (1)$$

By applying the angle bisector theorem in both $\triangle CDB$ and $\triangle ADC$ we get

$$\frac{AM}{MC} = \frac{AD}{DC} \quad \text{and} \quad \frac{BN}{NC} = \frac{BD}{DC}$$

and since $CD^2 = BD\cdot DA$, then we get the relation

$$AM\cdot BN = MC\cdot CN. \quad (2)$$

By (1) and (2) we get $MN^2 = 2AM\cdot BN$.

Solution 2, by Ivko Dimitrić

Denote the side lengths of $\triangle ABC$ by $a = BC$, $b = AC$, $c = AB$, as usual. The interior angle bisector from a vertex of a triangle divides the opposite side in the ratio of the two sides incident to that vertex. We also know that the triangles ACD, ABC are similar, having equal corresponding angles and hence the corresponding sides stand in the same ratio. Using these two results we have

$$\frac{AM}{MC} = \frac{AD}{DC} = \frac{AC}{BC} = \frac{b}{a} \quad\Longrightarrow\quad AM = \frac{b}{a}\,MC.$$

Then,

$$b = AM + MC = \left(\frac{b}{a}+1\right)MC \quad\Longrightarrow\quad MC = \frac{ab}{a+b}$$

and

$$AM = AC - MC = b - \frac{ab}{a+b} = \frac{b^2}{a+b}.$$

In the same manner, using similarity of $\triangle CBD$ and $\triangle ABC$ we get

$$NC = \frac{ab}{a+b} = MC \quad \text{and} \quad BN = \frac{a^2}{a+b}.$$

Consequently,

$$2 \cdot AM \cdot BN = 2 \cdot \frac{b^2}{a+b} \cdot \frac{a^2}{a+b} = 2\left(\frac{ab}{a+b}\right)^2 = MC^2 + NC^2 = MN^2,$$

by the Pythagorean Theorem for isosceles $\triangle CNM$.

OC475. Let $N > 1$ be an integer. Denote by x the smallest positive integer with the following property: there exists a positive integer y strictly less than $x-1$ such that x divides $N + y$. Prove that x is either p^n or $2p$, where p is a prime number and n is a positive integer.

Originally from 2018 Italy Math Olympiad, 4th Problem, Final Round.

We received 6 *submissions. We present the solution by UCLan Cyprus Problem Solving Group.*

Note that x satisfies the stated property if and only if $N \not\equiv 0, 1 \pmod{x}$.

Let $x = p_1^{r_1} \cdots p_k^{r_k}$ in its prime power decomposition. We may assume that $k \geq 2$ otherwise there is nothing to prove. Our aim is to show that $x = 2p$ for some prime p.

For each $1 \leq i \leq k$, let $x_i = p_i^{r_i}$. By the minimality of x, none of the x_i satisfies the stated property. Therefore $N \equiv 0, 1 \pmod{x_i}$ for each $1 \leq i \leq k$.

If $N \equiv 0 \pmod{x_i}$ for each $1 \leq i \leq k$ then (by the Chinese Remainder Theorem) $N \equiv 0 \pmod{x}$, a contradiction. Similarly we cannot have $N \equiv 1 \pmod{x_i}$ for each $1 \leq i \leq k$.

Without loss of generality we may assume that $N \equiv 0 \pmod{x_1}$ and $N \equiv 1 \pmod{x_2}$. In particular, $N \equiv 0 \pmod{p_1}$ and $N \equiv 1 \pmod{p_2}$.

We cannot have $N \equiv 0 \pmod{p_1 p_2}$ as then $N \equiv 0 \pmod{p_2}$, a contradiction. Also, we cannot have $N \equiv 1 \pmod{p_1 p_2}$ as then $N \equiv 1 \pmod{p_1}$, a contradiction. So the number $p_1 p_2$ also satisfies the property and therefore $x = p_1 p_2$. Assume now for contradiction that p_1, p_2 are both odd.

If N is even, then $N \not\equiv 0, 1 \pmod{2p_2}$. (As then either $N \equiv 0 \pmod{p_2}$, or $N \equiv 1 \pmod{2}$, both giving a contradiction.) Since $2 < p_1$, then $2p_2 < p_1 p_2 = x$, a contradiction as $2p_2$ satisfies the stated property.

If N is odd a similar argument shows that $x = 2p_1$ satisfies the stated property, again a contradiction. So the required result follows.

Notes on a Sum Problem

Bill Sands

Dedicated to the memory of Richard K. Guy.

For an integer $n > 1$, look at the following sequence of partial sums:

- $1,\ 1+2,\ \ldots,\ 1+2+\cdots+n,$
- $1+2+\cdots+n-(n+1),\ 1+2+\cdots+n-(n+1)-(n+2),$
  $\ldots,\ 1+2+\cdots+n-(n+1)-(n+2)-\cdots-2n,$
- $1+2+\cdots+n-(n+1)-(n+2)-\cdots-2n+(2n+1),$
  $1+2+\cdots+n-(n+1)-(n+2)-\cdots-2n+(2n+1)+(2n+2),\ \ldots,$
- $\ldots,$

where the first n natural numbers are added, the next n are subtracted, the next n added, and so on. We'll call this "the sequence of partial sums" or just "the sequence" for n. For instance, the sequence for $n = 3$ is:

$$1, 3, 6, 2, -3, -9, -2, 6, 15, 5, -6, -18, -5, 9, \ldots.$$

Question: for which natural numbers n does 0 eventually appear in its sequence of partial sums?

A special case of this problem appeared on the 2015/16 Alberta High School Mathematics Competition Part 2:

(a) Alya adds the following sequence of numbers together, one number at a time:

$$1, 2, -3, -4, 5, 6, -7, -8, 9, \ldots,$$

where the first two numbers are positive, the next two negative, the next two positive, and so on. Thus she gets the totals

$$1, 1+2, 1+2-3, 1+2-3-4, 1+2-3-4+5,$$

and so on. Prove that she will get zero infinitely often.

(b) Suppose instead Alya adds together the numbers

$$1, 2, 3, -4, -5, -6, 7, 8, \ldots,$$

where the first three numbers are positive, the next three negative, the next three positive, and so on. Prove that she will never get zero as a sum.

See `https://www.ualberta.ca/mathematical-and-statistical-sciences/outreach/alberta-high-school-math-competition` and click on "Question & Solution Archive", where you can find this contest and also contests from other years.

In this article, we will investigate the above question in general, getting a fairly satisfactory but not complete answer, with some questions at the end that others can tackle if they get interested.

We first establish two lemmas which will be needed later.

Lemma 1. For any k,

$$(kn+1)+(kn+2)+\cdots+(k+1)n-((k+1)n+1)-((k+1)n+2)-\cdots-(k+2)n = -n^2.$$

Proof. Immediate, since for all $i \in \{1, \ldots, n\}$, $(kn + i) - ((k + 1)n + i) = -n$. □

Lemma 2. Let $n \geq 3$, $t \geq 0$ be integers so that $n \mid t(t + 1)$ and $0.4 < t/n < 0.5$. If $n = kq$ for $q \geq 2$ a prime power and $k \leq 6$ an integer, then $n = 2m$ and $t = m - 1$ for some integer m.

Proof. Since $q \mid n$, we need $q \mid t$ or $q \mid (t + 1)$, so either t or $t + 1$ must be equal to sq for some positive integer s. If $t = sq$ for s a positive integer, then

$$\frac{s}{k} = \frac{t}{n} \in (0.4, 0.5),$$

which is impossible since $k \leq 6$. Thus it must be that $t + 1 = sq$, so

$$\frac{s}{k} - \frac{1}{qk} = \frac{sq - 1}{qk} = \frac{t}{n} \in (0.4, 0.5), \tag{1}$$

where $k \leq 6$. We consider the possible values of k one by one.

- If $k = 1$, then $\dfrac{s}{k} - \dfrac{1}{qk} = s - \dfrac{1}{q}$ cannot lie in the interval $(0.4, 0.5)$, since s is an integer and $q \geq 2$.
- Assume $k = 2$. If $s \geq 2$, then $\dfrac{s}{k} - \dfrac{1}{qk} \geq 1 - \dfrac{1}{2q} > \dfrac{1}{2}$, contradiction. Thus $s = 1$, which means $t = q - 1$ and $n = 2q$, as claimed.
- Assume $k = 3$. Then by (1) $s \geq 2$, so

  $$\frac{s}{k} - \frac{1}{qk} \geq \frac{2}{3} - \frac{1}{3q} \geq \frac{1}{2},$$

  which is impossible.
- Assume $k = 4$. By (1), $s \geq 2$. If $s \geq 3$, then

  $$\frac{s}{k} - \frac{1}{qk} \geq \frac{3}{4} - \frac{1}{4q} > \frac{1}{2},$$

  which is a contradiction. Thus $s = 2$ and $t = 2q - 1$, $n = 4q$, as claimed.

- Assume $k = 5$. By (1), $s \geq 3$. Then

$$\frac{s}{k} - \frac{1}{qk} \geq \frac{3}{5} - \frac{1}{5q} \geq \frac{1}{2},$$

impossible.

- Assume $k = 6$. By (1), $s \geq 3$. If $s \geq 4$, then

$$\frac{s}{k} - \frac{1}{qk} \geq \frac{2}{3} - \frac{1}{6q} > \frac{1}{2},$$

which is a contradiction. Thus $s = 3$ and $t = 3q - 1$, $n = 6q$, as claimed.

This proves Lemma 2. □

Now to our main question. Let $\mathcal{S}_0$ be the set of all positive integers n whose sequence of partial sums contains at least one zero. Suppose that $n \in \mathcal{S}_0$. So zero occurs somewhere in the sequence of partial sums for n. There are two cases.

Case 1: zero occurs while adding. That is, for some $k \geq 1$ and some $1 \leq t \leq n$,

$$1 + 2 + \cdots - 2kn + (2kn + 1) + \cdots + (2kn + t) = 0.$$

From Lemma 1, the sum $1 + 2 + \cdots - 2kn = -kn^2$, so we get that

$$kn^2 = \frac{t(4kn + t + 1)}{2}$$

which becomes

$$t(t + 1) = 2kn(n - 2t). \tag{2}$$

From (2) we get $t < n/2$ and also $n \mid t(t + 1)/2$.

Suppose that $n = 2m$ for some integer m, and put $t = m - 1$. Then from (2), $(m - 1)m = 4km \cdot 2$, so $k = (m - 1)/8$. Since k is an integer, we need

$$n = 2(8k + 1), \quad t = 8k,$$

and these give solutions to Case 1 for all integers $k \geq 1$. So

$$n \in \mathcal{S}_0 \textbf{ for all integers } n > 2 \textbf{ satisfying } n \equiv 2 \bmod 16.$$

Are there any other solutions in this case? If we put $t = \lambda n$ where $\lambda < 1/2$, then (2) becomes

$$k = \frac{\lambda(\lambda n + 1)}{2n(1 - 2\lambda)}.$$

Since $k \geq 1$, we need $\lambda^2 n + \lambda \geq 2n - 4n\lambda$, or $n(\lambda^2 + 4\lambda - 2) + \lambda \geq 0$. If $\lambda^2 + 4\lambda - 2 \leq -1/6$, then since $\lambda < 1/2$ and $n \geq 3$,

$$n(\lambda^2 + 4\lambda - 2) + \lambda < -\frac{3}{6} + \frac{1}{2} = 0;$$

thus we need $\lambda^2+4\lambda-2 > -1/6$, that is, $\lambda^2+4\lambda-11/6 > 0$, that is, $\lambda > \sqrt{35/6}-2$. Therefore $t/n = \lambda \in (\sqrt{35/6}-2, 1/2) \approx (0.415, 0.5)$.

Thus, if $n = kq$ for $q \geq 2$ a prime power and $k \leq 6$ an integer, Lemma 2 tells us that the above solution, namely $n = 2m$ and $t = m-1$ for some integer m, is the only solution to (2), and that n must be congruent to 2 mod 16 for there to be a solution in this case.

So we need only look at those integers n not of the form $n = kq$ for $q \geq 2$ a prime power and $k \leq 6$ an integer. For $n \leq 100$, say, these are:

$$56, \quad 60, \quad 63, \quad 70, \quad 72, \quad 77, \quad 84, \quad 88, \quad 90, \quad 91, \quad 99.$$

Of these, $n = 56$, 60, 72, 77, 88 and 91 have no integer t satisfying $n \mid t(t+1)$ and $\frac{t}{n} \in (\sqrt{35/6}-2, 1/2)$, and 70 and 84 have no integer t satisfying $n \mid t(t+1)/2$ and $\frac{t}{n} \in (\sqrt{35/6}-2, 1/2)$. Of the remainder, $n = 63$ permits $t = 27$, but then k given by (2) is not an integer. The same thing happens for $n = 90$ ($t = 44$) and $n = 99$ ($t = 44$). Thus there are no further solutions $n \leq 100$ in Case 1.

Case 2: zero occurs while subtracting. That is, for some $k \geq 0$ and some $1 \leq t \leq n$,

$$1 + 2 + \cdots + (2k+1)n - ((2k+1)n + 1) - \cdots - ((2k+1)n + t) = 0.$$

From Lemma 1, the sum $-(n+1) - (n+2) - \cdots - 2n + \cdots + (2k+1)n = kn^2$, so we get that

$$\frac{n(n+1)}{2} + kn^2 = \frac{t[2(2k+1)n + t + 1]}{2}$$

which becomes

$$t(t+1) = n[(2k+1)(n-2t) + 1] \tag{3}$$

and

$$k = \frac{2nt + t(t+1) - n(n+1)}{2n(n-2t)} \geq 0. \tag{4}$$

From (3) we get $t \leq n/2$ and also $n \mid t(t+1)$. If $t = n/2$, then from (3) we get $t(t+1) = n = 2t$, so $t = 1$ and $n = 2$. Equation (3) holds for all k in this case, so not only is $2 \in \mathcal{S}_0$, but its sequence of partial sums contains zero infinitely many times, namely

$$1+2-3 = 0, \quad 1+2-3-4+5+6-7 = 0, \quad 1+2-3-4+5+6-7-8+9+10-11 = 0, \ldots,$$

answering part (a) of the contest problem given earlier. Notice that $n = 2$ was excluded from Case 1, whereas every other positive integer congruent to 2 mod 16 was shown to belong to $\mathcal{S}_0$. But Case 2 shows that $2 \in \mathcal{S}_0$ nevertheless!

From now on we assume that $n > 2$, which implies $t < n/2$ from above. Again suppose that $n = 2m$ for some integer m, and put $t = m-1$. Then from (4),

$$k = \frac{4m(m-1) + (m-1)m - 2m(2m+1)}{4m \cdot 2} = \frac{m-7}{8} \ .$$

Since k is an integer, we need

$$n = 2(8k+7), \quad t = 8k+6,$$

and these give solutions to Case 2 for all integers $k \geq 0$. So

$$n \in \mathcal{S}_0 \textbf{ for all positive integers } n \equiv -2 \bmod 16.$$

To look for other solutions in this case, if we put $t = \lambda n$ where $\lambda < 1/2$, then

$$2nt+t(t+1)-n(n+1) = 2\lambda n^2+\lambda n(\lambda n+1)-n(n+1) = n[(\lambda^2+2\lambda-1)n+\lambda-1].$$

Since $\lambda^2 + 2\lambda - 1 \leq 0$ if $0 < \lambda \leq \sqrt{2}-1$, from (4) we must have $\lambda > \sqrt{2}-1$ and so $t/n = \lambda \in \left(\sqrt{2}-1, 1/2\right) \approx (0.414, 0.5)$.

Thus, as in Case 1, if $n = kq$ for $q \geq 2$ a prime power and $k \leq 6$ an integer, Lemma 2 gives that the only solution in this case is $n = 2m$ and $t = m-1$ for some integer m, which gave us $n \in \mathcal{S}_0$ for all positive integers $n \equiv -2 \bmod 16$.

From Case 1, among all integers $n \leq 100$, the only remaining candidates for membership in $\mathcal{S}_0$ are $n = 63, 70, 84, 90$ and 99. Of these, for $n = 63$ ($t = 27$) the k given by (4) is once again not an integer. The same thing happens for $n = 70$ ($t = 34$), $n = 90$ ($t = 44$) and $n = 99$ ($t = 44$). This leaves $n = 84$, which this time has the solution $t = 35, k = 0$. Thus $84 \in \mathcal{S}_0$.

The solutions $n = 2$, 14 and 84 are the first three in an infinite sequence of solutions, corresponding to $k = 0$ in Case 2, that is, all of these reach zero during the first series of subtractions: $1 + 2 + \cdots + n - (n+1) - \cdots - (n+t) = 0$ for some $t \in \{1, 2, \ldots, n\}$. This sequence $a_1 = 2, a_2 = 14, a_3 = 84, a_4 = 492, \ldots$ is given by the recurrence $a_n = 6a_{n-1} - a_{n-2} + 2$. See sequence A053141 in the *On-Line Encyclopedia of Integer Sequences* `https://oeis.org/`. As a consequence, $n = a_{13} = 3822685022$ in this sequence, which is congruent to -2 mod 16, will give zero two different times (at least): once in the first series of subtractions, and again (much later!) with $k = (a_{13} - 14)/16 = 238917813$ in Case 2. (Incidentally, this would be a very roundabout way of proving that 3822685022 is not of the form kq where q is a prime power and $k \leq 6$ is an integer!)

Putting all this together, the complete list of integers $n \leq 100$ in $\mathcal{S}_0$ is:

$$2,\ 14,\ 18,\ 30,\ 34,\ 46,\ 50,\ 62,\ 66,\ 78,\ 82,\ 84,\ 94,\ 98.$$

Curiously, this sequence is *not* in the *On-Line Encyclopedia of Integer Sequences*, which gives this author some hope that this material is not already known. Readers are invited to continue the sequence as far as they wish, finding any further members of $\mathcal{S}_0$ not congruent to ± 2 mod 16.

Here are some additional problems to consider. I have not thought about these problems; they might be really easy, or really hard, or something in between. But they are problems that naturally popped up, and who knows, you might be the first person to solve them!

1. Are there any positive integers greater than 2 with at least three zeroes in their sequence? The recurrence given above will generate infinitely many positive integers with at least two zeroes in their sequence, the first being 3822685022. No bets about whether there are others in between 2 and 3822685022!

