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MathemAttic

No. 29

The problems featured in this section are intended for students at the secondary school level.

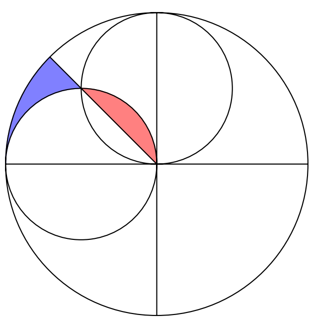
Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **January 30, 2022**.

MA141. *Proposed by Ed Barbeau.*

Determine all sets consisting of an odd number $2m+1$ of consecutive positive integers, for some integer $m \geq 1$ such that the sum of the cubes of the smallest $m+1$ integers is equal to the sum of the cubes of the largest m integers.

MA142. The sketch shown is from the files of Leonardo da Vinci. Two perpendicular diameters divide a circle into four parts. On each of these diameters a circle of half the diameter is drawn, tangent to the original circle and meeting at its centre. A radius to the large circle is drawn through the intersection points of these smaller circles. Show that the red and blue shaded regions are of the same area.



MA143. The integers from 1 to n are added to form the sum N and the integers from 1 to m are added to form the sum M, where $n > m+1$. If the difference between the two sums is $N - M = 2012$, then determine the value of $n + m$.

MA144. A game is played on a 7×7 board, initially blank. Betty Brown and Greta Green make alternate moves, with Betty going first. In each of her moves, Betty chooses any four blank squares which form a 2×2 block, and paints these squares brown. In each of her moves, Greta chooses any blank square and paints it green. They take alternate turns until no more moves can be made by Betty. Then Greta paints the remaining blank squares green. Which player, if either, can guarantee to be able to paint 25 or more squares in her colour, regardless of how her opponent plays?

MA145. Determine all integers n for which $n^3 - 3n + 2$ is divisible by $2n + 1$.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

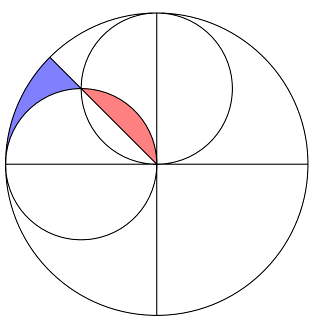
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 janvier 2022**.

MA141. *Proposé par Ed Barbeau.*

Déterminer tous les ensembles comprenant un nombre impair $2m + 1$ d'entiers positifs consécutifs, où m est un entier, $m \geq 1$, sachant qu'en plus la somme des cubes des $m + 1$ plus petits éléments de l'ensemble égale la somme des cubes des m plus grands éléments.

MA142. Le schéma suivant s'inspire de l'œuvre de Léonard de Vinci. Deux diamètres perpendiculaires divise un cercle en 4 parties égales. Sur chacun de ces diamètres on trace un cercle, tangent au cercle original, ces deux petits cercles se rencontrant au centre du cercle original. Un rayon du grand cercle passe par les points d'intersection des petits cercles. Démontrer que les régions en bleu et en rouge ont la même surface.



MA143. Pour $n > m + 1$, les entiers de 1 à n sont additionnés et donnent N, tandis que les entiers de 1 à m sont additionnés et donnent M. Si la différence entre ces sommes donne $N - M = 2012$, déterminer la valeur de $n + m$.

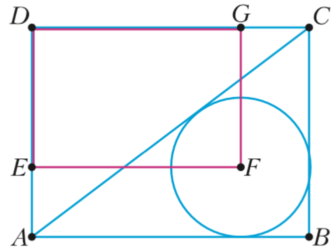
MA144. Au départ, les cases d'un échiquier de taille 7×7 sont en blanc. Par la suite, Bernadette Brun et Véronique Vert alternent à colorer certains carrés, Bernadette pour commencer. À chacun de ses coups, Bernadette choisit quatre carrés blancs formant un bloc 2×2 et les colore brun. À chacun de ses coups, Véronique choisit un carré blanc et le colore vert. Dès que Bernadette n'est plus capable de jouer, Véronique colore en vert tous les carrés encore blancs. La gagnante est celle qui colore au moins 25 carrés de sa couleur. Est-il possible pour une des deux joueuses de gagner, quelle que soit la stratégie de l'autre? Si c'est le cas, quelle joueuse?