2. Are there any integers besides 2 with infinitely many zeroes in their sequence?

3. Does $\mathcal{S}_0$ contain any *odd* integers?

4. Many variations of this theme are possible: here is one. Add up the numbers

$$1, 2, \ldots, 9, -10, -11, \ldots, -99, 100, 101, \ldots, 999, -1000, -1001, \ldots;$$

so all one-digit numbers are added, two-digit numbers subtracted, three-digit numbers added, and so on. Do you ever get zero?

Squares near products of consecutive integers

Ed Barbeau

To the memory of Richard Guy

Richard Guy was a mathematician of broad taste with an eye for what was rich and interesting. I hope that what follows is something that might catch his fancy. Let us start with two sets of observations:

$$4! + 1 = 5^2; \qquad 5! + 1 = 11^2; \qquad 6! + 3^2 = 27^2; \qquad 7! + 1 = 71^2;$$

and

$$3 \times 4 \times 5 \times 6 + 1 = 19^2; \qquad 4 \times 5 \times 6 \times 7 + 1 = 29^2; \qquad 5 \times 6 \times 7 \times 8 + 1 = 41^2.$$

In each case, we have a product of consecutive integers that differs from the next greater square by a square. The first set of equations seem fortuitous, but the second illustrates a general pattern since

$$(x-1)x(x+1)(x+2) = (x^2+x)(x^2+x-2) = (x^2+x-1)^2 - 1.$$

We will take a ramble through products of consecutive integers to see how common the phenomenon is.

For $m \geq 1$, let

$$f_{2m}(x) = (x-m+1)\cdots(x-1)x(x+1)\cdots(x+m-1)(x+m),$$

and

$$f_{2m+1}(x) = (x-m)\cdots(x-1)x(x+1)\cdots(x+m)$$

be defined for those x for which the product is positive, and for each $n \geq 2$, let

$$g_n(x) = \lceil \sqrt{f_n(x)} \rceil$$

be the smallest integer not less that the square root of $f_n(x)$. Therefore, $g_4(x) = x^2 + x - 1$.

It appears to be the case that a product of any number of consecutive positive integers can never be a square, a fact that is easy to establish for a product of 2, 3, 4 or 5 integers. However, a numerical check reveals that for 33 of the 70 integer values of x between 3 and 72, inclusive, $f_3(x) = (x-1)x(x+1)$ differs from the next greater square by a perfect square. This is remarkable, considering the sparseness of squares in the sequence of integers.

The difference between the product and the next square is often quite small. So we can examine solutions of the diophantine equation

$$y^2 = (x-1)x(x+1) + k^2 = x^3 - x + k^2$$

where k is a positive integer parameter. This admits a number of generic solutions given by

$$(x, |y|) = (-1, k), (0, k), (1, k), (k^2, k^3), (4k^2 - 1, k(8k^2 - 3)), (4k^2 + 1, k(8k^2 + 3)).$$

For individual values of k, additional solutions can be found. For example, $y^2 = x^3 - x + 1$ is satisfied by $(x, |y|) = (56, 419)$ and $y^2 = x^3 - x + 25$ is satisfied by $(x, |y|) = (3, 7), (7, 19), (8, 23), (13, 47), (32, 181)$. Indeed, non-generic solutions have been discovered when $1 \leq k \leq 8$ and $12 \leq k \leq 17$.

A useful device for obtaining additional solutions (x, y) when two are known is to determine the equation $y = ax + b$ of the line through two of them in the cartesian plane. This will intersect the graph of the equation in a third point whose abscissa is given by the cubic equation $(ax + b)^2 = x^3 - x + k^2$. This has three solutions that add to a^2, two of which are known; the third solution will be an integer when a is and will be the abscissa of another solution of the diophantine equation.

In the case of the product $f_4(x)$ of four consecutive integers, not only is $g_4(x)^2 - f_4(x)$ equal to a square (1), but also $(g_4(x) + 2)^2 - f_4(x)$ is square. Indeed,

$$(x^2 + x + 1)^2 - (x - 1)x(x + 1)(x + 2) = (2x + 1)^2.$$

As for the cubic, we can consider the diophantine equation

$$y^2 = (x - 1)x(x + 1)(x + 2) + k^2 = x^4 + 2x^3 - x^2 - 2x + k^2.$$

We know that when $k = 1$, there is a solution for any integer value of x, and when k is odd, there is at least one, namely

$$(x, y) = \left(\frac{1}{2}(k - 1), \frac{1}{4}(k^2 + 3)\right).$$

There are additional sporadic solutions such as

$$(k; x, |y|) = (13; 4, 23), (16; 6, 44), (27; 9, 93), (122; 14, 242).$$

What values of k admit solutions, and is it possible for there to be infinitely many solutions?

The case of five integers is similar to the case for three. The product $f_5(x) = (x - 2)(x - 1)x(x + 1)(x + 2)$ differs from the next greater square $g_5(x)^2$ by a square for 20 out of the 22 values of the integer x satisfying $3 \leq x \leq 24$. Nontrivial solutions of the diophantine equation $y^2 = f_5(x) + k^2$ have been found for $1 \leq k \leq 8$. Are there solutions for any positive value of k?

Likewise, the product $f_7(x)$ of seven consecutive integers differs from the next greater square for 18 out of the 21 values of x between 4 and 24 inclusive, as well as frequently for the second, third and fourth greater squares. The diophantine equation $y^2 = f_7(x) + 1$ has at least two solutions $(x, y) = (4, 71), (11, 4259)$

When there are six terms in the product, we get behaviour similar to that for four terms. We find that, for each small integer x, $f_6(x)$ differs from the next three greater squares by a square. For $x \geq 3$,

$$f_6(x) = (x-2)(x-1)x(x+1)(x+2)(x+3) = x^6 + 3x^5 - 5x^4 - 15x^3 + 4x^2 + 12x.$$

Define

$$u(x) = \frac{1}{2}(2x^3 + 3x^2 - 7x - 6);$$

$$v_0(x) = \frac{1}{2}(x^2 - 3x - 6); \qquad v_1(x) = \frac{1}{2}(x^2 + x + 4); \qquad v_2(x) = \frac{1}{2}(x^2 + 5x - 2).$$

Then

$$v_0(x)^2 = u(x)^2 - f_6(x); \qquad v_1(x)^2 = [u(x)+1]^2 - f_6(x); \qquad v_2(x)^2 = [u(x)+2]^2 - f_6(x).$$

Checking a table of values, we see that for x between 3 and 14 inclusive, $g_6(x) = u(x)$. So it would seem that, as in the case of a fourfold product, $g_6(x)$ is given by a polynomial generically.

However, further thought tells us that this cannot continue indefinitely. Note that $g_6(x) = u(x)$ if and only if

$$[u(x) - 1]^2 < f_6(x) = u(x)^2 - v_0(x)^2$$

if and only if

$$v_0(x)^2 - 2u(x) + 1 < 0.$$

The polynomial on the left side has degree 4 and positive leading coefficient, and so is eventually positive. Indeed, $f_6(15) = 13366080$, $g_6(15) = 3656$ and $u(15) = 3657$. It turns out that $f_6(15)$ differs from the next four greater squares by a square:

$$3656^2 - f_6(15) = 16^2; \qquad 3657^2 - f_6(15) = 87^2;$$
$$3658^2 - f_6(15) = 122^2; \qquad 3659^2 - f_6(15) = 149^2.$$

How often can $g_6(x)^2 - f_6(x)$ be square when $x \geq 15$?

Let us take stock. When n is even and exceeds 2, it appears to be the case that $f_n(x)$ can be written as the difference of squares of two polynomials, thus giving rise to solutions of the diophantine equation $y^2 = f_n(x) + k^2$ for infinitely many values of k.

We have to be careful, because of course any polynomial can be written as the difference of squares of two polynomials. If the polynomial $f(x)$ is factored in any way (including the trivial one) as $f(x) = p(x)q(x)$, then

$$f(x) = \left(\frac{p(x) + q(x)}{2}\right)^2 - \left(\frac{p(x) - q(x)}{2}\right)^2.$$

However, if may happen that the polynomials being squared do not take integer values when x is an integer. This happens with $f_2(x)$ and $f_3(x)$ for example. We

just have to note that $f_2(2) = f_3(2) = 6$, which cannot be written as the difference of two integer squares.

It appears that $f_5(x)$ and $f_7(x)$ cannot be written as the difference of squares of two polynomials mapping integers to integers. However, $f_8(x)$ has many such representations. To find them, since

$$f_8(x) = (x-3)(x-2)(x-1)x(x+1)(x+2)(x+4),$$

we can choose $p(x)$ to be the product of $k \geq 4$ linear factors and $q(x)$ the product of the remaining $8-k$ factors. Let $u(x) = \frac{1}{2}[p(x)+q(x)]$ and $v(x) = \frac{1}{2}[p(x)-q(x)]$. The polynomial $u(x)$ is equal to $g_8(x)$ if and only if $v(x)^2 - 2u(x) + 1 < 0$.

When $k \geq 5$, the degrees of $u(x)$ and $v(x)$ are both equal to k, and the polynomial on the left of the inequality is positive for x sufficiently large. When $k = 4$, since $p(x)$ and $q(x)$ have leading coefficient 1, the degree of $u(x)$ is 4 and that of $v(x)$ is at most 3. If $v(x)^2 - 2u(x) + 1$ were to be negative for infinitely many integer values of x, then the degree of $v(x)$ would have to be 0, 1 or 2. If its degree were 2, then its leading coefficient would have to be 1.

Let $p(x) = (x-3)x(x+2)(x+3)$ and $q(x) = (x-2)(x-1)(x+1)(x+4)$. The choice of factors for $p(x)$ and $q(x)$ is strategic, since the sum (2) and the sum of squares (22) for the sets $\{-3, 0, 2, 3\}$, $\{-2, -1, 1, 4\}$ are equal, and $p(x)$ and $q(x)$ differ in only their linear and constant terms. We obtain $u(x) = x^4 + 2x^3 - 9x^2 - 10x + 4$ and $v(x) = 8x + 4$. Since

$$v(x)^2 - 2u(x) + 1 = -2x(x-6)(x+1)(x+7) + 9,$$

we find that $g_8(x) = u(x)$ for $x \geq 7$.

Using the choices

$$(-3, 0, 1, 4; -2, -1, 2, 3),\ (-3, -1, 2, 4; -2, 0, 1, 3),\ (-3, -2, 3, 4; -1, 0, 1, 2)$$

to obtain the factors $p(x)$ and $q(x)$, we obtain the representations

$$\begin{aligned} f_8(x) &= u(x)^2 - (8x+4)^2 = [u(x)+2]^2 - (2x^2+2x+6)^2 \\ &= [u(x)+8]^2 - (4x^2+4x-12)^2 = [u(x)+32]^2 - (8x^2+8x-36)^2. \end{aligned}$$

Compare this with the fourfold product $f_4(x)$.

We are left with a few questions. Can $f_n(x)$ be written as the difference of squares of two polynomials taking integer values on the integers (perhaps in several ways) if and only if n is even and greater than 2? For what values of the positive integer k is the diophantine equation $y^2 = f_n(x) + k^2$ solvable? For what values of n is the diophantine equation $y^2 = f_n(x) + 1$ solvable? More mysteriously, if there any way to account for the frequency of $g_n(x)^2 - f_n(x)$ being square?

Noting that $(x-1)(x+1)+1 = x^2$, we can see if products of consecutive integers of the same parity exhibit interesting properties with respect to squares. Remarkably, for integers x between -2 and 16 inclusive, the product $(x-2)x(x+2)$ differs from the next greater square by a square. There is no discernable pattern, and the chain

is broken by $15\times17\times19 = 4845$ which differs from 4900 by 55. When k is a positive integer parameter, the diophantine equation

$$y^2 = (x-2)x(x+2) + k^2 = x^3 - 4x + k^2$$

seems to have an abundance of solutions. With little effort, nontrivial solutions have been found for 17 of the first 25 values of k. Similarly, the product $(x-4)(x-2)x(x+2)(x+4)$ differs from the next larger square by a square when $x = 1$ and $5 \le x \le 14$, with the first breaks in the pattern at $x = 15, 19, 23, 27$. The product of n consecutive integers of the same parity is given by $2^n f_n(x/2)$, so that the foregoing results for expressing $f_n(x)$ as a difference of polynomial squares can be transferred to the new environment.

We can be more far-ranging in our exploration of products. For example, take an odd product of consecutive integers, but leave the middle one out. We have for example:

$$(x-2)(x-1)(x+1)(x+2) = (x^2-2)^2 - x^2;$$
$$(x-3)(x-2)(x-1)(x+1)(x+2)(x+3) = (x^3-7x)^2 - 6^2;$$
$$(x-4)(x-3)(x-2)(x-1)(x+1)(x+2)(x+3)(x+4) = (x^4-15x^2+24)^2 - (10x)^2.$$

Leaving out the middle two terms of an even product gives, for instance,

$$(x-2)(x-1)(x+2)(x+3) = (x^2+x-4)^2 - 2^2,$$
$$4(x-3)(x-2)(x-1)(x+2)(x+3)(x+4) = (2x^3+3x^2-19x-10)^2 - (x^2+x-26)^2.$$

In all but the last instance, the first term on the right is the least square exceeding the left side when x is large.

There is a connection with the Tarry-Escott problem. For each positive integer m, this asks for two distinct sets $\{a_0, a_1, \ldots, a_m\}$ and $\{b_0, b_1, \ldots, b_m\}$ of $m+1$ integers for which

$$a_0^k + a_1^k + \cdots + a_m^k = b_0^k + b_1^k + \cdots + b_m^k$$

for $k = 0, 1, 2, \ldots, m$. For $m = 2$ and 3, the pairs $(\{-3,1,2\}, \{-2,-1,3\})$ and $(\{-11,-3,3,11\}, \{-9,-7,7,9\})$ fill the bill. When the condition is satisfied, for $1 \le k \le m$, the kth symmetric functions of the $\{a_i\}$ and $\{b_i\}$ are equal, and the polynomial

$$\prod_{i=0}^{m}(x-a_i) - \prod_{i=0}^{m}(x-b_i)$$

is constant. Thus

$$\prod_{i=0}^{m}(x-a_i) \times \prod_{i=0}^{m}(x-b_i) = u(x)^2 - c^2$$

for some polynomial $u(x)$ of degree $m+1$ and constant c. For example,

$$(x-11)(x-9)(x-7)(x-3)(x+3)(x+7)(x+9)(x+11) = (x^4-130x^3+2529)^2 - 1440^2.$$

Squares with Large Digit Average

Matthieu Dufour and Silvia Heubach

1 Introduction

In the world of recreational mathematics, an area that was near and dear to Richard Guy's heart, many problems involve properties of integers and their multiples, squares, or powers, respectively. These types of puzzles or problems have natural appeal because they are easy to state, easy to understand, and for the most part, rarely require more than the four basic arithmetic operations and basic properties of numbers to understand. Some of these puzzles can be solved by reasoning alone, others require some programming to find answers or conjectures that then need to be proved properly. Often, it is not immediately obvious from the statement of the problems whether there is an easy solution or whether it is a hard problem to solve. Here is an example, which is listed as Problem F24 in a compilation of unsolved problems edited by Richard Guy [1]:

> Which integers have squares that contain at most two different digits?

An easy answer is that there are infinitely many such integers, namely 10^k, 2×10^k, and 3×10^k for any $k>0$. The more interesting question is whether there exist any other squares with at most two different digits that do not end in a zero. The first few examples of such integers are easy to find – the integers from one to nine that result in two-digit squares. A computer search turns up a total of 24 such squares, with the two largest being

$$3114^2 = 9696996 \quad \text{and} \quad 81619^2 = 6661661161$$

(see `https://oeis.org/A018884`). A very convincing probabilistic argument, which is not a proof, seems to indicate that there are no others. The enterprising reader can investigate similar questions regarding squares that have exactly three different digits, and so on. Many more squares problems can be found at [2].

2 Squares With Large Digit Average

The question we will investigate is a somewhat related property of squares, namely the average of their digits, which we will refer to as the *digit average* and denote by $DA(n)$. As we have seen from the introductory problem, some squares have very low digit average, for example the sequence of infinitely many squares that have only two digits, many of them zeros. In fact, the digit average of $(10^\ell)^2$ equals $\dfrac{1}{2\ell+1}$. Thus, we can make the digit average arbitrarily low.

What about the other extreme? Can we find arbitrarily large integers whose squares have high digit averages? The maximal digit average of any integer can

be at most nine, namely for an integer that consists of all nines, such as 999. The only such square is 9, because the k-digit number 99.....99 for $k \geq 2$ is congruent to 3 mod 4 (since it is one less than a multiple of 100), and this remainder cannot occur for a square. So, how close to the ideal can we get as the number of digits in the integer we square gets larger and larger? And which integers would produce the squares with large digit average?

Claim 1. For $k \geq 1$, we have that

$$(2\underbrace{9\ldots9}_{k}8\underbrace{3\ldots3}_{k})^2 = 8\underbrace{9\ldots9}_{k-1}8\underbrace{9\ldots9}_{k}82\underbrace{7\ldots7}_{k-1}\underbrace{8\ldots8}_{k}9 \tag{1}$$

$$(2\underbrace{9\ldots9}_{k}8\underbrace{3\ldots3}_{k}27)^2 = 8\underbrace{9\ldots9}_{k-1}8\underbrace{9\ldots9}_{k+1}89\underbrace{7\ldots7}_{k-1}9\underbrace{8\ldots8}_{k}929 \tag{2}$$

with digit averages of $8.25 - \dfrac{5.75}{4k+3}$ and $8.25 - \dfrac{3.5}{4k+6}$, respectively.

While it would be sufficient to list only one of these two sequences to answer our question, we present both of these sequences here. The first one has a simpler structure (and usually, simpler answers are preferred), while the second one has a slightly faster convergence to 8.25. For a given $\varepsilon > 0$, we need $k > \dfrac{23}{16\varepsilon} - \frac{3}{4}$ for the first sequence to have a digit average of $8.25 - \varepsilon$, while we only need $k > \dfrac{14}{16\varepsilon} - \frac{3}{2}$ for the second sequence. We now prove our claim for the first sequence.