MA145. Déterminer tous les entiers n tels que $n^3 - 3n + 2$ est divisible par $2n + 1$.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(4), p. 170–172.

MA116. Let $ABCD$ and $DEFG$ be two rectangles so that the point E lies on the side AD, the point G lies on the side CD and the point F is the incenter of $\triangle ABC$. What is the ratio of the area of $ABCD$ and the area of $DEFG$?



Originally Problem 14 from Savin Contest, Kvant 2020(11-12), proposed by A. Andreeva and M. Panov.

We received 11 submissions, 9 of which were correct and complete. We present the solution obtained independently by two high school students Emilian Sega (Monta Vista High School) and Vishak Srikanth (Stanford Online High School).

Let $AB = a$, $BC = b$, and $AC = c$. From the Pythagorean Theorem in $\triangle ABC$, $a^2 + b^2 = c^2$.

Let r be the inradius of triangle ABC. It is well-known that $r = \dfrac{a+b-c}{2}$. We then have:

$$\frac{[ABCD]}{[DEFG]} = \frac{ab}{(a-r)(b-r)} = \frac{ab}{\left(a - \dfrac{a+b-c}{2}\right)\left(b - \dfrac{a+b-c}{2}\right)}$$
$$= \frac{4ab}{(c+a-b)(c-a+b)} \quad (\star)$$

Since $c^2 = a^2 + b^2$, it follows that

$$(c+a-b)(c-a+b) = [c+(a-b)][c-(a-b)] = c^2-(a-b)^2 = c^2-a^2+2ab-b^2 = 2ab\,.$$

Substituting in $(\star)$ we obtain:

$$\frac{[ABCD]}{[DEFG]} = \frac{4ab}{2ab} = 2.$$

MA117. For which natural numbers n can the set $\{1, 2, \ldots, n\}$ be partitioned into two subsets so that the sum of numbers in one subset equals to the product of the numbers in the other subset?

Originally Problem 15 from Savin Contest, Kvant 2020(11-12), proposed by V. Letzko.

We received 6 submissions, of which only 1 was correct and complete as the majority of the solvers forgot to consider the case $n = 1$. We present the solution by the Missouri State University Problem Solving Group.

Such subsets can be found for any $n \neq 2, 4$. We will denote the set we take the sum over by S and the set we take the product over by P. If $n = 1$, we take $S = \{1\}$ and $P = \{\}$ and recall that the product over an empty index set is 1. Clearly no solution is possible when $n = 2$. When $n = 3$, we take $S = \{1, 2\}$ and $P = \{3\}$. Exhaustive enumeration shows there is no solution for $n = 4$. If $n = 2k + 1$ with $k > 1$, we take

$$P = \{1, k, 2k\}$$

and S to be its complement. Then

$$\begin{aligned}\sum_{i\in S} i &= \frac{n(n+1)}{2} - 1 - k - 2k \\ &= (2k+1)(k+1) - 3k - 1 = 2k^2 = 1 \cdot k \cdot (2k) = \prod_{i\in P} i.\end{aligned}$$

If $n = 2k$ with $k > 2$, we take

$$P = \{1, k-1, 2k\}$$

and S to be its complement. Then

$$\begin{aligned}\sum_{i\in S} i &= \frac{n(n+1)}{2} - 1 - (k-1) - 2k \\ &= k(2k+1) - 3k = 2k^2 - 2k = 1 \cdot (k-1) \cdot (2k) = \prod_{i\in P} i.\end{aligned}$$

MA118. Can you colour all natural numbers using exactly 7 colours so that the product of any two (not necessarily distinct) numbers of the same colour results in a number of that same colour? For example, if 3 and 4 and coloured red, then 9, 12 and 16 must also be coloured red.