Proof. The first task is to write the integer in a way that is friendly for squaring, so we define the notion of the *position* of a digit, which is the power of ten it corresponds to. Thus, the rightmost digit has position zero, and then the positions increase from right to left. An integer consisting of k repetitions of the digit d that end at position a and are followed by zeros has the value

$$\underbrace{d\ldots d}_{k}\underbrace{0\ldots0}_{a} = d\cdot 10^a(\underbrace{1\ldots1}_{k}) = d\cdot 10^a(10^k-1)/9. \tag{3}$$

For example, the sequence of four 5s in 34555589 has value $\dfrac{5}{9}\cdot 10^2(10^4-1)$. We will also use that

$$(90x^2-5x-1)^2 = 8100x^4 - 900x^3 - 155x^2 + 10x + 1. \tag{4}$$

Now we are ready to prove the expression for the square in equation (1). We show the identities that were used in the derivation underneath the equal signs. Any

other identities result from rearrangements and basic algebra.

$$\begin{aligned}
&(2\underbrace{9\ldots9}_{k}8\underbrace{3\ldots3}_{k})^2 \\
&\quad\underset{(3)}{=} \left(2\cdot10^{2k+1}+10^{k+1}(10^k-1)+8\cdot10^k+\tfrac{1}{3}(10^k-1)\right)^2 \\
&\quad= \left(30\cdot10^{2k}-\tfrac{5}{3}\cdot10^k-\tfrac{1}{3}\right)^2 = \left(\frac{1}{3}\left[90\cdot(10^k)^2-5\cdot10^k-1\right]\right)^2 \\
&\quad\underset{(4)}{=} \frac{1}{9}\left[8100\cdot10^{4k}-900\cdot10^{3k}-155\cdot10^{2k}+10^{k+1}+1\right] \\
&\quad= 900\cdot10^{4k}-100\cdot10^{3k}-\tfrac{155}{9}\cdot10^{2k}+\tfrac{10}{9}\cdot10^k+\tfrac{1}{9} \\
&\quad= \underbrace{(800+100)}_{900}\cdot10^{4k}+\underbrace{(800+100-1000)}_{-100}\cdot10^{3k}+\underbrace{\left(\tfrac{7}{9}+82-100\right)}_{-155/9}\cdot10^{2k} \\
&\qquad\quad+\underbrace{\left(-\tfrac{70}{9}+\tfrac{80}{9}\right)}_{10/9}\cdot10^k+\underbrace{\left(9-\tfrac{80}{9}\right)}_{1/9} \\
&\quad= 8\cdot10^{4k+2}+10^{3k+3}(10^{k-1}-1)+8\cdot10^{3k+2}+10^{2k+2}(10^k-1) \\
&\qquad\qquad+82\cdot10^{2k}+7\cdot10^{k+1}(10^{k-1}-1)/9+8\cdot10(10^k-1)/9+9 \\
&\quad\underset{(3)}{=} 8\underbrace{9\ldots9}_{k-1}8\underbrace{9\ldots9}_{k}82\underbrace{7\ldots7}_{k-1}\underbrace{8\ldots8}_{k}9.
\end{aligned}$$

To compute the digit average, we simply multiply each digit that occurs by its frequency and divide by the total number of digits to obtain

$$\begin{aligned}
DA(2\underbrace{9\ldots9}_{k}8\underbrace{3\ldots3}_{k}{}^2) &= \frac{2\cdot1+7(k-1)+8(k+3)+18k}{4k+3} = \frac{33k+19}{4k+3} \\
&= \frac{8.25(4k+3)-5.75}{4k+3} = 8.25-\frac{5.75}{4k+3}.
\end{aligned}$$

The proof for the second sequence follows using the same steps: use the equation $(9000x^2-500x-19)^2 = 81\cdot10^6x^4-9\cdot10^6x^3-92000x^2+19000x+361$ instead of (4) and work from both sides of the equation. □

This result tells us that we can achieve a digit average as close as we want to 8.25 (from below), so the question now becomes whether we can do better than this value. We will answer this question in the next section.

3 Can we do better?

Having looked at countless squares with reasonably high digit average, we are convinced that there is no other sequence that generates infinitely many squares with digit average above 8.25. Empirical evidence (which is not a proof) from computer programming output strongly suggests that any squares with digit average above 8.25 are bound to be "sporadic", that is, they are not numerous and do not have a particular structure. Below we present an argument why we are strongly convinced there are nevertheless infinitely many of them with digit average above 8.25 and below a threshold value t_0 that is around 8.3.

First, let's assume that M is a very large finite set and A and B are subsets of M, with magnitudes $|M| = m$, $|A| = a$, and $|B| = b$. Now suppose that the values of a and b can be determined exactly using a counting argument, while the value of $c = |A \cap B|$ cannot be determined explicitly, and that the size of M makes a brute force enumeration using a computer program infeasible. We know that $\max\{a + b - m, 0\} \le c \le \min\{a, b\}$. If this range of possible values for c is large, then knowing this interval does not provide much information at all. However, if properties A and B are "independent" in a probabilistic sense, that is, knowing that an integer n belongs to A provides no (or very little) information about whether n belongs to B, and vice-versa, then probability p_{AB} for an element to be in $A \cap B$ is simply the product of the two probabilities p_A and p_B. Assuming strong independence between A and B gives that the expected number of elements in $A \cap B$ is given by

$$c \approx m \cdot p_{AB} = m \cdot p_A \cdot p_B = m \cdot \frac{a}{m} \cdot \frac{b}{m} = \frac{ab}{m}.$$

Now, given an integer n, let M be the set of integers with n digits (so $m = 9 \cdot 10^{n-1}$), B be the set of perfect squares in M, and A_t be the set of integers in M for which $DA(n) \ge t$. The number of elements in B is easily calculated as

$$b = \lfloor 10^{n/2} \rfloor - \lfloor 10^{(n-1)/2} \rfloor \approx (\sqrt{10} - 1)10^{(n-1)/2}.$$

Note that the difference between the actual value and the approximation quickly tends to zero as n increases, so we will use equality instead of $\approx$ in any equations relating to the value of c. Substituting the values of b and m into the equation for c and simplifying leads to

$$c = c(t, n) = \frac{(\sqrt{10} - 1)}{9} 10^{-(n-1)/2} \cdot a(t, n),$$

where $a(t, n) = |A_t|$. While there is no explicit formula for $a(t, n)$, it can be determined recursively using the following algorithm.

Let $g(s, n)$ count the integers with n or fewer digits whose digit sum is s, where we write every such integer with exactly n digits by adding extra zeros if needed.

For example, if $n = 6$, 13 is written as 000013. To evaluate $g(s, n)$, we use the following recursion that is obtained from conditioning on the last digit:

$$g(s, 1) = 1 \text{ for } s = 0, 1, \ldots, 9, \text{ and } g(s, 1) = 0 \text{ otherwise;}$$
$$g(s, k) = \sum_{r=0}^{9} g(s - r, k - 1) \text{ for } 2 \leq k \leq n.$$

Once $g(s, n)$ is calculated, the number $h(s, n)$ of integers with n digits or fewer whose digit sum is at least s is simply given by

$$h(s, n) = \sum_{r=s}^{9n} g(r, n)$$

because the largest such number consists of n nines, making the digit sum $9n$. The number of integers $e(s, n)$ that have exactly n digits and whose digit sum is at least s is then computed as

$$e(s, n) = h(s, n) - h(s, n - 1).$$

Because our goal is to have the digit average to be larger than t, we need $s/n \geq t$ or equivalently, $s \geq nt$. Thus,

$$a(t, n) = e(\lceil nt \rceil, n).$$

These functions can now be programmed, for example in Python, to investigate the behavior of $c(t, n)$ for different values of t and n. One notices quickly that for fixed n, the values of $c(t, n)$ exhibit a behavior typical of percolation. Specifically, there exists a threshold value $t_0(n)$ at which the behavior of $c(t, n)$ changes. For values of t slightly smaller than $t_0(n)$, the values of $c(t, n)$ are quite large, and if t exceeds $t_0(n)$, then $c(t, n)$ becomes very small. Remarkably, this threshold $t_0(n)$ is very stable when increasing the value of n (= number of digits), showing that there is a threshold t_0 that is independent of n for large enough values of n.

So how can we pin down the value of t_0? Recall that $c(t, n)$ gives the expected number of squares that have a digit average that exceeds t when the square has n digits. For the threshold $t_0(n)$ to exist, we need to have at least one such square, and the threshold would occur exactly when we go from having many squares to not having any such square. That is, at the threshold, we would have $c(t_0, n) = 1$, or, equivalently, $\log(c(t_0, n)) = 0$. Table 1 shows the values of $\log(c(t_0, n))$ for t-values near the observed threshold for $n = 500$, $n = 5000$, and $n = 10000$. The values of t in the table are incremented in steps of either 1/500, 1/5000, or 1/10000, respectively, because these are the increments for the corresponding digit averages. By linear interpolation of the two logarithm values closest to zero we obtain observed thresholds of 8.2981, 8.29986, and 8.30006, which shows that the threshold does not depend on n very much. This computational evidence leads to the question of the limit of the threshold values as n grows without bound. How close is it to 8.3? This is a rather hard question to answer since we only

t	$\log(c(t,500))$	t	$\log(c(t,5000))$	t	$\log(c(t,10000))$
8.292	2.681	8.2992	2.9263	8.2997	3.2297
8.294	1.801	8.2994	2.04085	8.2998	2.3438
8.296	0.920	8.2996	1.15524	8.2999	1.4579
8.298	0.037	8.2998	0.26946	8.3000	0.57186
8.300	-0.847	8.3000	-0.61649	8.3001	-0.3143
8.302	-1.733	8.3002	-1.50261	8.3002	-1.2005
8.304	-2.621	8.3004	-2.38890	8.3003	-2.0867

Table 1: Table of values of $\log(c(t,500))$, $\log(c(t,5000))$ and $\log(c(t,10000))$ for t-values near observed thresholds.

can compute the values of $c(t,n)$ for finite values of n and the formulas used to compute $a(t,n)$ are recursive.

An easier question is to find squares whose digit average is 8.25 or above, because we already know that we can get as close to 8.25 from below as we want with our two specialized sequences. To find such integers, our program selected promising integers and computed the digit average of their squares for integers with up to 20 digits, an otherwise prohibitively time consuming task. We found

$$707106074079263583^2 = 499998999999788997978888999589997889$$
$$941800402941090273 13^2 = 886987998979999999989898498 6998979999969,$$

with digit averages of 8.25 and 8.275, respectively. We challenge the readers to find additional squares with digit average of 8.25 and above, and maybe even break the current record. If you find such an integer, send it together with your name and location or affiliation to `sheubac@calstatela.edu` for posting on Silvia's website `http://www.calstatela.edu/faculty/silvia-heubach`. Happy hunting!

Both authors last saw Richard Guy at an MSRI workshop on Combinatorial Games in honor of Elwyn Berlekamp's 75th birthday. Richard, then 98, was still actively engaged in doing mathematics. We believe that if we had posed this problem to him, he would have started to think about how to tackle it. What an inspiration!

References

[1] R. K. Guy, *Unsolved Problems in Number Theory*, 2nd Edition, 1994, page 262. Springer-Verlag, New York

[2] Goodman, Len and Weisstein, Eric W., *Square Number*. From MathWorld–A Wolfram Web Resource. `https://mathworld.wolfram.com/SquareNumber.html`

An introduction of the problem of finding the chromatic number of the plane (I)

Veselin Jungić

1 Introduction

In their influential book *Unsolved Problems in Geometry*, Hallard Croft, Kenneth Falconer, and Richard Guy included the chromatic number of the plane problem:

Problem 1. What is the smallest number of sets ("colours") with which we can cover the plane in such a way that no two points of the same set are a unit distance apart? [1]

This smallest number is called the *chromatic number of the plane* and it is often denoted by χ. Croft, Falconer, and Guy described the problem as "fascinating" and claimed that it "combines the ideas from set theory, combinatorics, measure theory and distance geometry." As Alexander Soifer has established [9, 10], the problem was originally posted in 1950 by Edward Nelson, a graduate student at the University of Chicago at the time. Since then the problem, because of its simple and easily understandable statement, but also because of its evasiveness, has attracted the attention of some of the most prominent mathematicians as well as that of recreational mathematicians. Over the years, this widespread interest resulted in various generalizations of the problem. For a detailed account of the history of the problem, the people involved, and some of the generalizations of the problem, see [9, 10]

Regardless of all these efforts, until very recently (this will be discussed in a sequel to this article), the lower bound for χ, established by Nelson in the 1950s [10] remained unchanged.

As Croft, Falconer, and Guy observed in their presentation of [1], even though the problem is stated in the geometrical terms (plane, points, distance) it easily translates in a graph theory question. Consider the graph $G = (V, E)$ in which the set of vertices V is the set of the points in the plane. Two vertices are adjacent, i.e. connected by an edge $e \in E$, if and only if the distance between the vertices/points equals to 1. The question is to find the chromatic number of the graph G, i.e. the smallest number of colours sufficient for colouring the set V so that no two adjacent vertices are coloured by the same colour.

In 1951, Nicolaas de Brujin and Paul Erdős [3] proved that, if one assumes the axiom of choice, then the chromatic number of an infinite graph is equal to the maximum chromatic number of its finite subgraphs. This fact means that finding the chromatic number of the plane is "purely finite in character." [1]

In what follows, through a series of exercises and examples, we will try to provide to the reader a long–lasting taste of the problem of finding the chromatic number of the plane.

Exercise 1. Prove that $\chi \geq 3$, in other words prove that if the points in the plane are each coloured by one of two given colours then there must exist two points that are at a unit distance and coloured by the same colour.

2 The Moser spindle

In the May 1961 issue of the *Canadian Math Bulletin*, Leo and William Moser posted their solution to the following problem:

Problem 2.

(a) Prove that every set of six points in the plane can be coloured in three colours in such a way that no two points a unit distance apart have the same colour.

(b) Show that in (a) six cannot be replaced by seven. [6]

The problem itself was proposed by the Moser brothers probably around the late 1950s. Both Leo and William Moser were prominent Canadian mathematicians. [7, 8] We know that Richard Guy and Leo Moser knew each other and shared their passion for problem solving. For example, in his celebrated article "Strong Law of Small Numbers" from 1988, Guy writes:

> This example [#8], as well as example 5., was first shown to me by Leo Moser, a quarter of a century ago. [4]

This means that it is almost certain that Richard Guy was aware of at least a version of the problem of finding the chromatic number of the plane in the early 1960s. The reader has probably already realized that the solution to Problem 2 establishes that $\chi \geq 4$.

Next, we follow Leo and William Moser's solution [6] to Problem 2. Let two points which are a unit distance apart be called *friends*, otherwise they are *strangers*. If a finite set of points can be coloured by k colours so that no pair of friends have the same colour we say that this set permits a proper k-colouring. In the rest of this note we assume that any set of four or more points is a subset of a plane.

Exercise 2. Show that four points in the plane cannot be friends to each other and that two points cannot have three common friends.

Exercise 3. Show that any set of four points permits a proper 3-colouring.

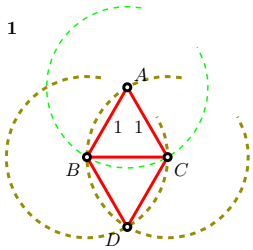
Exercise 4. Show that any set of five points permits a proper 3-colouring.

Exercise 5. Show that any set of six points permits a proper 3-colouring.

Example 1. Below we demonstrate a construction of a unit distance graph on seven vertices known as the Moser spindle.

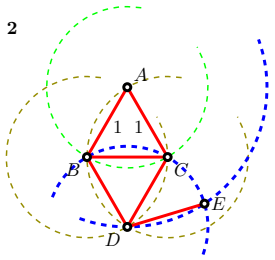
Start by choosing a point A in the plane and then draw a circle with the centre at A and radius 1. Denote this circle by c_1. Choose a point B on the circle c_1. Draw the line segment $\overline{AB}$.

Step 1



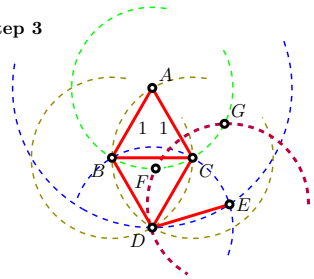
Next, draw a circle with the centre at B and radius 1. Denote this circle by c_2. Let C be an intersection point of c_1 and c_2. Draw a circle, call it c_3, with the centre at C and radius 1. Observe that the point A belongs to both c_2 and c_3. Let D be the other intersection point of c_2 and c_3. Draw the line segments $\overline{AC}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$. Observe that all those line segments are of length 1.

Step 2



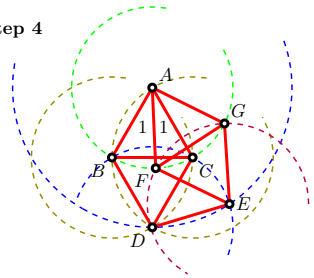
Draw a circle, call it c_4, with the centre at D and radius 1. Draw a circle with the centre at A and passing through the point D. Denote this circle by c_5. Next, choose a point E in the intersection of c_4 and c_5. Draw the line segment $\overline{DE}$. Observe that $|\overline{DE}| = 1$.

Step 3



Draw a circle with the centre at E and radius 1. Denote this circle by c_6. Let F and G be the intersection point of c_1 and c_6.

Step 4

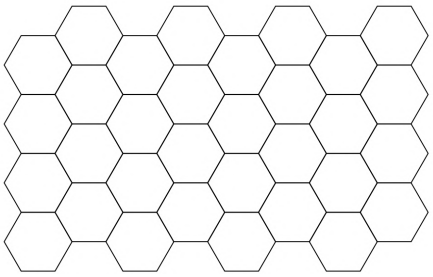


Draw the line segments $\overline{AF}$, $\overline{AG}$, $\overline{EF}$, and $\overline{EG}$. Observe that all those line segments are of length 1.
The Moser spindle appears!

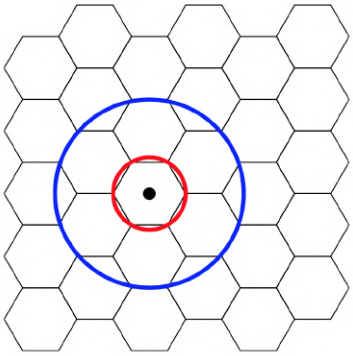
Exercise 6. Use the Moser spindle to prove that $\chi \geq 4$.

3 What about the upper bound?

Example 2. It is possible to cover the plane with congruent regular hexagons:



Exercise 7. Suppose that each regular hexagon in Example 2 is with the side length equal to $\frac{2}{5}$. Choose one of the hexagons and call it h_1. Let c_1 be the circumscribed circle of the hexagon h_1. Let c_2 be the smallest circle that contains h_1 and all six hexagons that share a side with h_1. What is the diameter of c_1? What is the diameter of c_2?



Exercise 8. Use Example 2 and Exercise 7 to show that $\chi \leq 7$.

Therefore, from Exercise 6 and Exercise 8 it follows that:

Theorem 3. $4 \leq \chi \leq 7$.