Originally Problem 5 from Savin Contest, Kvant 2020(10), proposed by M. Evdokimov.

We received 3 submissions, of which 2 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

This can be done using k colours for any positive integer k. The case $k = 1$ is trivial. For $k > 1$, let $p_1, \ldots, p_{k-1}$ be distinct primes. For each $i = 1, \ldots, k-1$ define the set

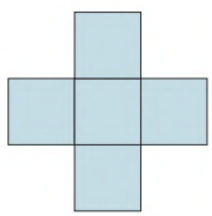
$$S_i = \left\{ p_i^j \middle| \, j \geq 1 \right\}.$$

Colour S_i with the i^{th} colour, and

$$T = \mathbb{N} \setminus \left(\cup_{i=1}^{k-1} S_i \right)$$

with the k^{th} colour. The S_i are clearly closed under multiplication. Suppose $x, y \in T$, but $xy \notin T$. Then $xy \in S_i$ for some i, which means $xy = p_i^j$ for some $j > 0$. This forces $x = p_i^{\ell}$ and $y = p_i^{j-\ell}$ for some $0 \leq \ell \leq j$. One of ℓ or $j - \ell$ must be non-zero, without loss of generality say $\ell \neq 0$. Then $x \in S_i$ contradicting the fact that $x \in T$. Therefore T is also closed under multiplication.

MA119. Alice and Bob are playing tic-tac-toe on an infinite grid. The winner is declared when they place their sign over 5 squares in the shape of a plus. If Alice goes first, can Bob always prevent her from winning?



Originally Problem 6 from Savin Contest, Kvant 2020(10), proposed by D. Ivanov.

We received 2 submissions, of which 1 was correct and complete. We present the solution by Richard Hess.

Bob can always prevent Alice from picking 5 unit squares in the cross shape. He does this by partitioning the infinite grid into 2×2 unit squares. When Alice picks a unit square in one of these 2×2 squares Bob answers by picking a vacant unit square in the same 2×2 square. Alice can never pick more than two unit squares in any of Bob's 2×2 squares if Bob follows this strategy. Any cross shape in the grid will have 3 unit squares in one of Bob's 2×2 squares; thus, Bob can prevent Alice from winning this game.

MA120. With grid paper and pencil, it is easy to draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths; for example, the so-called Egyptian triangle with side 3, 4 and 5 will do. Can you draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths, but so that none of its sides follows grid lines?

Originally Problem 18 from Savin Contest, Kvant 2021(1).

We received 6 submissions, all correct. We present the solution by the Missouri State University Problem Solving Group.

The answer is yes. Let (a, b, c) and (p, q, r) with $aq \neq bp$ be Pythagorean triples, that is $a^2 + b^2 = c^2, p^2 + q^2 = r^2$, and $a, b, c, p, q, r \in \mathbb{Z}^+$. The complex numbers 0, a, and bi form the vertices of a right triangle with integer side lengths. If we multiply each of these by $p + qi$, we will still have a right triangle with integer side lengths, since this multiplication corresponds to a rotation followed by a dilation by a factor of r. Performing this operation and converting to cartesian coordinates gives the vertices

$$A = (0, 0), B = (ap, aq), C = (-bq, bp).$$

Clearly neither $\overrightarrow{AB}$ nor $\overrightarrow{AC}$ is parallel to the coordinate axes. Since

$$\overrightarrow{CB} = (ap + bq, aq - bp),$$

and we chose p and q so that $aq - bp \neq 0$, this is not parallel to the coordinate axes either.

Although the argument above shows the side lengths are integers, we will also verify this directly:

$$\begin{aligned} AB &= a\sqrt{p^2 + q^2} \\ &= ar \\ AC &= b\sqrt{p^2 + q^2} \\ &= br \\ BC &= \sqrt{(ap + bq)^2 + (aq - bp)^2} \\ &= \sqrt{a^2 + b^2}\sqrt{p^2 + r^2} \\ &= cr. \end{aligned}$$

For example, taking $(a, b) = (3, 4)$ and $(p, q) = (4, 3)$, we obtain

$$A = (0, 0), B = (12, 16), C = (-12, 9)$$

giving a right triangle with sides of length 20, 15, and 25.