It should be mentioned that, in addition to Exercise 6 and Exercise 8, there are other ways to establish the bounds presented in Theorem 3. See, for example, [9, 10].

4 Remarks

As often happens in mathematics, an object created to answer a particular question may be used later in a completely different mathematical environment. For example, if the Moser spindle is considered a graph on seven vertices then its planar embedding – i.e. a drawing of the graph so that no edge crosses any other

edge – requires that one edge is drawn longer than the others. This implies that the Moser spindle is not a so-called matchstick graph! For more details about this part of the life of the Moser spindle, see, for example, [11] .

Going in a different direction, we have found the Moser spindle prove to be an interesting geometrical object on its own. We hope that the reader would be curious to search for the measures of the various angles contained in the Moser spindle, as well as the non-unit distances between the vertices. As we will see in the sequel, some of these geometric properties played important roles in the proof that $\chi \geq 5$.

But the main reason for this note is to serve as another, perhaps modest, tribute to the many contributions to mathematics made by Richard Guy and his collaborators and friends, including Leo Moser. Richard's entire professional life was about collaboration and sharing everything he knew about mathematics, as well as his love for it, with anyone who would listen: from pioneers who marked the development of mathematics in the 20th century, like Paul Erdős and John Conway, to champions of the promotion of mathematics to the general public, such as Martin Gardner, to adults whose very first positive experience with mathematics was a conversation with Richard, as the author of this note witnessed firsthand.

Richard also left us with a warning that we must be careful when making definite conclusions about mathematical phenomena based on limited data, regardless of its quantity or if it was computer-generated. Here is Richard's Strong Law of Small Numbers as a reminder that one should not rush to generalize patterns that appear among "small numbers":

> There aren't enough small numbers to meet the many demands made of them. [4]

5 Hints and solutions

Exercise 1. Consider an equilateral triangle.

Exercise 2. Observe that the existence of four points that are friends with each other would imply the existence of a triangle inscribed in the unit circle with all its sides equal to 1. A mutual friend of two friends, say A and B, must belong to the intersection of circles with their centres at A and B and radii equal 1.

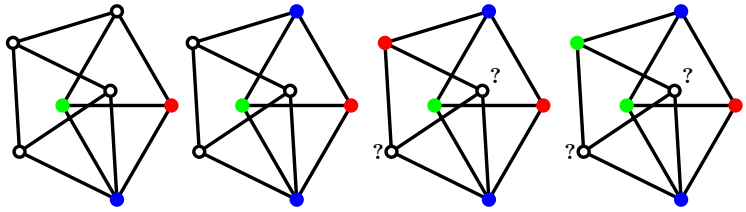
Exercise 3. Say that A and B are friends. Use two colours to colour those two points. Only one of C and D can be a mutual friend to A and B, say C. Use the third colour to colour C. How should we colour the point D?

Exercise 4. Observe that from the second part of Exercise 2 it follows that not all of the points A, B, C, D, and E can have exactly three friends each. Suppose that the point A does not have three friends. Say that A has two friends, B and C. Use Exercise 3 to properly 3-colour points B, C, D, and E. How should we

colour the point A? If A has four friends then those four points belong to a circle with the centre at A and we can colour them properly with two colours.

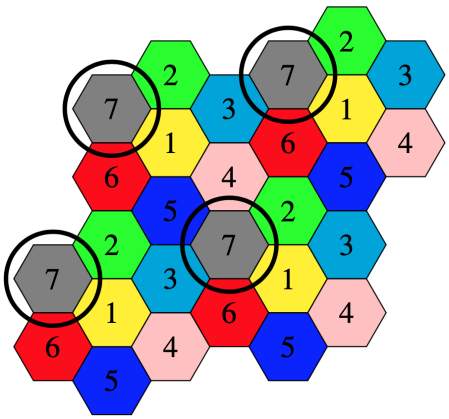
Exercise 5. Let A, B, C, D, E, and F be six points in the plane. If there is a point with two or four or five friends then we can use a similar argument as in Exercise 4. Suppose that all points have exactly three friends. Say that A is a friend with B, C, and D. Observe that A, E, and F have a common friend, say B. If E and F are strangers then there is a proper 2-colouring of the six points. If E and F are friends, consider the following two cases: (1) B is the only common friend for E and F; (2) E and F have another common friend.

Exercise 6. Use three colours and try to avoid colouring two friends with the same colour.



Exercise 7. $\frac{4}{5}$ and $\frac{2\sqrt{7}}{5}$.

Exercise 8. In 1961, Hugo Hadwiger proposed [5] the following 7-colouring of a tessellation of the plane by regular hexagons, with diameter d, $\frac{4}{5} < d < 1$. Observe that each hexagon is surrounded by hexagons of a different colour.



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De l'utilité d'une vieille curiosité grecque

Claude Goutier

1 Introduction

On doit principalement aux Grecs de l'Antiquité, les fondements de la géométrie et de la théorie des nombres. Cette dernière contient des problèmes simples et faciles à comprendre, mais dont la solution peut être très élaborée ou encore avoir échapé à la sagacité de très nombreux mathématiciens depuis des siècles.

Euclide a développé une géométrie qui porte son nom, démontré l'infinitude des nombres premiers et donné une formule décrivant les nombres parfaits pairs.

Dans l'édition du *Scientific American* de mars 1968, Martin Gardner écrit:

> "One would be hard put to find a set of whole numbers with a more fascinating history and more elegant properties, surrounded by greater depths of mystery — and more totally useless — than the perfect numbers ..."

Traduction libre: On serait bien en peine de trouver un ensemble de nombres entiers avec une histoire plus fascinante et des propriétés plus élégantes, mais tout aussi inutiles, que les nombres parfaits ...

Dans cet article, nous allons examiner sous la forme d'un parcours historique, la relation inattendue qui existe entre les nombres parfaits et les formes universelles qui produisent des nombres de Carmichael, et trouver une utilité à ces nombres parfaits.

2 Les nombres parfaits

Un nombre est dit parfait [1, B1] s'il est égal à la somme de ses diviseurs stricts (c'est-à-dire autre que lui-même). Par exemple, les diviseurs stricts de 6 sont 1, 2 et 3. Puisque $6 = 1 + 2 + 3$, c'est un nombre parfait.

Au troisième siècle avant notre ère, Euclide nous a donné une formule suffisante pour un nombre parfait pair: $(2^p - 1)2^{p-1}$ lorsque $2^p - 1$ est un nombre premier. Au dix-huitième siècle, Euler a démontré que tous les nombres parfaits pairs sont nécessairement de cette forme.

Le nombre p doit lui aussi être premier car sinon

$$2^{(ab)} - 1 = (2^a - 1)(1 + (2^a) + (2^a)^2 + ... + (2^a)^{b-1})$$

est un nombre composé.

On peut se poser quelques questions à propos de ces nombres parfaits. Sont-ils en nombre infini? Existe-t-il des nombres parfaits impairs? Sont-ils utiles?

On ignore la réponse aux deux premières questions. Quant à la troisième, c'est justement la raison de cet article.

Dans le tableau qui suit, G est un nombre parfait, n son rang, p et m_p des paramètres dans la formule d'Euclide, d le nombre de diviseurs stricts suivi de ceux-ci.

n	p	m_p	G	d	diviseurs stricts
1	2	3	6	3	1 2 3
2	3	7	28	5	1 2 4 7 14
3	5	31	496	9	1 2 4 8 16 31 62 124 248
4	7	127	8128	13	1 2 4 8 16 32 64 127 254 508 1016 2032 4064
5	13	8191	33550336	25	1 2 4 8 16 32 64 ... 4193792 8387584 16775168
...					
51	82589933	—	—	165179865	La marge est trop étroite pour tous les contenir

2.1 Les nombres de Mersenne

Le nombre $2^p - 1$ qui apparaît dans la formule d'Euclide est appelé nombre de Mersenne [1, A3]. On n'en connaît très peu, le cinquante et unième a été trouvé le 7 décembre 2018. Étant donné leur taille, la plupart ont été découverts en utilisant des ordinateurs, surtout grâce au "Great Internet Mersenne Prime Search" [2], un projet de recherche distribuée. Le programme de recherche, très optimisé, est aussi utilisé pour tester de manière intensive le processeur central des ordinateurs. La forme très particulière des nombres de Mersenne est utile dans certains programmes de génération de nombres aléatoires. C'est ainsi que l'on retrouve le "Mersenne Twister" utilisé dans certains jeux sur des consoles vidéo.

Les nombres de Mersenne fournissent aussi des exemples de très grands nombres premiers. En effet, lorsque p vaut 82589933, le nombre de chiffres de $2^p - 1$ est de $p \cdot log_{10}2 + 1$, soit $82589933 \times 0.30103 + 1 = 24862048$ chiffres. On voit que les nombres de Mersenne, contrairement à ceux de Fermat, ont une certaine utilité, mais attendons la suite.

3 Les nombres de Carmichael

Avant d'aborder les formes universelles, il faut examiner quelques concepts supplémentaires étudiés au fil du temps.

3.1 1640: Le petit théorème de Fermat

Fermat avance que pour tout entier a, le nombre $a^n - a$ est un multiple de n lorsque n est un nombre premier.

Cette propriété ne permet malheureusement pas de déterminer avec certitude si un nombre est premier. En effet, il existe des nombres dits pseudo-premiers pour lesquels la propriété est vérifiée pour plusieurs entiers a. Par exemple: 341 qui n'est pas premier ($341 = 11 \times 31$) satisfait la condition du petit théorème de Fermat puisqu'il divise $2^{341} - 2$.

Il existe surtout des nombres composés, les nombres de Carmichael, qui satisfont la propriété pour toutes les valeurs de l'entier a. Qui plus est, il a été démontré qu'il en existe une infinité [3].

Ces nombres de Carmichael [1, A13] peuvent être caractérisés d'une autre manière.

Exercise 3.1.1. Quelles sont les valeurs de a qui satisfont toujours la relation et qui, pour cette raison, ne sont d'aucune utilité pour distinguer les nombres composés des nombres premiers?

3.2 1899: Le critère de Korselt

Le critère de Korselt est une autre façon de caractériser les nombres de Carmichael.

$$N = p_1 \dots p_d \text{ avec } 2 < p_1 < p_2 < \dots < p_d \text{ et } d \geq 3, \quad (p_i - 1) \quad \text{divise} \quad (N - 1).$$

Autrement dit, un nombre de Carmichael est composé d'au moins 3 facteurs premiers impairs distincts. Chacun de ces facteurs doit aussi satisfaire une condition de divisibilité.

Nous allons remplacer le nombre 1 qui apparaît dans la formule précédente, par un paramètre δ. En faisant cela, nous traiterons simultanément les nombres de Carmichael et les nombres de Lucas-Carmichael [1, A13].

Sans nous attarder, disons simplement que les nombres de Lucas-Carmichael sont définis par le critère de Korselt lorsque δ vaut -1.

On montre facilement qu'un nombre de Carmichael peut s'écrire de la façon suivante: $N = p_i(1 + k_i(p_i - 1))$. Mais en utilisant δ, on obtient les deux formules:

$$N = p_i(1 + k_i(p_i - \delta)), \quad (N - \delta) = (p_i - \delta)(1 + k_i p_i).$$

On voit alors très clairement le rôle joué par δ et le caractère commun des nombres de Carmichael et de Lucas-Carmichael. Ceci illustre aussi le critère de Korselt, en explicitant le produit de facteurs premiers et la condition de divisibilité.

Exercise 3.2.1. Démontrez les deux formules précédentes et réalisez qu'il s'agit d'une seule et même équation présentée de deux façons différentes.

3.3 1939: Les formes universelles

Jack Chernick a étudié ce qu'il appelle des formes universelles donnant des nombres de Carmichael dès que chaque facteur est premier. L'avantage de ces formes, une fois établies, est qu'il suffit de vérifier que tous les facteurs sont premiers.

La plus simple et la plus connue est:

$$(6m+1)(12m+1)(18m+1),$$

m étant un entier positif non nul. En posant $m=1$ dans la forme précédente, on obtient: (6+1)(12+1)(18+1). On peut vérifier que les facteurs sont premiers et que 6, 12 et 18 divisent bien 1728 égal à $2^6 3^3$. Rappelons que tous les facteurs doivent être premiers. Si $m=2$, le deuxième facteur $(24+1)$ n'est pas premier. Chernick donne d'autres formes universelles, par exemple: $(10m+7)(20m+13)(50m+31)$.

Exercise 3.3.1. Démontrez que cette dernière forme est universelle et trouvez la première valeur de m satisfaisante.

4 Une forme universelle parfaite

Nous allons maintenant construire une forme universelle parfaite:

$$N = (g_1 Gm+\delta)\cdots(g_d Gm+\delta)$$

en nous assurant qu'elle satisfasse le critère de Korselt.

On doit avoir au moins trois facteurs, donc $d \geq 3$. Chaque facteur doit être distinct des autres. Les g_i seront distincts et G sera le plus grand commun diviseur des valeurs $g_i G$. Le paramètre m est un paramètre libre permettant d'obtenir plusieurs nombres distincts à partir d'une même forme. Le paramètre δ permet de traiter simultanément les nombres de Carmichael et de Lucas-Carmichael. Finalement, chaque facteur $(g_i Gm+\delta)$ doit être premier.

Pour satisfaire la condition de divisibilité, il faut que:

$$N-\delta = (g_1 Gm+\delta)\ldots(g_d Gm+\delta)-\delta$$

soit divisible par chacun des $(g_i Gm+\delta)-\delta$.

En développant le produit et en regroupant les termes, on obtient:

$$N-\delta = (\prod_{1\leq i\leq d} g_i)G^d m^d + \cdots + \delta^{d-2}(\ldots)G^2m^2 + \delta^{d-1}(g_1+\cdots+g_d)Gm + (\delta^d-\delta).$$

En considérant que la valeur absolue de δ vaut 1, le terme constant $\delta^d-\delta$ s'annule dès que d est impair. Cette condition est nécessaire lorsque δ vaut -1.

Il faut aussi que l'avant-dernier terme $\delta^{d-1}(g_1+\cdots+g_d)Gm$ soit divisible par chacun des $(g_i Gm)$. C'est a dire que g_i divise $(g_1+\cdots+g_d)$.

Nous supposons donc que G est un nombre parfait égal à la somme de ses diviseurs stricts g_i.

Chacun des $(g_i Gm)$ divise bien les termes précédents puisque les g_i sont des diviseurs de G.

Exercise 4.0.1. Quelle condition supplémentaire faut-il imposer à m lorsque la forme universelle parfaite est dérivée d'un nombre parfait impair?

4.1 Une propriété des nombres parfaits

Il nous reste à montrer que d, le nombre de facteurs stricts d'un nombre parfait, est impair.

Nous distinguons deux cas. Si le nombre parfait est pair, nous utilisons la formule d'Euclide $(2^p - 1)2^{p-1}$ et pour s'implifier l'écriture, posons $m = 2^p - 1$.

Énumérons tous les diviseurs en formant deux suites: la première est celle des puissances consécutives de deux: 1, 2^1, ..., 2^{p-1}, la deuxième sera le produit de m et de chacun des termes de la première suite, soit: m, $m2^1$, ..., $m2^{p-1}$. Le nombre total des diviseurs est pair puisqu'on a deux suites de même longueur. En omettant le nombre parfait lui-même, on obtient le nombre impair de diviseurs stricts, soit $2p - 1$.

Si le nombre parfait est impair, tous ses diviseurs stricts doivent être impairs et pour obtenir une somme impaire (c'est-à-dire le nombre parfait lui-même), le nombre de termes doit être impair.

Remarquons qu'en généralisant le critère de Korselt pour inclure les nombres de Lucas-Carmichael, nous avons mis en évidence cette propriété des nombres parfaits.

5 Résumé et quelques exemples

En résumé, nous avons utilisé les diviseurs stricts d'un nombre parfait pour construire une forme universelle parfaite.

Il faut montrer que la construction est valide et que l'on peut trouver des valeurs de m qui rendent tous les facteurs premiers.

Plutôt que de donner des exemples qui deviennent vite volumineux, voici un programme pour évaluer une forme dans le calculateur gp de pari/gp[4]:

```
parfait=[6,28,496];  stricts=[[1,2,3],[1,2,4,7,14],[1,2,4,8,16,31,62,124,248]];
S(d)={if (d == 1,"C",d == -1,"D","?")};  /* == operateur de comparaison */

forme (n, d, m) =  /* n = rang du nombre parfait, d = delta, m = variable */
                   /* ! indique un facteur non premier */
{
  G=parfait[n];  g=stricts[n];  nb=matsize(g)[2];
  N=1; for (i=1,nb, N=N*(g[i]*G*m+d));  /* produit des facteurs de la forme */
  printf ("%s %d = ", S(d), N);
  for (i=1,nb, printf(" %d%s", (g[i]*G*m+d), if (isprime (g[i]*G*m+d),"","!")));
  print ("       m = ", m);
}
```

Les valeurs satisfaisantes de m pour les trois premiers nombres parfaits sont données dans les tableaux qui suivent lorsque δ vaut respectivement 1 et -1:

n	6	28	496
1	1	2136	3494274080
2	6	2211	59586796500
3	35	4071	65403950535
4	45	5106	90497573145
5	51	5430	102867466065
6	55	9000	183907014270
7	56	10656	187988501295
8	100	17655	205676959950
9	121	18315	221913416340

n	6	28	496
1	1	219	10405375365
2	4	3354	54995354820
3	5	3870	75970448190
4	14	4884	128038936200
5	15	7824	141170974350
6	29	15525	204686064660
7	39	24825	207758902890
8	40	34914	288204274995
9	49	42669	294525612150

Pour le nombre parfait suivant, 8128 avec ses 13 diviseurs, la première valeur satisfaisante de m semble hors de portée. Finalement après 26 jours de calcul sur 98 processeurs, 216818853118725 est la première valeur satisfaisante de m.

À titre d'exemple dans gp, forme(1,1,1) et forme(2,-1,219) afficheront

```
C 1729 =  7 13 19    m = 1

D 6794971926001106598911 =  6131 12263 24527 42923 85847   m = 219
```

6 Ajout de facteurs supplémentaires

On peut ajouter à volonté λ facteurs supplémentaires à une forme universelle parfaite. Chacun étant de la forme: $s_j = (2^j G^2 m + \delta)$, $0 \leq j < \lambda$ et m doit être divisible par $2^{\lambda-1}$.