Robo Creativity

The name of each robot consists of a string of at least two letters chosen from C, D, E, F, P, Q, R and S. The initial letter of a robot defines its function. Let x denote the part of the robot's name after the initial letter. Then the following rules apply.

- The robot Cx creates the robot x.
- The robot Px creates (or produces) the robot xx.
- The robot Dx destroys the robot x.
- The robot Qx destroys (or quashes) the robot xx.
- The robot Ex is the worst enemy of the robot x.
- The robot Rx is an enemy of (or rejects) the robot xx.
- The robot Fx is the best friend of the robot x.
- The robot Sx is a friend of (or supports) the robot xx.

1. Find a robot that creates itself.
2. Find a robot that destroys itself.
3. Find a robot that is a friend of itself.
4. Find a robot that creates its best friend.
5. Find a robot that is a friend of its worst enemy.
6. Find a robot that is the best friend of one of its enemies.
7. Find two robots which create each other.
8. Find two robots which destroy each other.
9. Find two robots such that the first creates the second and the second destroys the first. (Find two solutions.)
10. Find two robots such that the first creates the second and the second is a friend of the first.
11. Find two robots such that the first is the best friend of the second and the second destroys the worst enemy of the first.
12. Find two robots such that the first creates the best friend of the second, while the second destroys the worst enemy of the first. (Find two solutions.)

Find the answers on the last page of this issue.

By Andy Liu.

PROBLEM SOLVING VIGNETTES

No. 19

Shawn Godin

Thinking Like a Mathematician

As a student of mathematics, it is easy to get a false sense of what mathematics is. We are taught theorems and techniques and we spend endless hours of homework practicing, honing our skills. Even those who venture into the world of mathematics competitions may be misled. In competitions, students do run into more questions that are genuine "problems" rather than just "exercises", but the process is the same: read and understand the problem, come up with and execute a plan, get an answer and check its reasonableness, then go on to the next problem. It is easy to think that a mathematician sits in his office, with a large list of problems to solve, and works through them one by one. However, for a mathematician, in many cases finding the answer to a question is where the real fun begins!

Let's consider a problem to illustrate. The following problem is from the 2020 *Canadian Mathematical Gray Jay Competition* (CMGC). The CMGC is a multiple choice competition from the CMS for elementary school students. The first competition was written on Thursday October 8, 2020 and the second will be written Thursday November 18, 2021.

11. *Alice types the fraction* $\frac{30}{37}$ *into an online calculator and it calculates the decimal form to thousands of decimal places. What is the sum of the first* 2020 *digits after the decimal?*
(A) *6060* **(B)** *6061* **(C)** *6062* **(D)** *6063* **(E)** *6064* **(F)** *6065*

Division yields

$$\frac{30}{37} = 0.810\,810\,810\,8\cdots = 0.\overline{810}$$

so the fraction is periodic with period 3. Since we are interested in the first 2020 digits after the decimal and $2020 \div 3 = 673\frac{1}{3}$, we can deduce that the first 2020 digits will be 673 copies of 810, followed by one 8. Since $673\times(8+1+0)+8 = 6065$, the answer is **(F)**, and we can go on to #12.