En partant de la forme universelle parfaite: $N = (1 \cdot 6m + \delta)(2 \cdot 6m + \delta)(3 \cdot 6m + \delta)$, on ajoute successivement: $(2^0 \cdot 36m + \delta)$ $(2^1 \cdot 36m + \delta)$ $(2^2 \cdot 36m + \delta)$ ...

Notons en passant la relation: $s_{j+1} = 2s_j - \delta$ et que dans le cas des nombres de Lucas-Carmichael ($\delta = -1$), λ doit être pair.

Exercise 6.0.1. Démontrez que les formes universelles parfaites étendues satisfont le critère de Korselt.

7 Nombres premiers de Sophie Germain

Dans le cadre de ses recherches sur le grand théorème de Fermat, Sophie Germain [1, B48] a étudié une paire de nombres premiers reliés de la façon suivante: $p_2 = 2p_1 + 1$. Le premier nombre, p_1, est appelé nombre premier de Sophie Germain. Le deuxième nombre, p_2, est appelé nombre premier sûr.

L'efficacité de certains algorithmes de factorisation repose sur la présence de nombreux petits facteurs de $N - 1$. Ces nombres premiers sûrs sont donc une entrave à ces algorithmes.

Pour les nombres de Lucas-Carmichael, les facteurs supplémentaires qui satisfont $s_{j+1} = 2s_j - \delta$ sont justement de la forme des nombres premiers de Sophie Germain, puisque δ vaut -1.

8 De nouvelles questions

Les quelques exemples précédents ont montré qu'il existe des formes universelles parfaites dont tous les facteurs sont premiers.

Étant donnée une suite d'expressions $(Hg_im + d)$, où H, le plus grand commun diviseur des g_i (relativement premiers entre eux) et d sont donnés, on peut se poser les questions suivantes à propos des variables:

Quelles sont les conditions sur H, g_i et d, pour qu'il y ait au moins une chance que les termes soient premiers? Existe-t-il au moins un m qui rende tous les termes premiers? Existe-t-il une infinité de valeurs de m satisfaisantes?

9 Conclusion

À partir des nombres parfaits que nous ont légués les Grecs de l'Antiquité, nous avons exploré quelques facettes de la théorie des nombres qui regorge de problèmes parfois très simples et de questions souvent sans réponse. Ce qui est d'ailleurs le sujet du livre *Unsolved problems in Number Theory* de Richard K. Guy plusieurs fois cité dans cet article.

Nous avons surtout trouvé une utilité aux nombres parfaits en construisant une forme universelle parfaite à partir de leurs diviseurs stricts.

Pour terminer, comme dans le *Timeo Danaos et dona ferentes* de Virgile, méfiez-vous des cadeaux des Grecs anciens, ils vous réservent encore bien des surprises.

10 Remerciements

Je tiens à remercier les éditeurs de la revue *Crux Mathematicorum* qui m'ont offert l'occasion d'écrire cet article et tout particulièrement mon amie Claudie Cardin qui a patiemment revisé plus d'une fois le texte et m'a fait part de ses très judicieuses remarques.

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Every positive integer is a sum of four squares

Michael Barr

1 Introduction

Ever since I took a course in number theory many decades ago, I have been fascinated by the theorem that every positive integer is the sum of four squares. The proof I learned was rather complicated. We present here a much more elegant argument that is due to the Fields Medal winning mathematician Alan Baker (1939–2018). It was originally proved by Joseph-Louis Lagrange (1736–1813), Italian/French mathematician. Actual birth name: Giuseppe Lodovico Lagrangia.

Here is an outline of the main steps of the argument.

Step 1. If a and b are each a sum of 4 squares so is their product ab. This reduces the argument to showing it for primes.

Step 2. If p is prime, then there is a $k < p$ for which kp is a sum of four squares.

Step 3. If $k > 1$, then there is an $l < k$ such that lp is a sum of four squares.

The first two steps are standard; Baker's argument applies to the third step.

2 Modular arithmetic

Modular arithmetic is crucial to the argument. It will be familiar to many of you. Nonetheless there are one or two things in this development that may be new to you.

Let m be a positive integer. We say of (not necessarily positive) integers x and y that $x \equiv y \pmod{m}$ when m divides $x - y$. For example $5 \equiv 16 \pmod{11}$, but also $-5 \equiv 17 \pmod{11}$ since 11 divides $-5 - 17$. Note that $x \equiv 0 \pmod{m}$ iff m divides x. When $x \equiv y \pmod{m}$ holds, we will say that x **is congruent to** y mod m.

Here are a couple of easily derived properties of modular arithmetic that we will be using, generally without comment.

1. $x \equiv x \pmod{m}$.
2. If $x \equiv y \pmod{m}$ then $y \equiv x \pmod{m}$.
3. If $x \equiv y \pmod{m}$ and $y \equiv z \pmod{m}$ then $x \equiv z \pmod{m}$.
4. If $x \equiv y \pmod{m}$ and $x' \equiv y' \pmod{m}$ then $x + x' \equiv y + y' \pmod{m}$ and $xx' \equiv yy' \pmod{m}$. In particular, $x \equiv y \pmod{m}$ implies $x^2 \equiv y^2 \pmod{m}$.

For any m, the numbers $0, 1, \ldots, m-1$ constitutes a **complete set of residues mod** m. This means that every number is congruent to one and only one number in that set. For a positive number x you simply divide by m and use the remainder. If $x < 0$, you can still define division with non-negative remainder.

But this is not the only possible complete set of residues. For example, mod 11, the numbers $0, 2, 4, 6, \ldots, 20$ are another one. For example, $7 \equiv 18 \pmod{11}$. For our purposes, the set $-5, -4, \ldots, 0, \ldots, 5$ is a complete set of residues mod 11. For any odd m, the set $-(m-1)/2, \ldots, 0, \ldots, (m-1)/2$ is called the absolutely least residues mod m. All the residues in that set have absolute value less than $m/2$.

3 Step 1, the product identity

This identity will be used twice, once to show S4S-1 and again in Step 3.

$$\begin{aligned}&(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2)\\ &= (ax+by+cz+dw)^2 + (ay-bx-cw+dz)^2\\ &+(az+bw-cx-dy)^2 + (aw-bz+cy-dx)^2\end{aligned} \tag{PI}$$

This is somewhat tediously proved by direct multiplication. For example the cross term $2axby$ coming from squaring the first term on line 2 cancels the term $2ay(-bx)$ coming from squaring the second term. Eventually, all the cross terms cancel and only terms like a^2x^2 remain.

If you know about quaternions (you do **not** have to know about quaternions to read this paper), then it may interest you to know that the particular patterns of plus and minus signs comes from the product

$$(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(x + y\mathbf{i} + z\mathbf{j} + w\mathbf{k})$$

But even knowing about quaternions doesn't prove the formula above. In fact the formula is what is needed to prove that quaternionic absolute value preserves products.

What this formula does do is reduce the question at hand to showing that every prime is the sum of four squares. Since $2 = 1^2 + 1^2 + 0^2 + 0^2$ is a sum of four squares we can confine ourselves to odd primes.

Exercise. The quaternions are a 4 dimensional associative, but non-commutative, algebra spanned by $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$, such that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

Derive the rest of the multiplications from these equations.

4 Proof of Step 2

We begin with an important property of prime numbers: *If a prime divides a product, then it divides one of the factors.* This is standard and can be found in any discussion of the greatest common divisor.

Let p be an odd prime.

As x ranges over the set of $(p+1)/2$ numbers $0, 1, \ldots, (p-1)/2$, I claim that no two of the elements of the set of all x^2 are congruent mod p. In fact, if $0 \leq x \leq (p-1)/2$, $0 \leq x' \leq (p-1)/2$, and $x^2 \equiv x'^2 \pmod{p}$, then p divides

$$x^2 - x'^2 = (x - x')(x + x').$$

But then p must divide either $x - x'$ or $x + x'$ and both the sum and difference are less than p and so one (or both) must be 0, which is possible only if $x = x'$. Thus those $(p+1)/2$ numbers are all distinct mod p. In a similar way, if y ranges over the set of numbers $0, 1, \ldots, (p-1)/2$, the $(p+1)/2$ numbers $-1 - y^2$ are all distinct. Since the two sets of numbers altogether total $p+1$ numbers at least one pair of them must be congruent. Since both sets are individually distinct mod p, there must be a pair x, y for which $x^2 \equiv -1 - y^2 \pmod{p}$ which is to say that $x^2 + y^2 + 1$ is a multiple of p, say kp. Since x and y are non-negative, k cannot be 0 and since x and y are less than $p/2$,

$$x^2 + y^2 + 1 \leq p^2/2,$$

so that $k < p$.

5 Proof of Step 3

Suppose now that $a^2 + b^2 + c^2 + d^2 = kp$ with $1 < k < p$. We first consider the case that k is an even number. In that case, either all four of the numbers a, b, c, d are even, or all are odd, or two of them are even and two odd. In the last case, assume that a and b are both even and c and d both odd. In all three cases, we compute

$$\begin{aligned}
&\left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2 + \left(\frac{c+d}{2}\right)^2 + \left(\frac{c-d}{2}\right)^2 \\
&= \frac{a^2 + 2ab + b^2 + a^2 - 2ab + b^2 + c^2 + 2cd + d^2 + c^2 - 2cd + c^2}{4} \\
&= \frac{2a^2 + 2b^2 + 2c^2 + 2d^2}{4} = \frac{k}{2}p
\end{aligned}$$

and so $\ell = k/2$ proves Step 3 in this case. Now suppose that $k > 1$ is odd. Let x be the absolutely least residue of $a \pmod{k}$. That is $a \equiv x \pmod{k}$ and $|x| < k/2$. Similarly, let y, z, w be the absolutely least residues of b, c, d, respectively mod k.

Since $x^2+y^2+z^2+w^2 \equiv a^2+b^2+c^2+d^2 \pmod{k}$ and the latter is kp, it follows that $x^2+y^2+z^2+w^2$ is also a multiple of k, say lk. Since all four terms are less than $k^2/4$, it follows that $l < k$. Clearly,

$$(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+w^2) = lk^2p$$

We will next show that every term used in the product identity, PI above, is divisible by k.

$$\begin{aligned}
ax+by+cz+dw &\equiv a^2+b^2+c^2+d^2 = kp \equiv 0 \pmod{k}\\
ay-bx-cw+dz &\equiv ab-ba-cd+dc = 0 \pmod{k}\\
az+bw-cx-dy &\equiv ac+bd-ca-db = 0 \pmod{k}\\
aw-bz+cy-dx &\equiv ad-bc+cb-da = 0 \pmod{k}
\end{aligned}$$

It now follows that

$$\begin{aligned}
&\left(\frac{ax+by+cz+dw}{k}\right)^2 + \left(\frac{ay-bx-cw+dz}{k}\right)^2\\
+&\left(\frac{az+bw-cx-dy}{k}\right)^2 + \left(\frac{aw-bz+cy-dx}{k}\right)^2\\
&= lp
\end{aligned}$$

It now follows that p is the sum of four squares.

6 Sums of two and three squares

What integers are sums of two squares? There is a product identity that is much easier than the one for four squares (exercise: find it) which basically reduces this question to primes. The oddest prime of all, namely 2, is a sum of two squares. It is an easy exercise to show that no prime $p \equiv 3 \pmod{4}$ is a sum of two squares. It is somewhat harder to show that every prime $p \equiv 1 \pmod{4}$ is a sum of two squares. From these observations, you can readily see the forward implication: *The number n is a sum of two squares iff every prime divisor $p \equiv 3 \pmod{4}$ divides n an even number of times.* Thus 162 is a sum of two squares (since 3 divides it four times) while 54 isn't.

The situation with three squares is not so simple. By inspection, we can see that 7 is the first number not the sum of three squares and the next one is $15 = 3 \times 5$, which shows that there can be no product identity for three squares. After 15 comes 23, which might cause you to conjecture that the only exceptions are $n \equiv 7 \pmod{8}$. Until you come to 28, another exception. The final answer was given by French mathematician Adrien-Marie Legendre (1752–1833). *The number n is a sum of three squares iff it is not of the form $4^e(8k+7)$.* You may enjoy showing the "only if" part.

Some memories of Richard Guy

Alex Fink

Richard Guy was not only a friend to me, but a mentor and a formative mathematical influence since I met him in the late 1990s. I write this to portray a few pieces of his attitude to mathematics and life, as I saw them.

Richard's long dedication to mentorship and young people in mathematics was remarkable. He was mentoring and writing joint publications with students until nearly the end – my erstwhile record, with him, for largest age gap between coauthors didn't have a chance of surviving, but his role in the record was secure! For just as long he pitched in to University of Calgary's outreach efforts, such as the marking effort for the Calgary Junior Math Contest and appearances at the weekly Math Nite for highschoolers (and younger: this was where I met him, around grade five).

As a researcher, Richard's dedication to following his own trails in his research, regardless of the popular opinion of the mathematical subareas in question, was refreshing. To Richard math *was* recreation, so an area being characterised as "recreational mathematics" was no dismissal like it is for so many. Richard was fond of telling how combinatorics as a whole lay in disrepute when he got his start in research, midcentury, but ascended to near-respectability by the turn of the century and made it by his last decade. I'm delighted to see his tastes vindicated.

Unsurprisingly given this, Richard's work was spread over many areas of mathematics. While I was working with him as an undergraduate he encouraged such breadth in me too. He would suggest I accompany him to conferences even somewhat outside my area: his explanation was that even if I didn't understand the talks I'd absorb some of the language and be better prepared for the next one. In retrospect I'm sure it was also a nudge about networking, given Richard's own immense mathematical networks.

Richard's mathematical explorations were driven by hands-on play with examples, and if a computer could be coaxed to produce the examples, all the better. When I paid a visit to Richard in his office in his nineties and beyond, he'd always have open a terminal window or four in which he had the number theory software PariGP computing so-called aliquot sequences. This also illustrates one of his tenets of research—always have multiple balls in the air. Not only does this give you somewhere to turn when stuck, but while you're focussing on one project, there will still be a little part of your mind at work on the others.

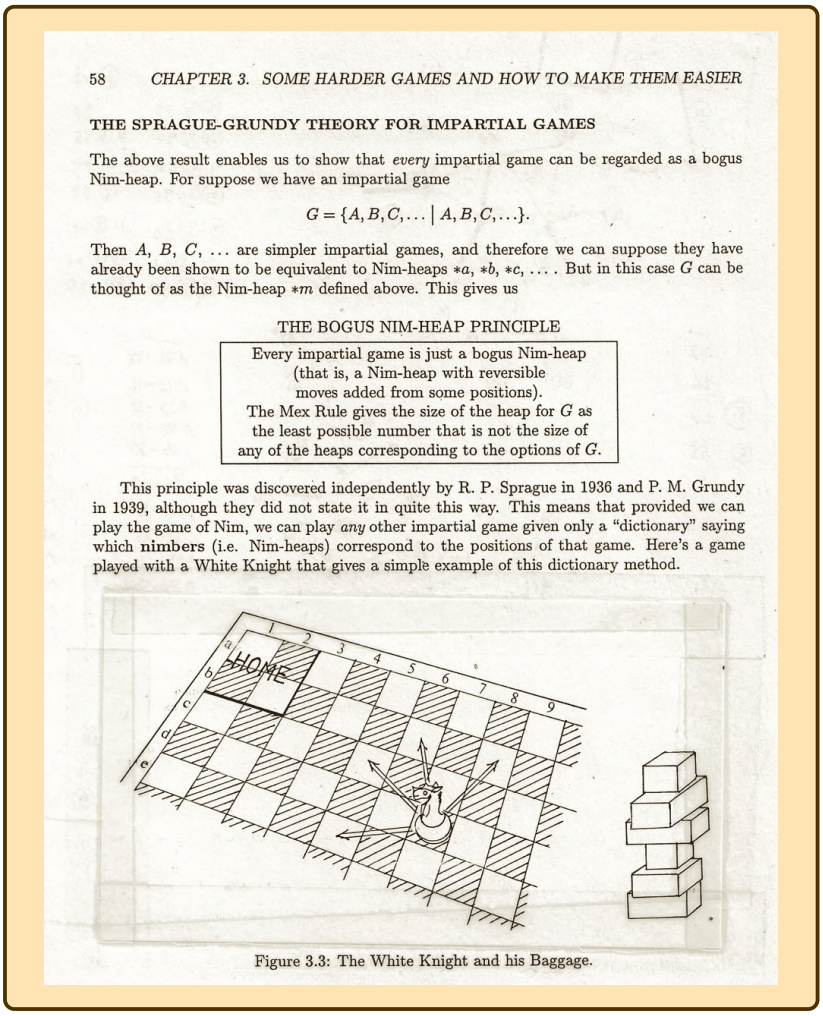
An aliquot sequence is obtained by iterating the function mapping a natural number n to the sum of all its proper divisors (i.e. excluding n itself). So the aliquot sequence beginning with 12 runs $12, 16, 15, 9, 4, 3, 1, 0$, and there it runs out. Dickson conjectured in 1913, refining Catalan (1888), that this behaviour was representative: every aliquot sequence is bounded, either reaching zero or becoming periodic [3]. But Richard thought this behaviour was only the small numbers

up to their usual tricks (cf. [5]); with John Selfridge in 1975 [7] he made the "counter-conjecture" that aliquot sequences starting at even numbers escape to infinity except at a set of asymptotic density zero. In amassing evidence for the counter-conjecture, rather than automating computations of sequences and letting the machine churn away on them, Richard would only give the command to compute one term at a time. He'd want to see every term and copy it to a text file, where he could take in the gently rolling shapes formed by the varying numbers of digits, and he'd mark it up with a factorisation to see the dance of the "drivers", powers of two times certain other small primes, which when they appeared as factors stuck around awhile and reliably guided the sequence up or down.

As this shows, once Richard had examples to hand they would need to be set down in the way best allowing visual extraction of insight. Whatever visualisation he settled on would be precisely laid out and lovingly drawn. Euclidean geometry is, of course, given to this. For triangle geometry experiments Richard kept a coordinatised drawing of a Generic Triangle on large graph paper, following the idea on which Clark Kimberling's Encyclopedia of Triangle Centers [8] is built: if it's true in one random enough example, it's always true (this is not just a heuristic but can be firmed up into a rigorous fact!) Richard, of course, picked as his generic triangle one that had the nicest Diophantine properties he could muster. This was a longstanding passion: his old posters of Euclidean geometry from his teacher days stayed atop his bookshelves, never far away. Here are a few more memorable examples from my research with him: the honeycombs on unrolled tori that were Cayley graphs associated to actions of Conway's "extraversion" of a triangle on associated objects – draw in, say, the angle bisectors (two lines per angle!) of a triangle, and observe how they are permuted when you continuously swap the places of two vertices [6]; and the maps in the style of Conway's topographs, trivalent plane graphs with numbers labelling the faces [2], from when we were playing with Markov's equation $x^2 + y^2 + z^2 = 3xyz$ in number fields.

The charming graphics in the original edition of *Winning Ways* [1] were drawn and lettered by Richard, and he made sure they were retained in the second edition. After the second edition's appearance I'd sometimes get papers from Richard whose back sides had been used in assembling the second edition, with the old graphics cut out and carefully taped into the spaces left for them. You see, after serving that purpose these pages had become One Sided Paper, not to be discarded but to be saved for use of its remaining side in the OSPital!

Richard was an early adopter of computers. Though the computer age was in full swing by the time I met him, there were still plenty of signs. The first I noticed was that his email address was in the Computer Science department. This was because his computing account at the U of C predated the fission of Computer Science and Math into separate departments, and naturally, when they did split, the former took the computers. Richard's standard tools for computer graphics were LaTeX packages `epic` and `eepic`, which freed the user from the harshest restrictions of the old `picture` environment (now you could have lines of arbitrary slope!) but were still distinctly closer to the metal than modern standards like `TikZ`. He was impressively fluent at the required coordinatisation.



58 CHAPTER 3. SOME HARDER GAMES AND HOW TO MAKE THEM EASIER

THE SPRAGUE-GRUNDY THEORY FOR IMPARTIAL GAMES

The above result enables us to show that *every* impartial game can be regarded as a bogus Nim-heap. For suppose we have an impartial game

$$G = \{A, B, C, \ldots \mid A, B, C, \ldots\}.$$

Then A, B, C, ... are simpler impartial games, and therefore we can suppose they have already been shown to be equivalent to Nim-heaps $*a$, $*b$, $*c$, ... But in this case G can be thought of as the Nim-heap $*m$ defined above. This gives us

THE BOGUS NIM-HEAP PRINCIPLE

Every impartial game is just a bogus Nim-heap
(that is, a Nim-heap with reversible
moves added from some positions).
The Mex Rule gives the size of the heap for G as
the least possible number that is not the size of
any of the heaps corresponding to the options of G.

This principle was discovered independently by R. P. Sprague in 1936 and P. M. Grundy in 1939, although they did not state it in quite this way. This means that provided we can play the game of Nim, we can play *any* other impartial game given only a "dictionary" saying which **nimbers** (i.e. Nim-heaps) correspond to the positions of that game. Here's a game played with a White Knight that gives a simple example of this dictionary method.

Figure 3.3: The White Knight and his Baggage.

A sample page of *Winning Ways*. One of the first moments in Richard's research career was his independent rediscovery and development of the Sprague-Grundy theory, noticing it applied more generally than had been seen before. Richard Nowakowski has written this moment up: `https://notes.math.ca/en/article/richard-guy-and-game-theory/`

A glimmer of the excitement Richard must have felt in the mechanical computer days came to the surface during one of his visits to my parents' home. My folks had Richard over for a number of dinners, especially during my visits back after I'd moved out, and his good appetite and fondness for just one more drink were reliable sources of cheer. Anyway, I had in my old bedroom some of my uncle Archie's mechanical calculators, including a Curta, an accumulator machine where the addend could be shifted in place value by turning the body, nicknamed the

"pepper grinder" for its cylindrical shape with crank on top. During one visit a group of us – Richard, me, my partner, my dad – fell to an online investigation of the Curta calculator's online mechanisms, and Richard was rapt throughout.

Richard was nimble with wordplay. He'd make sure that just the right name was assigned to a mathematical concept, or title to a paper. He shared this disposition with many in his circles, including the other authors of *Winning Ways*, and one of the standard anecdotes about the book is a perfect illustration. The three of them invented many combinatorial games for the book. If a game couldn't be given the right name right away, it went in the "games without names" file, to later be married to a name that, on its coining, was sent to the "names without games" file. Andrew Bremner shared the wordplay bug, and Richard and I took some hikes with him in the Rockies; on the drives to and from, typical entertainments were naming words that contained the letters on a licence plate as subsequences, or doing the Saturday NYT crossword without writing anything down. Both of them were stiff competition! It was fitting that at one of Richard's birthday celebrations at the U of C (his 90th?), one of the impromptu games the attendees fell to was presentation of spontaneous limericks celebrating math and Richard.

Richard, and his wife Louise, were both staunch pacifists and proponents of nuclear disarmament, and my favourite memory thereof fits here. While corresponding with Andrew about six-letter anagrams to illustrate my and his paper exploring the exceptional S_5 subgroup of S_6 [4], Richard was delighted to discover that the permutation taking ANDREW to END WAR lay in the subgroup, enough so that we used it to close the article.

Aspects of writing other than *le mot juste* were important to Richard too, and some of his most memorable advice to me was on the subject of writing. When at work on a project, his advice was to write liberally at every stage: "the easiest way to edit is to cut things out". He was also always mindful of the audience he was writing for, and was careful not to making assumptions of background knowledge that might alienate some of this audience. Hence he'd never call anything "well-known" without the follow-up "... to those who well know it".

Richard belonged to the Department of Mathematics and Statistics at the U of C since the beginning of its existence in an autonomous university, and by my day was certainly a fixture. Here's a fact that echoes the central place Richard had for me in the department: his personal library was indexed together with the department's general library in a single catalog, and if the book you sought was in Richard's collection, you could just knock on his personal door onto the library stacks. Richard was also famed in the university as a whole: my dad enjoys telling of being invited by Richard to a Chancellor's Club evening soirée, where Richard introduced him to both Robert Thirsk, Canadian astronaut and then Chancellor, as well as Elizabeth Cannon, then President. He moved easily amongst them all and was respected by all of them.

Richard's abiding love for Louise kept him steady and lifted his spirits all the time I knew him, before and after her passing in 2010. A window with Louise's photo

was the only constant presence on Richard's office computer, as terminals with computations and data files and papers were opened and closed. After 2010 he'd bring a framed version of her photo to accompany him on big outings, like the Calgary Tower Climb or the Alpine Club of Canada's opening of the Louise and Richard Guy Hut.

The story of the Hut and both Guy's passion for mountaineering and the ACC is well told by Chic Scott [9]. But Richard couldn't be contained by the mountains. The Calgary Tower Climb, up the 802 steps from ground level to the tower's observation deck, saw Richard's participation every year until, amazingly, 2019 (by which point it had become the Bow Building Climb). And he walked his 3km commute to the office well into his 90s, finishing of course by climbing the stairs to the fourth floor.

Richard never wanted to impose on anyone, in spite of his age. Getting around Calgary in his last years, after walking to the office became an impracticality, he was grateful for rides from many friends but stuck it out a long time taking transit. My parents, who held season tickets for the same nights as Richard, would give him lifts home from Vertigo Theatre in downtown Calgary, an offer made more palatable by having my grandfather to drop off too so they could say it was no extra burden for Richard to come along. Richard loved the theatre and maintained his subscription even through the 2019/20 season: he was clearly an optimist!

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Cyclic Subtraction Set Games

Silvia Heubach, Melissa A. Huggan,
Richard J. Nowakowski, Craig Tennenhouse

Problem

Richard K. Guy always pushed the boundaries of combinatorial games, particularly impartial games. He established combinatorial games as a subject with his invention of an infinite class of Taking-and-Breaking games. These are called *Octal* games since the rules are described by an octal code. With Elwyn Berlekamp and John Conway, he wrote *Winning Ways*, which established Partizan Combinatorial Game Theory as an independent subject. However, Richard returned to impartial games with a primer entitled *Fair Game*. With Alex Fink, he published a tantalizing problem called the 'The Number-Pad game' (and its solution). The game is played as follows: Initially, the score is 0. With a key-pad, as you would find on a phone or calculator, the first player presses a number and '+', after that the next player pushes a different button in the same row or column as the last number and then '+', and so on. In this version, the number-pad does not include 0. The first player to go over 30 loses. One can change this to subtracting from 30, wherein the first player to go below 0 loses.

The generalization of the Sprague-Grundy theory, the major tool to solve impartial combinatorial games, to cover this type of problem is not particularly nice, but in special cases, like the Number-Pad game, it does reduce to 'obvious' bookkeeping. We propose a similar impartial game that Richard would have liked.

The Cyclic Subtraction Game: The board has two components consisting of a heap, say of size n, and the repeating sequence $(1, 2, 3, 1, 2, 3, \ldots)$. In general, the next player takes away whatever number is at the beginning of the sequence from the heap and then deletes the first term of the sequence. The first player who cannot move loses.

For example, with heap size 8 the game would be

$$8, (1, 2, 3 \ldots) \to 7, (2, 3, 1, \ldots) \to 5, (3, 1, 2, \ldots) \to 2, (1, 2, 3 \ldots) \to 1, (2, 3, 1 \ldots)$$

whereupon the game stops because there is only one token in the heap but the next player must remove two tokens.

This is a boring game since the play is fixed once the sequence and heap size are given, so let's make it more exciting. We will use the convention of calling the players Richard and Louise (like Richard and his wife).

The Cyclic Subtraction Game Version 2: Louise is allowed to select the heap size or the starting number in the sequence. Richard then chooses the other. Louise then starts the game. Can Louise win? Can she win if the first player who cannot

move is the winner? To make the game non-trivial, the heap size must be at least six.

The Multi-heap Cyclic Subtraction Game: Suppose the original game is played with two (or more) heaps where a player gets to choose in which heap to play. Who wins? We do not have an answer. We leave that to any enterprising gamester.

Solution

The solution to Version 2 of the Cyclic Subtraction game requires the analysis of the original boring game. Impartial games have just two types of positions. A Next player winning position, i.e., one in which the next player has a winning move, is represented by $\mathcal{N}$. A position in which the player to move is bound to lose is represented by $\mathcal{P}$, short for Previous player win. We lay out the situation in a $3 \times n$ table, where the columns give the heap size and the first, second and third row, respectively, indicate that the first term in the sequence is 1, 2, or 3. Let (r, n) denote the position in row r and heap size n, respectively. In this game, a player has exactly one choice of position to move to. The heap size will be decreased by r, and the row value will change to the next term in the sequence. The rules to generate the table recursively are:

- The entry for $(1, n)$ is $\mathcal{P}$ if $(2, n-1)$ exists and is $\mathcal{N}$, else it is $\mathcal{N}$;
- The entry for $(2, n)$ is $\mathcal{P}$ if $(3, n-2)$ exists and is $\mathcal{N}$, else it is $\mathcal{N}$;
- The entry for $(3, n)$ is $\mathcal{P}$ if $(1, n-3)$ exists and is $\mathcal{N}$, else it is $\mathcal{N}$.

If the position referenced does not exist, that means there is no move, so the player loses.

Table 2: Outcomes for the Cyclic Subtraction Game

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$
2	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$
3	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{P}$	$\mathcal{P}$	$\mathcal{P}$

The columns for heaps of size 12, 13, and 14 are the same as those for sizes 0, 1, and 2. Since the rules only look back at most 3 columns, the columns now repeat forever. We need to only look at the first 12 columns and work modulo 12 for the heap size.

If Louise chooses a starting number, then Richard can always find a heap size where the entry is $\mathcal{P}$. Louise has to move and loses. For example, if she says 2 then Richard can give a heap size that is 0, 1, 5, 8, 9, or 10 modulo 12. However, if Louise chooses the heap size to be 6 and 11 modulo 12, then whatever Richard chooses as the start of the sequence, the position is a next player win and Louise, as the next player, wins.

Note that if they were playing first-player-who-cannot-move wins, then Louise can still win by choosing heap sizes 0 or 5 modulo 12.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

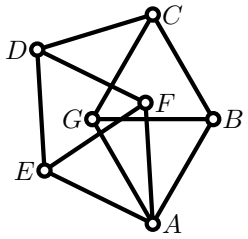
To facilitate their consideration, solutions should be received by **December 15, 2020**.

4571. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

What is the smallest square integer expressible as the product of three distinct nonzero integers in arithmetic progression?

4572. *Proposed by Veselin Jungić. Dedicated in memoriam to Richard K. Guy.*

In 1961, Canadian mathematicians Leo and William Moser introduced a geometric object consisting of seven vertices and eleven line segments of the unit length. This object is now known as the Moser spindle: see p. 390-396 of this issue for more details.



In the Moser spindle, find the measure of the angle $\angle GAF$.

4573. *Proposed by J. Chris Fisher.*

For any triangle ABC let γ be the circle through A and B that surrounds the incircle α and is tangent to it, while β is a circle inside the triangle that is tangent to the sides AC and BC. Then β is externally tangent to γ if and only if it is also tangent to the line parallel to (but not equal to) AB that is tangent to the incircle.

This result was conjectured following the solution of Honsberger problem H4 [2018: 143-144], which related H4 to Problem 2.6.4 in H. Fukagawa and D. Pedoe, Japanese Temple Geometry Problems: San Gaku*, The Charles Babbage Research Centre (1989) page 37.*

4574. *Proposed by George Apostolopoulos.*

Let $x_1, \ldots, x_n$ be positive real numbers with $x_i < 64$ such that $\sum_{i=1}^{n} x_i = 16n$. Prove that

$$\sum_{i=1}^{n} \frac{1}{8 - \sqrt{x_i}} \geq \frac{n}{4}.$$

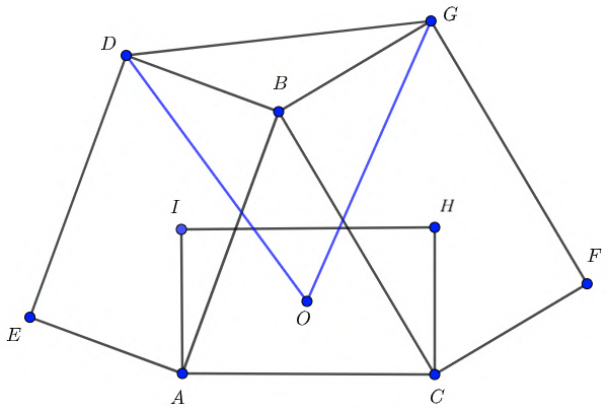
4575. *Proposed by Nguyen Viet Hung.*

Determine the coefficient of x in the following polynomial

$$\left(1+\binom{n}{0}x\right)\left(1+\binom{n}{1}x\right)^2\left(1+\binom{n}{2}x\right)^3\cdots\left(1+\binom{n}{n}x\right)^{n+1}.$$

4576. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABDE$, $BCFG$ and $ACHI$ be three similar rectangles as given in the figure. Suppose $\frac{AB}{AE}$ is constant and let O be the center of $ACHI$. Show that $OD = OG$ and $\angle GOD$ is constant when A and C are fixed but B can move.



4577. *Proposed by Nikolai Osipov.*

For any integer k, solve the equation

$$xy^2 + (kx^2 + 1)y + x^4 + 1 = 0$$

in integers x, y.

4578. *Proposed by Ed Barbeau. Dedicated in memoriam to Richard K. Guy.*

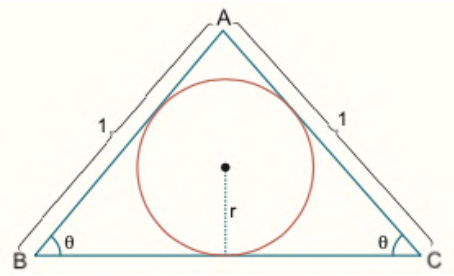
Suppose that $\{a, b, c\}$ and $\{u, v, w\}$ are two distinct sets of three integers for which $a + b + c = u + v + w$ and $a^2 + b^2 + c^2 = u^2 + v^2 + w^2$. What is the minimum possible value assumed by $|abc - uvw|$?

4579. *Proposed by George Stoica.*

Let $a, b, c \in \mathbb{Z}^*$ such that $\dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b} \in \mathbb{Z}$. Prove that $\dfrac{ab}{c}, \dfrac{bc}{a}, \dfrac{ca}{b} \in \mathbb{Z}$.

4580. *Proposed by Alpaslan Ceran.*

In an isosceles triangle ABC with $AB = AC = 1$, find the length of BC which maximizes the inradius.



. . . . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 décembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

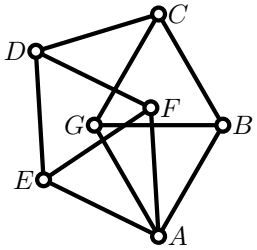
4571. *Proposeé par Ed Barbeau. En mémoire - Richard K. Guy.*

Quel est le plus petit entier carré pouvant être écrit comme le produit de trois entiers non nuls en progression arithmétique?

4572. *Proposeé par Veselin Jungić. En mémoire - Richard K. Guy.*

En 1961, les mathématiciens Canadiens Leo et William Moser ont proposé l'étude d'un objet géométrique ayant sept sommets et onze arêtes de longueur un. Cet objet est maintenant connu sous le nom *fuseau de Moser* (p. 390-396).

Déterminer la mesure de l'angle $\angle GAF$ dans le fuseau de Moser indiqué.



4573. *Proposeé par J. Chris Fisher.*

Pour un triangle ABC, soit γ le cercle qui passe par A et B, puis qui entoure le cercle inscrit α en lui étant tangent. Soit aussi β un cercle quelconque à l'intérieur du triangle et tangent aux côtés AC et BC. Alors β est tangent à γ à son extérieur si et seulement si β est tangent à la ligne, distincte de AB, qui est parallèle à AB et tangente au cercle inscrit.