At this point a mathematician may pause to ask, and try to answer, some questions. That is, they would create their own investigation based on the problem at hand. A first question might be "what is the nature of the decimal representations of fractions with denominator 37?" Calculating the first few, we start to see some patterns:

$$\begin{aligned}\frac{1}{37} &= 0.027\,027\,0\cdots = 0.\overline{027}\\ \frac{2}{37} &= 0.054\,054\,0\cdots = 0.\overline{054}\\ \frac{3}{37} &= 0.081\,081\,0\cdots = 0.\overline{081}\\ \frac{4}{37} &= 0.108\,108\,1\cdots = 0.\overline{108}\end{aligned}$$

The first few decimal representations are periodic with period 3, just like the fraction in the problem. Taking a minute to think, we can convince ourselves that since $\frac{1}{37}$ is periodic, with period 3, then the other fractions will just be multiples of it. This seems to make sense as

$$\begin{aligned}54 &= 2 \times 27\\ 81 &= 3 \times 27\\ 108 &= 4 \times 27\\ 810 &= 30 \times 27.\end{aligned}$$

That is, the *repetend* (the repeating part of the decimal) of the decimal representation of $\frac{k}{37}$, with $0 < k < 37$ seems to be $27k$ (with possibly a 0 added to the front if $37k < 100$). The place where this seems to break down is at $\frac{37}{37} = 1$ and yet $27 \times 37 = 999$. That means our pattern suggests, $\frac{37}{37} = 0.\overline{999} = 0.999\,999\,9\ldots$ so there must be something wrong . . . or is there?

Let's examine this strange number and let $x = 0.999\,999\,9\ldots$. If we multiply by 10 and do a little algebra, we get

$$\begin{aligned}10x &= 9.999\,999\,9\ldots\\ x &= 0.999\,999\,9\ldots\end{aligned}$$

Subtracting these two equations yields $9x = 9$, or $x = 1$. It seems we have shown that $0.999\,999\,9\cdots = 1$! This turns out to be true. As such, many numbers have *two* decimal representations. For example

$$\begin{aligned}1.999\,999\,9\ldots &= 2,\\ 0.499\,999\,9\ldots &= 0.5,\\ 3.141\,499\,999\,9\ldots &= 3.1415.\end{aligned}$$

How do we deal with this? Mathematicians, being practical, would just use the "easiest" representation of a number, but keep in mind what we discovered so that if anything like that shows up again, we know how to handle it.

Next, a mathematician may ask themselves "what do other periodic decimals with period 3 look like?". Let's start with the decimal

$$\frac{4}{37} < 0.123\,123\,1\cdots < \frac{5}{37},$$

which is not one of the fractions we have investigated. Using the idea we used above (mathematicians love reusing ideas and techniques in new situations) when evaluating $0.999\,999\,9\ldots$, if we let $x = 0.123\,123\,1\ldots$, then

$$\begin{aligned} 1000x &= 123.123\,123\,1\ldots \\ x &= \quad 0.123\,123\,1\ldots \end{aligned}$$

it follows that

$$0.123\,123\,1\cdots = \frac{123}{999}$$

It would seem that if we have any decimal of the form $0.\overline{ABC}$, where A, B, and C are digits in the three-digit repetend, then

$$0.\overline{ABC} = \frac{ABC}{999}$$

where ABC is a three-digit number with digits A, B, and C. I will leave the details of this for your entertainment.

Next, we noticed that the original fractions we were considering had denominator 37, not 999. We also noticed that $27 \times 37 = 999$, so we could write them as fractions with denominator 999 since 37 is a factor of 999. We could then conjecture that fractions whose denominator is a factor of 999 are periodic with period 3. The factors of 999 are $1, 3, 9, 27, 37, 111, 333, 999$. Looking at the decimal representations of fractions with numerator 1 and whose denominator is a factor of 999, we get

$$\begin{aligned} \frac{1}{1} &= 1 & \qquad \frac{1}{999} &= 0.\overline{001} \\ \frac{1}{3} &= 0.\overline{3} & \frac{1}{333} &= 0.\overline{003} \\ \frac{1}{9} &= 0.\overline{1} & \frac{1}{111} &= 0.\overline{009} \\ \frac{1}{27} &= 0.\overline{037} & \frac{1}{37} &= 0.\overline{027} \end{aligned}$$

Our conjecture is false, in general, but seems to be true in many cases. If we rewrite

$$\frac{1}{3} = 0.\overline{3} = 0.333\,333\,3\cdots = 0.\overline{333}$$

and similarly $\frac{1}{9} = 0.\overline{111}$ we see that these two only fail because they are repeating with a *smaller* period. However, these two decimal representations can be thought of as periodic with period 3. Usually, where we talk about **the** period of something that is periodic we mean the *smallest* period. However, if we reexamine the definition of what it means to be periodic, then something that is periodic with period 3 is also periodic with period 6, 9, 12, ...