Cette conjecture fait suite à la solution au probléme H4 de Honsberger [2018: 143-144], faisant le lien entre H4 et le probléme 2.6.4 dans H. Fukagawa et D. Pedoe, Japanese Temple Geometry Problems. *San Gaku, The Charles Babbage Research Centre (1989) page 37.*

4574. *Proposeé par George Apostolopoulos.*

Soient $x_1, \ldots, x_n$ des nombres réels positifs tels que $x_i < 64$ et $\sum_{i=1}^{n} x_i = 16n$. Démontrer que

$$\sum_{i=1}^{n} \frac{1}{8 - \sqrt{x_i}} \geq \frac{n}{4}.$$

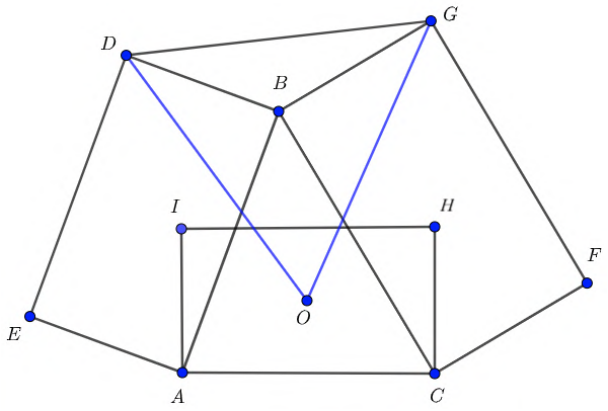
4575. *Proposeé par Nguyen Viet Hung.*

Déterminer le coefficient de x dans le polynôme

$$\left(1 + \binom{n}{0} x\right) \left(1 + \binom{n}{1} x\right)^2 \left(1 + \binom{n}{2} x\right)^3 \cdots \left(1 + \binom{n}{n} x\right)^{n+1}.$$

4576. *Proposeé par Dao Thanh Oai et Leonard Giugiuc.*

Soient $ABDE$, $BCFG$ et $ACHI$ trois rectangles similaires, tels qu'illustrés. Fixons le ratio $\frac{AB}{AE}$ et considérons O, le centre de $ACHI$. Démontrer que $OD = OG$ et que $\angle GOD$ est constant lorsque A et C sont fixes et que B est libre de bouger.



4577. *Proposeé par Nikolai Osipov.*

Soit k un entier quelconque. Déterminer les solutions entières x, y à l'équation

$$xy^2 + (kx^2 + 1)y + x^4 + 1 = 0.$$

4578. *Proposeé par Ed Barbeau. En mémoire - Richard K. Guy.*

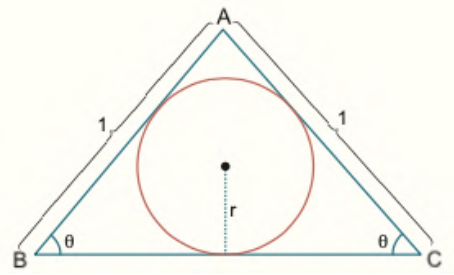
Soeint $\{a, b, c\}$ et $\{u, v, w\}$ deux ensembles distincts, chacun consistant de trois entiers, respectant $a+b+c = u+v+w$ et $a^2+b^2+c^2 = u^2+v^2+w^2$. Déterminer la valeur minimale de $|abc - uvw|$.

4579. *Proposeé par George Stoica.*

Soient $a, b, c \in \mathbb{Z}^*$ tels que $\dfrac{ab}{c} + \dfrac{bc}{a} + \dfrac{ca}{b} \in \mathbb{Z}$. Démontrer que $\dfrac{ab}{c}, \dfrac{bc}{a}, \dfrac{ca}{b} \in \mathbb{Z}$.

4580. *Proposeé par Alpaslan Ceran.*

Pour un triangle isocèle ABC tel que $AB = AC = 1$, déterminer la longueur de BC qui maximise le rayon du cercle inscrit.



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(3), p. 123–128.

4521. *Proposed by Robert Frontczak.*

Let $m \in \mathbb{N}$, define the sequence $a_n (n \geq 0)$ by $a_0 = m$, $a_1 = a_2 = \cdots = a_m = 1$ and $a_n = \sqrt{a_{n-m-1} \cdot a_{n-m}}$ for $n \geq m+1$. Determine $\lim_{n\to\infty} a_n$.

We received 8 submissions of which 3 were correct and complete. A common error was to miscalculate the pattern of the sequence and arrive at an incorrect limit. We present the solution of Walther Janous, slightly modified.

For $n \geq 0$ define $z_n = \log_m a_n$ (note that $a_n > 0$ for all n, so this is well-defined); that is, $a_n = m^{z_n}$. We get the linear recursion $z_0 = 1$, $z_1 = \cdots = z_m = 0$ and $z_n = \frac{1}{2}(z_{n-m} + z_{n-m-1})$ for $n \geq m+1$. This recurrence relation has characteristic polynomial $p(x) = x^{m+1} - \frac{1}{2}x - \frac{1}{2}$, which factors as $(x-1)(x^m + x^{m-1} + \cdots + x + \frac{1}{2})$. Clearly $x = 1$ is a solution of $p(x) = 0$.

We will show that all the roots of $p(x)$ which are different from 1 lie in the interior of the unit disk, and the equation $p(x) = 0$ has no multiple solutions.

If λ is any root of $p(x)$ then it satisfies $2\lambda^{m+1} = \lambda + 1$, so

$$2|\lambda|^{m+1} = |\lambda + 1| \leq |\lambda| + 1. \tag{1}$$

If $|\lambda| > 1$ then $|\lambda| + 1 < |\lambda|^{m+1} + 1$, which combined with the above inequality gives us $2|\lambda|^{m+1} < |\lambda|^{m+1} + 1$, that is, $|\lambda|^{m+1} < 1$, a contradiction. On the other hand, if $|\lambda| = 1$ then (1) implies that $|\lambda + 1| = |\lambda| + 1$, which can only happen when $\lambda = 1$. Therefore, for $\lambda \neq 1$ we must have $|\lambda| < 1$.

Now suppose λ is a multiple root of $p(x)$. Then it is also a root of $p'(x)$, that is, it satisfies $(m+1)\lambda^m - \frac{1}{2} = 0$. Substituting for λ^m in $p(\lambda) = 0$ we would then get $\lambda \cdot \frac{1}{2(m+1)} - \frac{\lambda}{2} - \frac{1}{2} = 0$, which would give $\lambda = -\frac{m+1}{m}$. However, this λ is outside the unit disk and hence cannot be a solution to $p(x) = 0$.

Let $\lambda_0, \ldots, \lambda_m$ be the roots of $p(x)$, with $\lambda_0 = 1$. Since these roots are all distinct, the recurrence relation given by z_n has solution

$$z_n = \alpha_0 + \alpha_1 \lambda_1^n + \cdots + \alpha_m \lambda_m^n \text{ for } n \geq 0,$$

where the coefficients α_k are determined by the initial values of the sequence. Note that $|\lambda_k| < 1$ for $k \neq 0$ implies that $\lim\limits_{n\to\infty} z_n = \alpha_0$, so in order to calculate the desired limit it suffices to calculate α_0.

The values α_k are the solutions to the following system of equations, obtained from $z_0 = 1$, $z_1 = \cdots = z_m = 0$,

$$\begin{cases} \alpha_0 + & \alpha_1 + & \cdots + & \alpha_m & = 1 \\ \alpha_0 + & \alpha_1\lambda_1 + & \cdots + & \alpha_m\lambda_m & = 0 \\ \alpha_0 + & \alpha_1\lambda_1^2 + & \cdots + & \alpha_m\lambda_m^2 & = 0 \\ & \vdots & & & \\ \alpha_0 + & \alpha_1\lambda_1^m + & \cdots + & \alpha_m\lambda_m^m & = 0. \end{cases}$$

We use Cramer's Rule to calculate

$$\alpha_0 = \frac{\begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \lambda_1 & \dots & \lambda_m \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \lambda_1^m & \dots & \lambda_m^m \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \dots & 1 \\ 1 & \lambda_1 & \dots & \lambda_m \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \lambda_1^m & \dots & \lambda_m^m \end{vmatrix}}$$

The denominator is a Vandermonde determinant and thus equals

$$\prod_{0 \le k < j \le m} (\lambda_j - \lambda_k).$$

Expanding the determinant in the numerator down the first column we get another Vandermonde determinant, namely

$$\begin{vmatrix} \lambda_1 & \dots & \lambda_m \\ \vdots & \vdots & \vdots \\ \lambda_1^m & \dots & \lambda_m^m \end{vmatrix} = \lambda_1 \dots \lambda_m \begin{vmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ \lambda_1^{m-1} & \dots & \lambda_m^{m-1} \end{vmatrix} = \lambda_1 \dots \lambda_m \prod_{1 \le k < j \le m} (\lambda_j - \lambda_k).$$

Therefore

$$\alpha_0 = \frac{\lambda_1 \dots \lambda_m}{\prod_{k=1}^{m} (\lambda_k - \lambda_0)}.$$

Recall $\lambda_0 = 1$ and $\lambda_1, \dots, \lambda_m$ are the roots of $q(x) = x^m + x^{m-1} + \cdots + x + \frac{1}{2}$. Hence

$$\lambda_1 \cdot \ldots \cdot \lambda_m = \frac{(-1)^m}{2},$$

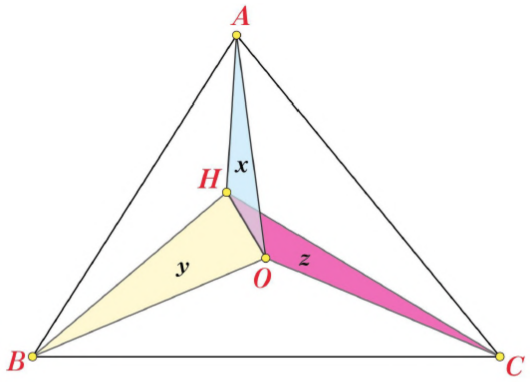
$$\prod_{k=1}^{m} (\lambda_k - 1) = (-1)^m q(1) = (-1)^m (m + \frac{1}{2}),$$

and we get $\alpha_0 = \frac{1}{2m+1}$. Therefore $\lim_{n\to\infty} z_n = \frac{1}{2m+1}$ and $\lim_{n\to\infty} a_n = m^{\frac{1}{2m+1}}$.

4522. *Proposed by Miguel Ochoa Sanchez, Leonard Giugiuc and Kadir Altintas.*

Let ABC be an acute triangle with orthocenter H and circumcenter O. Denote Area(AHO)=x, Area(BHO)=y and Area(CHO)=z. Prove that

$$2(x^2y^2 + y^2z^2 + z^2x^2) = x^4 + y^4 + z^4.$$



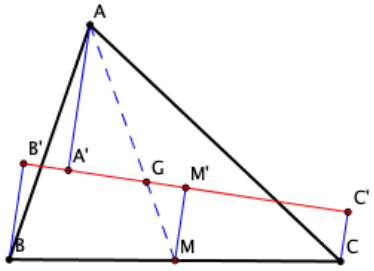
We received 24 submissions, all correct, and feature a composite of the solutions submitted independently by Ivko Dimitrić and by the UCLan Cyprus Problem Solving Group.

We shall prove a result that has the given problem as a very special case:

Let ℓ be an arbitrary line containing the centroid G of an arbitrary triangle ABC, and choose any two points O' and H' on ℓ. Set $[AH'O'] = x$, $[BH'O'] = y$, and $[CH'O'] = z$. Prove that

$$2(x^2y^2 + y^2z^2 + z^2x^2) = x^4 + y^4 + z^4.$$

In the original formulation (with $O' = O$ and $H' = H$), the line OH through the circumcenter and the orthocenter is the Euler line of a triangle, which contains the centroid G.



We begin by recalling the familiar identity,

$$2(x^2y^2+y^2z^2+z^2x^2)-(x^4+y^4+z^4) = (x+y+z)(-x+y+z)(x-y+z)(x+y-z).$$

In detail:

$$\begin{aligned}
&2(x^2y^2+y^2z^2+z^2x^2) = x^4+y^4+z^4\\
&\iff (x^2+y^2-z^2)^2-4x^2y^2=0\\
&\iff (x^2+y^2-z^2-2xy)(x^2+y^2-z^2+2xy)=0\\
&\iff ((x-y)^2-z^2)((x+y)^2-z^2)=0\\
&\iff (x-y-z)(x-y+z)(x+y-z)(x+y+z)=0.
\end{aligned}$$

So it is enough to show that the sum of two of the areas is equal to the other. And since $AO'H', BO'H', CO'H'$ have $O'H'$ as a common side, the proof comes down to showing that the distance from $\ell = O'H'$ to one of the vertices is equal to the sum of its distances to the other two vertices. Note first that because G is in the interior of the triangle, ℓ intersects two of its sides (or all three if it passes through a vertex). As in the figure, we assume that ℓ meets the sides through A, and we let M be the midpoint of BC and A', B', C', M' be the projections of A, B, C, M on ℓ. The right triangles $AA'G$ and $MM'G$ are similar, and we know that $\frac{AG}{GM} = 2$; therefore $\frac{AA'}{MM'} = 2$. But MM' is the midline of the trapezoid $B'BCC'$, whence $2MM' = BB' + CC'$, and we conclude that $AA' = BB' + CC'$, which completes the proof.

Editor's comments. The result concerning lines through centroid G (used in the featured solution) is certainly familiar to those of us who teach calculus. Dimitrić observes that because G is the center of gravity of the vertices of a triangle, any triangle, considered as three point masses, would balance with G on the x-axis, which implies that the y-coordinates of the vertices would sum to zero.

4523*. *Proposed by Leonard Giugiuc.*

Let n be a natural number such that $n \geq 2$. Further, let $\{a_1, a_2, \ldots, a_n\} \subset [0,1]$ and $\{b_1, b_2, \ldots, b_n\} \subset [1, \infty)$ such that

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = n+1.$$

Prove that

$$\frac{1}{n}(n^2+1) \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 \leq n+3.$$

We received 9 correct solutions. One submission proved only one inequality and two others were incorrect. The most efficient solutions were essentially as follows.

Let $u = \sum_{k=1}^{n} a_k$ and $v = \sum_{k=1}^{n} b_k$. By either the Root-Mean-Square and Arithmetic Mean or Cauchy-Schwarz inequality, we have that

$$\sum_{k=1}^{n} a_k^2 \geq \frac{u^2}{n} = \frac{(n+1-v)^2}{n} \qquad \text{and} \qquad \sum_{k=1}^{n} b_k^2 \geq \frac{v^2}{n}.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 &\geq \frac{1}{n}[(n+1)^2 - 2v(n+1) + 2v^2] \\ &= \frac{n^2+1}{n} + \frac{2}{n}[n - v(n+1) + v^2] \\ &= \frac{n^2+1}{n} + \frac{2}{n}[(v-n)(v-1)].\end{aligned}$$

Since $v \geq n \geq 1$, the sum of the squares is not less than $(n^2+1)/n$ and equality occurs when all the a_k are equal, all the b_k are equal, $v = n$ and $u = 1$. Thus, we obtain the left inequality with equality if and only if $a_k = 1/n$ and $b_k = 1$ for each k.

For each k, let $b_k = c_k + 1$. Then $0 \leq a_k, c_k \leq 1$ $a_k^2 \leq a_k$, $c_k^2 \leq c_k$ and

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} c_k = 1.$$

Then

$$\begin{aligned}\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 &= \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} (c_k+1)^2 \\ &= \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} c_k^2 + 2\sum_{k=1}^{n} c_k + n \\ &\leq \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} c_k + 2\sum_{k=1}^{n} c_k + n \\ &= n + 1 + 2\sum_{k=1}^{n} c_k \leq n+3.\end{aligned}$$

Equality occurs if and only if $\sum_{k=1}^{n} c_k = 1$ and each of a_k and c_k is either 0 or 1. In this case, all variables vanish except for one c_k equal to 1. For example, we have equality when $a_1 = \cdots = a_n = 0; b_1 = \cdots = b_{n-1} = 1; b_n = 2$.

4524. *Proposed by Lorian Saceanu.*

Let x, y, z be non-negative real numbers at most one of which is zero. Prove that if

$$x^2 + y^2 + z^2 = 2(xy + yz + xz),$$

then

$$5 \leq (x+y+z)\left(\sum_{cyclic} \frac{1}{y+z}\right) \leq \frac{27}{5}$$

and determine when equality holds for either bound.

We received 21 submissions, all correct. We present the solution by Oliver Geupel.

Due to the homogeneity, we may assume that $x+y+z=1$. Then:

$$xy+yz+zx=\frac{x^2+y^2+z^2+2(xy+yz+zx)}{4}=\frac{(x+y+z)^2}{4}=\frac{1}{4}, \tag{1}$$

and

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+zx)=1-\frac{1}{2}=\frac{1}{2}. \tag{2}$$

From (1) and (2), we get

$$\begin{aligned}(x+y+z)\left(\sum_{\text{cyclic}}\frac{1}{y+z}\right)&=\frac{(x+y)(x+z)+(y+z)(y+x)+(z+x)(z+y)}{(x+y)(y+z)(z+x)}\\&=\frac{x^2+y^2+z^2+3(xy+yz+zx)}{(x+y+z)(xy+yz+zx)-xyz}\\&=\frac{\frac{1}{2}+3\left(\frac{1}{4}\right)}{\frac{1}{4}-xyz}=\frac{\frac{5}{4}}{\frac{1}{4}-xyz}\geq 5\end{aligned} \tag{3}$$

with equality holding if and only if $xyz=0$. Hence by the given constraint, exactly one of x,y,z is zero.

To prove the upper bound, consider the polynomial

$$f(t)=(t-x)(t-y)(t-z)=t^3-t^2+\frac{1}{4}t-xyz,$$

which has three real zeros and a local maximum at $t=1/6$. Since

$$f\left(\frac{1}{6}\right)=\left(\frac{1}{6}\right)^3-\left(\frac{1}{6}\right)^2+\frac{1}{24}-xyz=\frac{1}{54}-xyz,$$

we have

$$0\leq f\left(\frac{1}{6}\right)=\frac{1}{54}-xyz,$$

which together with (3) then yields

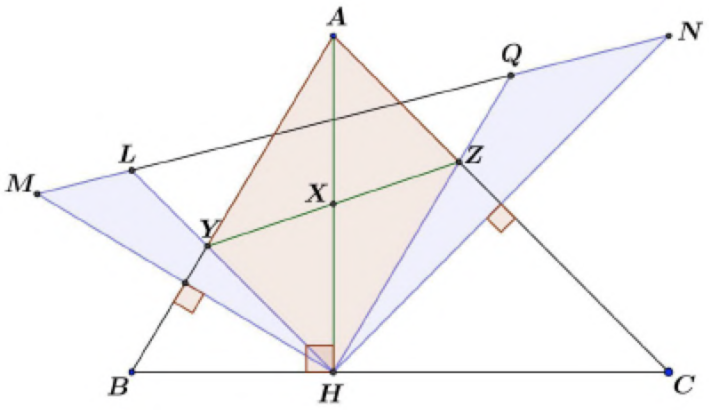
$$(x+y+z)\left(\sum_{\text{cyclic}}\frac{1}{y+z}\right)=\frac{5/4}{1/4-xyz}\leq\frac{5/4}{1/4-1/54}=\frac{27}{5}.$$

Finally, the equality holds if and only if the local maximum at $t=1/6$ is 0; that is, if two of the roots are equal to $1/6$ and the third one is $2/3$.