Finally, if we recall from earlier that $1 = 0.999\,999\,9\cdots = 0.\overline{9} = 0.\overline{999}$ we see that we were actually correct, as long as we realize when we say "period 3", it isn't necessarily the smallest period of the decimal representation.

You may have also noticed something else in the decimal representations written above. That is, if $a \times b = 999$ where a and b are positive integers, then

$$\frac{1}{a} = 0.\overline{B} \quad \text{and} \quad \frac{1}{b} = 0.\overline{A}$$

where A is the three digit number formed by the digits of a, with possibly one or two zeroes appended to the front to make it three digits in length (and similarly with B and b). That is, if $a = 27$, then $b = 37$, $A = 027$, and $B = 037$.

We may think we are done with periodic decimals with period 3, but what about things like $0.5\overline{197}$ and $0.29\overline{197}$? Clearly, these are not numbers that can be written as a proper fraction with denominator 999. However, if we play with things we will find

$$\begin{aligned} 0.5\overline{197} &= 0.5 + 0.0\overline{197} \\ &= 0.5 + \frac{1}{10} \times 0.\overline{197} \\ &= \frac{5}{10} + \frac{197}{9990} \\ &= \frac{5192}{9990} \end{aligned} \qquad \begin{aligned} 0.29\overline{197} &= 0.29 + 0.00\overline{197} \\ &= 0.29 + \frac{1}{100} \times 0.\overline{197} \\ &= \frac{29}{100} + \frac{197}{99\,900} \\ &= \frac{29\,168}{99\,900} \end{aligned}$$

We will find that if a decimal is periodic with period 3, where n decimals occur after the decimal point *before* the first collection of repeating decimals, then the number can be written as a fraction with denominator 999×10^n. I will leave the exploration and verification of this to the reader.

We seem to have completely investigated numbers whose decimal representation is periodic with period 3. Where do we go from there? The natural place would be to describe all periodic decimals. Many of you may already know the decimal representations that have period 1:

$$\begin{array}{lll} \frac{1}{9} = 0.111\cdots = 0.\overline{1} & \frac{2}{9} = 0.222\cdots = 0.\overline{2} & \frac{3}{9} = \frac{1}{3} = 0.333\cdots = 0.\overline{3} \\ \frac{4}{9} = 0.444\cdots = 0.\overline{4} & \frac{5}{9} = 0.555\cdots = 0.\overline{5} & \frac{6}{9} = \frac{2}{3} = 0.666\cdots = 0.\overline{6} \\ \frac{7}{9} = 0.777\cdots = 0.\overline{7} & \frac{8}{9} = 0.888\cdots = 0.\overline{8} & \end{array}$$

Recalling our earlier work we can generate some fractions that are periodic with period 1 where the repeating decimals start later in the representation. For example:

$$\begin{aligned} \frac{17}{90} &= 0.188\,888\cdots = 0.1\overline{8} \\ \frac{851}{900} &= 0.945\,555\cdots = 0.94\overline{5} \\ \frac{4431}{9000} &= 0.492\,333\cdots = 0.492\overline{3} \end{aligned}$$