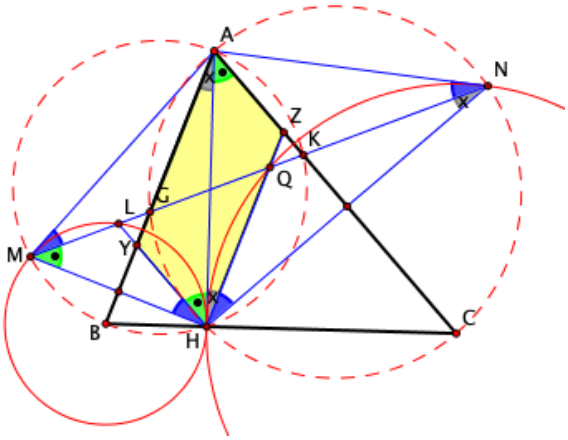
Dropping our initial hypothesis $x+y+z=1$, we then conclude that the equality holds if and only if two of the numbers x,y,z are equal while the third one is four times the others.

4525. *Proposed by Julio Orihuela and Leonard Giugiuc.*

Let H be the foot of the altitude from vertex A to side BC of the acute triangle ABC; let the circle with center B and radius BH meet the perpendicular from H to AB again at M, and the circle with center C and radius CH meet the perpendicular from H to AC again at N. Moreover, let the line MN meet the first circle again at L and the second circle again at Q, and finally, let Y be the point where HL intersects AB and Z the point where HQ intersects AC. Prove that $AYHZ$ is a parallelogram and $\angle MHL = \angle QHN$.



We received 21 submissions, all of which were correct, and we feature the solution by Theo Koupelis.



By construction, AH is the common tangent to the circles (B, BH) and (C, CH). By symmetry about the lines AB and AC, AM and AN are also tangents to the circles (B, BH) and (C, CH), respectively; thus

$$AM = AH = AN.$$

From tangency, $\angle MHL = \angle AMN$, and $\angle QHN = \angle MNA$. But triangle AMN

is isosceles, and therefore $\angle AMN = \angle MNA$, and thus $\angle MHL = \angle QHN$, as desired.

Let G, K be the points where MN meets the sides AB, AC, respectively. Again by symmetry about the lines AB and AC,

$$\angle GHA = \angle AMG = \angle AMN = \angle MNA = \angle KNA = \angle AHK.$$

As a consequence, M, A, K, H and N, A, G, H are concyclic point sets, and therefore,

$$\angle KAH = \angle KMH \quad \text{and} \quad \angle HAG = \angle HNG.$$

However, from tangency we have

$$\angle KMH = \angle LMH = \angle LHA \quad \text{and} \quad \angle HNG = \angle HNQ = \angle AHQ.$$

Therefore $\angle KAH = \angle LHA$, (which makes $AZ||YH$) and $\angle HAG = \angle AHQ$, (which makes $AY||HZ$), and thus $AYHZ$ is a parallelogram.

Editor's comment. There is no need to require the given triangle ABC to be acute. The featured solution remains valid for an arbitrary triangle ABC if directed angles are used. For those readers familiar with properties of the orthic triangle (whose vertices are the feet of the altitudes of ΔABC), it is easy to show that GHK is the orthic triangle of the given triangle ABC (where G and H are the points where the given line MN meets the sides AB and AC, as in the featured solution).

4526. *Proposed by Michel Bataille.*

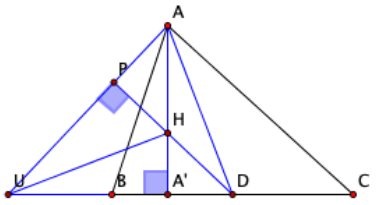
Let ABC be a scalene, not right-angled triangle with orthocenter H and let D, E, F be the midpoints of BC, CA, AB, respectively. Points U, V, W, respectively on the lines BC, CA, AB, are such that AU, BV, CW are perpendicular to HD, HE, HF (respectively). Prove that U, V, W are collinear.

We received 16 solutions, all correct, and we feature three of them.

Solution 1, by Titu Zvonaru.

Let A' be the foot of the altitude from A, P be the foot of the perpendicular from D to AU, and set $x = UB$. We shall use directed distances along the sides of ΔABC, taking x to be positive when B lies between U and C. We shall use familiar formulas for parts of a triangle, namely

$$AH = 2R\cos A, \quad BH = 2R\cos B, \quad BA' = c\cos B,$$
$$BD = \frac{a}{2}, \quad \text{and} \quad AD^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2).$$



Because right triangles UHP and UDP share the base UP, while right triangles PHA and PDA share the base PA, we have

$$UD^2 - DP^2 = UH^2 - HP^2 \quad \text{and} \quad AD^2 - DP^2 = AH^2 - HP^2.$$

Subtracting these equations we obtain

$$\begin{aligned} UD^2 - AD^2 &= UH^2 - AH^2 \\ &= (UA'^2 + HA'^2) - AH^2 = (x + BA')^2 + (BH^2 - BA'^2) - AH^2. \end{aligned}$$

It follows that

$$\left(x + \frac{a}{2}\right)^2 - \frac{2b^2 + 2c^2 - a^2}{4} = (x + c\cos B)^2 + 4R^2\cos^2 B - c^2\cos^2 B - 4R^2\cos^2 A,$$

or

$$ax - 2cx\cos B = \frac{1}{2}(b^2 + c^2 - a^2) + 4R^2(\cos^2 B - \cos^2 A). \tag{1}$$

The cosine law combined with the sine law says that

$$\frac{1}{2}(b^2 + c^2 - a^2) = bc\cos A = 4R^2 \sin B \sin C \cos A$$

which, together with other familiar formulas turns equation (1) into successively,

$$\begin{aligned} (\sin A - 2\sin C\cos B)x &= 2R(\cos A(\sin B\sin C - \cos A) + \cos^2 B) \\ (\sin(B + C) - 2\sin C\cos B)x &= 2R(\cos A(\sin B\sin C + \cos(B + C)) + \cos^2 B) \\ (\sin B\cos C - \sin C\cos B)x &= 2R\cos A(\cos B\cos C + \cos^2 B) \\ (\sin(B - C))x &= 2R\cos B(\cos A\cos C - \cos(A + C)) = 2R\sin A\cos B\sin C. \end{aligned}$$

We have, finally,

$$UB = x = \frac{2R\sin A\cos B\sin C}{\sin(B - C)}, \quad \text{and}$$

$$UC = x + a = \frac{2R\sin A\cos B\sin C}{\sin(B - C)} + 2R\sin A = \frac{2R\sin A\sin B\cos C}{\sin(B - C)};$$

hence,

$$\frac{UB}{UC} = \frac{\tan C}{\tan B}.$$

Similarly, $\frac{VC}{VA} = \frac{\tan A}{\tan C}$ and $\frac{WA}{WB} = \frac{\tan B}{\tan A}$. The product of these three fractions equals 1, so that by the converse of Menelaus' theorem, U, V, W are collinear.

Solution 2 is a joint solution by Corneliu Manescu-Avram and Zlota Nicusor.

Choose a cartesian system of coordinates with H as origin and A on the y-axis. We can take, without loss of generality, B and C on the line $y = -1$. Then we have $B(2b, -1)$ and $C(2c, -1)$, say with $b < c$, while

$$AB : y + 1 = 2c(x - 2b) \quad \text{and} \quad AC : y + 1 = 2b(x - 2c).$$

It follows that

$$A(0, -4bc - 1),$$

so that

$$D(b + c, -1), \quad E(c, -2bc - 1), \quad \text{and} \quad F(b, -2bc - 1).$$

We deduce that

$$AU : y + 4bc + 1 = (b + c)x, \quad BV : (2bc + 1)(y + 1) = c(x - 2b),$$

$$CW : (2bc + 1)(y + 1) = b(x - 2c);$$

therefore

$$U\left(\frac{4bc}{b+c}, -1\right), \quad V\left(\frac{2bc(4bc+1)}{4b^2c+2b-c}, \frac{(c-2b)(4bc+1)}{4b^2c+2b-c}\right),$$
$$W\left(\frac{2bc(4bc+1)}{4bc^2+2c-b}, \frac{(b-2c)(4bc+1)}{4bc^2+2c-b}\right).$$

Points U, V, W are collinear if and only if

$$\begin{vmatrix} x_U & y_U & 1 \\ x_V & y_V & 1 \\ x_W & y_W & 1 \end{vmatrix} = 0.$$

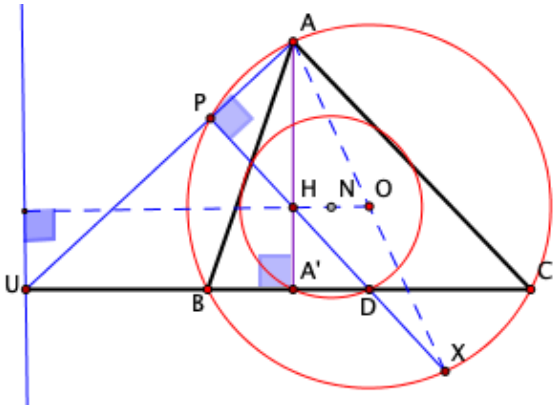
Eliminating the denominators, we must verify that

$$\begin{vmatrix} 4bc & -(b+c) & b+c \\ 2bc(4bc+1) & (c-2b)(4bc+1) & 4b^2c+2b-c \\ 2bc(4bc+1) & (b-2c)(4bc+1) & 4bc^2+2c-b \end{vmatrix} = 0.$$

But it is clear that the sum of the last two rows equals $4bc+1$ times the first row, which completes the proof.

Solution 3 is a composite of the solutions by Jiahao Chen and by J. Chris Fisher.

As in the figure, let A' be the foot of the altitude to BC, P be the point where HD intersects AU, and define X to be one end of the diameter AX of the circumcircle.



Because both BX and CH are perpendicular to AB, the lines are parallel. Similarly $BH||CX$, whence quadrilateral $BXCH$ is a parallelogram. This implies that the midpoint D of the diagonal BC lies on the other diagonal HX so that X is on the line containing H, D, and P. Because $\angle XPA = 90°$, P must lie on the circumcircle; consequently, the power of U with respect to the circumcircle satisfies

$$UA \cdot UP = UC \cdot UB. \tag{2}$$

But the right triangles AUA' and DUP are similar, whence

$$UA \cdot UP = UD \cdot UA'. \tag{3}$$

Because the nine-point circle of ΔABC contains the points A' and D, this last equation says that $UA \cdot UP$ is also the power of U with respect to the nine-point circle. Equations (2) and (3) together, therefore, imply that U lies on the radical axis of the two circles. Similarly, V and W also must lie on this radical axis; that is, U, V, W are collinear as claimed.

Related consequences. It is easily seen that U lies also on $B'C'$, V on $C'A'$ and W on $A'B'$, where A', B', C' are the feet of the altitudes to the lines BC, CA, AB, respectively. For example, U is the radical center of the circumcircle, nine-point circle, and the circle whose diameter is AH (because the last circle contains the points P, C', and B'). This observation leads to an alternative proof of the collinearity of U, V, W: Since the triangles ABC and $A'B'C'$ are perspective from the point H, the intersections of corresponding sides of the triangles, namely

$$U = BC \cap B'C', \quad V = CA \cap C'A', \quad W = AB \cap A'B',$$

are collinear on the Desargues line. This line is sometimes called the *orthic axis* of triangle ABC (because $A'B'C'$ is the orthic triangle). Note, further, that also the triangles DEF and XYZ are perspective from H (where X, Y, Z are points where HD, HE, HF, respectively, meet the circumcircle), so that their corresponding sides meet in collinear points

$$EF \cap YZ, \quad FD \cap ZX \quad DE \cap XY.$$

Moreover, by using circles such as $EFYZ$ it is easily seen that also these three points lie on the line UVW.

Editor's comments. The radical axis of two circles is always perpendicular to the line joining the centers; here the radical axis of the circumcircle and nine-point circle is perpendicular to the Euler line (which is the line joining the circumcenter O to the nine-point center N, and which also contains the orthocenter H). Several readers commented that care should be taken when interpreting this result given an isosceles triangle ABC. The Euler line would then contain a vertex. So, for example, should $AB = AC$, the Euler line would be AD; in this case the radical axis VW would be the line through A parallel to BC, and U would be the point at infinity common to those two lines. Should ΔABC be equilateral, then the circumcircle and nine-point circle would be concentric so that technically, there

would be no radical axis: the points U, V, W would be collinear on the line at infinity. The problem's restriction to nonright triangles is superfluous: for example, with a right angle at A, both V and W would coincide with A, and the radical axis would be the common tangent to the circumcircle and nine-point circle at A with U the point where that tangent intersects BC (possibly at infinity).

4527. *Proposed by George Stoica.*

Let $n \geq 4$ be a positive integer. Prove that the roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients satisfy $|a_{n-2}|, |a_{n-1}| \leq |a_n| \leq |a_0|$, cannot be all real.

We received eight correct solutions and one incorrect solution. The successful solutions were variants of the following.

Suppose a polynomial of arbitrary degree n has its roots $r_1, \ldots, r_n$ all real and satisfies the stated conditions on the coefficients. Then

$$\begin{aligned} 1 &\leq \left(\frac{|a_0|}{|a_n|}\right)^{2/n} = (r_1^2 r_2^2 \cdots r_n^2)^{\frac{1}{n}} \leq \frac{1}{n}(r_1^2 + r_2^2 + \cdots + r_n^2) \\ &= \frac{1}{n}\left[(r_1 + r_2 + \cdots + r_n)^2 - 2\left(\sum_{1 \leq i < j \leq n} r_i r_j\right)\right] = \frac{1}{n}\left[\left(\frac{a_{n-1}}{a_n}\right)^2 - \frac{2a_{n-2}}{a_n}\right] \\ &\leq \frac{1}{n}\left[\left(\frac{|a_{n-1}|}{|a_n|}\right)^2 + \frac{2|a_{n-2}|}{|a_n|}\right] \leq \frac{3}{n}, \end{aligned}$$

whereupon $n \leq 3$. Thus, the polynomial cannot have degree exceeding 3 and satisfy the other conditions.

However, the polynomials $x^2 - x - 1$ and $x^3 + x^2 - x - 1$ show that the situation is possible for degrees 2 and 3.

4528. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a rectangle situated in a plane $\mathcal{P}$. Find

$$\min_{M \in \mathcal{P}} \left(\frac{MA + MC}{MB + MD}\right).$$

We received 11 solutions, 9 of which were correct. We present the solution by UCLan Cyprus Problem Solving Group.

Suppose that the rectangle has dimensions $AB = CD = a$ and $AD = BC = b$.

By Ptolemy's inequality we have

$$(AC)(MB) \leqslant (MC)(AB) + (MA)(BC) = a(MC) + b(MA)$$

and

$$(AC)(MD) \leqslant (MC)(AD) + (MA)(DC) = b(MC) + c(MA)\,.$$

Therefore $(AC)(MB+MD) \leqslant (a+b)(MA+MC)$. Thus,

$$\frac{MA+MC}{MB+MD} \geqslant \frac{AC}{a+b} = \frac{\sqrt{a^2+b^2}}{a+b}\,.$$

Equality can occur if $M=A$ or $M=C$.

Note: In fact, the above are the only cases of equality. Let ω be the circumcircle of $ABCD$. In the first application of Ptolemy's inequality, equality occurs if and only if M belongs on the arc AC of ω which contains D. In the second application equality occurs if and only if M belongs on the arc AC of ω which contains B. So equality occurs in both cases if and only if $M=A$ or $M=C$.

4529. *Proposed by George Apostolopoulos.*

Let a, b, c be the side-lengths of a triangle. Prove that

$$\frac{2a+b}{a+c} + \frac{2b+c}{b+a} + \frac{2c+a}{c+b} \geq \frac{9}{2}.$$

We received 36 submissions, all correct. We present an amalgamation of similar solutions by Michel Bataille and Gayen Subhankar, modified slightly by the editor.

Note first that the equality below can be checked readily:

$$(a+c)(2a+b) + (b+a)(2b+c) + (c+b)(2c+a) = 2(a+b+c)^2.$$

Let L denote the left-hand side of the given inequality. Furthermore, let

$$\begin{aligned} &x_1 = 2a+b,\ x_2 = 2b+c,\ x_3 = 2c+a,\\ &y_1 = (a+c)(2a+b),\ y_1 = (b+a)(2b+c),\ y_1 = (c+b)(2c+a). \end{aligned}$$

Then

$$x_1+x_2+x_3 = 3(a+b+c) \quad \text{and} \quad y_1+y_2+y_3 = 2(a+b+c)^2.$$

Since

$$\frac{x_1^2}{y_1} = \frac{2a+b}{a+c},\ \frac{x_2^2}{y_2} = \frac{2b+c}{b+a} \quad \text{and} \quad \frac{x_3^2}{y_3} = \frac{2c+a}{c+b},$$

we have by Titu's Lemma, which is a special case of Cauchy-Schwarz Inequality, that

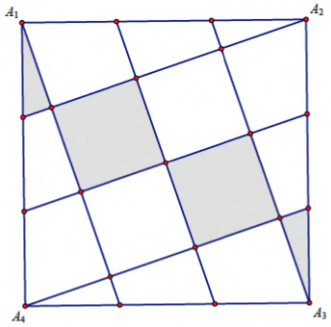
$$\begin{aligned} L &= \frac{(2a+b)^2}{(a+c)(2a+b)} + \frac{(2b+c)^2}{(b+a)(2b+c)} + \frac{(2c+a)^2}{(c+b)(2c+a)}\\ &= \frac{x_1^2}{y_1} + \frac{x_2^2}{y_2} + \frac{x_3^2}{y_3} \geq \frac{(x_1+x_2+x_3)^2}{y_1+y_2+y_3}\\ &= \frac{9(a+b+c)^2}{2(a+b+c)^2} = \frac{9}{2}, \end{aligned}$$

completing the proof.

Editor's note. Bataille pointed out that the inequality actually holds for all positive numbers a, b and c.

4530. *Proposed by Arsalan Wares.*

Let A be a square with vertices A_k, $k = 1, 2, 3, 4$. On each side of A, mark 2 points which divide the side into 3 equal parts. These 8 points and the vertices of A are connected to one another, dividing A into 16 disjoint regions, as shown in the figure. Determine the ratio of the area of the shaded regions to the area of A.



We received 33 submissions, out of which 28 were correct and complete. We present the solution by Joel Schlosberg.

The square A can be dissected into 60 mutually congruent triangles, with the shaded regions being made up of 14 of them, so the ratio of their total area to that of A is 7/30.