To investigate representations that are periodic with period 2, we will look at an example $0.121\ 212\ 1\cdots = 0.\overline{12}$. Following our procedure from earlier, if we let x represent our number we get $100x = 12.\overline{12}$, and hence, after subtracting the original value, $99x = 12$ so $x = \frac{12}{99} = \frac{4}{33}$. The common thread seems to be the denominators 9, 99, 999, 9999, ..., and their factors. For the period 2 case we have

$$\begin{array}{lll} 1 \times 99 = 99 & \dfrac{1}{1} = 1 = 0.\overline{99} & \dfrac{1}{99} = 0.\overline{01} \\ 3 \times 33 = 99 & \dfrac{1}{3} = 0.\overline{33} & \dfrac{1}{33} = 0.\overline{03} \\ 9 \times 11 = 99 & \dfrac{1}{9} = 0.\overline{11} & \dfrac{1}{11} = 0.\overline{09} \end{array}$$

so we see that if a factor of 99 is our denominator, the fraction is periodic with period 2 (not necessarily the smallest period) and if $a \times b = 99$ then $\frac{1}{a} = 0.\overline{B}$ and $\frac{1}{b} = 0.\overline{A}$ where A and B are as defined earlier (refined for the period 2 case).

At this point in our investigation we can make a few conjectures. Noting that $\overbrace{999\ldots9}_{k\ 9\text{s}} = 10^k - 1$, we conjecture:

1. The periodic representation of $\frac{k}{10^n-1}$, where $k < 10^n - 1$, is periodic with period n (not necessarily the smallest).
2. The repetend of the decimal representation of $\frac{k}{10^n-1}$ is just the digits of k possibly with a number of zeros attached to the left to make the repetend n digits long.
3. If $d \mid (10^n - 1)$ then fractions with denominator d are periodic with period n (not necessarily the smallest).
4. If $a \times b = 10^n - 1$ then $\frac{1}{a} = 0.\overline{B}$ and $\frac{1}{b} = 0.\overline{A}$ where A and B are n digit repetends, that when treated as whole numbers satisfy $A = a$ and $B = b$ (i.e. they may have some "leading zeros").

If we think about it we can see that 1 implies 3, while 1 and 2 implies 4 (I will leave the details to the reader), so if we can prove the first two conjectures the last two must be true.

Playing with an example might give us a sense of what is going on. Consider $x = \frac{1234}{9999}$, then

$$\begin{aligned} 9999x &= 1234 \\ 10^4x - x &= 1234 \\ 10^4x &= 1234 + x \\ x &= \frac{1234}{10^4} + \frac{x}{10^4} \end{aligned}$$

Now, this equation isn't really useful in this form, as it defines x in terms of itself. However, if we consider this a recursive definition, we can substitute the equation back into itself over and over to get

$$\begin{aligned} x &= \frac{1234}{10^4} + \frac{\frac{1234}{10^4} + \frac{x}{10^4}}{10^4} = \frac{1234}{10^4} + \frac{1234}{10^8} + \frac{x}{10^8} \\ &= \frac{1234}{10^4} + \frac{1234}{10^8} + \frac{1234}{10^{12}} + \frac{x}{10^{12}} \\ &= \frac{1234}{10^4} + \frac{1234}{10^8} + \frac{1234}{10^{12}} + \frac{1234}{10^{16}} + \frac{x}{10^{16}} \\ &\vdots \\ &= \frac{1234}{10^4} + \frac{1234}{10^8} + \frac{1234}{10^{12}} + \frac{1234}{10^{16}} + \cdots = 0.\overline{1234} \end{aligned}$$

Now what we have done isn't really a proof, but we probably start to believe that conjectures 1 and 2 are indeed true. Applying the method we used earlier to turn repeating decimals into fractions we can construct a more solid proof. This is left to the reader.

The interesting thing that ends up happening is the more questions we answer, the more questions arise. As we investigate, if one question seems too hard, we can ask an easier related question or some other question. It is amazing the places you can end up if you follow this process. Next time you answer a question, ask your own questions and see how far you can take it!

Below are a few further investigations based on what we have done so far.

1. Determine under what conditions would a fraction have a terminating decimal like $\frac{1}{2} = 0.5$ and $\frac{57}{80} = 0.7125$.

2. Show that all primes except 2 and 5 divide many numbers of the form $10^n - 1$ (modular arithmetic will be useful). This can be used to deduce which fractions will yield periodic decimal representations.

3. Notice that $10^n - 1 = 9 \times \overbrace{111\ldots11}^{n\text{ 1s}}$, where $\overbrace{111\ldots11}^{n\text{ 1s}}$ is a n-digit *repunit* number (we talked about them in an earlier column [2019:45(6), p. 313-317]). Determine under what conditions will a repunit number be a factor of another repunit number.

4. Does our strange method of solving an equation work in general? That is, for the equation $9x = 18$, if we successively rewrote it as $10x = 18 + x$ and then $x = 1.8 + \frac{x}{10}$, would the recursive method give the correct result? Under what conditions would it work for a linear equation? (Note, a similar method can be used for quadratic equations that will lead to continued fraction expansions of some quadratic irrationals. If interested, you may want to research continued fractions.)

OLYMPIAD CORNER

No. 397

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **January 30, 2022**.

OC551. A semicircle k with diameter PQ is given. A chord BC of fixed length d is constructed on it, the endpoints of which are different from the points P and Q. From point B draw a ray so that this ray cuts the diameter PQ at a point A such that $\angle PAB = \angle QAC$. Prove that the magnitude of $\angle BAC$ does not depend on the position of the chord BC on the semicircle k.

OC552. Let a and b be two distinct positive real numbers. Consider the equation

$$\lfloor ax + b \rfloor = \lfloor bx + a \rfloor,$$

where $\lfloor y \rfloor$ denotes the integer part of the real number y. Prove that there exists an interval of length at least

$$\frac{1}{\max\{a, b\}}$$

all of whose points are solutions of the given equation.

OC553. Determine all the pairs of integers (a, b) such that $a^2 + 2b^2 + 2a + 1$ is a divisor of $2ab$.

OC554. Let $ABCD$ be a rectangle and let $E \in CD$ and $F \in AD$. The perpendicular line through point E to line FB intersects line BC at point P and the perpendicular line through point F to line EB intersects line AB at point Q. Prove that points P, D, Q are collinear.

OC555. Let $p > 3$ be a prime. Let K be the number of permutations $(a_1, a_2, \ldots, a_p)$ of $\{1, 2, \ldots, p\}$ such that

$$a_1a_2 + a_2a_3 + \cdots + a_{p-1}a_p + a_pa_1$$

is divisible by p. Prove $K + p$ is divisible by p^2.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 janvier 2022**.

OC551. Soit k un demi cercle de diamètre PQ. On y construit une corde BC d'une longueur prédéterminée d, dont les extrémités sont différentes de P et Q. À partir du point B, on trace un rayon qui rencontre le diamètre PQ en un point A tel que $\angle PAB = \angle QAC$. Démontrer que la mesure de l'angle $\angle BAC$ ne dépend pas des points où les extrémités de BC sont situées sur le demi cercle k.

OC552. Soient a et b deux nombres réels distincts positifs et soit

$$\lfloor ax + b \rfloor = \lfloor bx + a \rfloor,$$

une équation, où $\lfloor y \rfloor$ dénote la partie entière du nombre réel y. Démontrer qu'il existe un intervalle de longueur au moins

$$\frac{1}{\max\{a, b\}},$$

dont tous les points sont solutions de l'équation donnée.

OC553. Déterminer toutes les paires d'entiers (a, b), telles que a^2+2b^2+2a+1 est un diviseur de $2ab$.

OC554. Soit $ABCD$ un rectangle; soient aussi $E \in CD$ et $F \in AD$. La perpendiculaire vers la ligne FB au point E rencontre la ligne BC au point P; la perpendiculaire vers la ligne EB au point F rencontre la ligne AB au point Q. Démontrer que P, D et Q sont alignés.

OC555. Soit $p > 3$ un nombre premier. Aussi, soit K le nombre de permutations $(a_1, a_2, \ldots, a_p)$ de $\{1, 2, \ldots, p\}$ telles que

$$a_1a_2 + a_2a_3 + \cdots + a_{p-1}a_p + a_pa_1$$

est divisible par p. Démontrer que $K + p$ est divisible par p^2.

OLYMPIAD CORNER SOLUTIONS

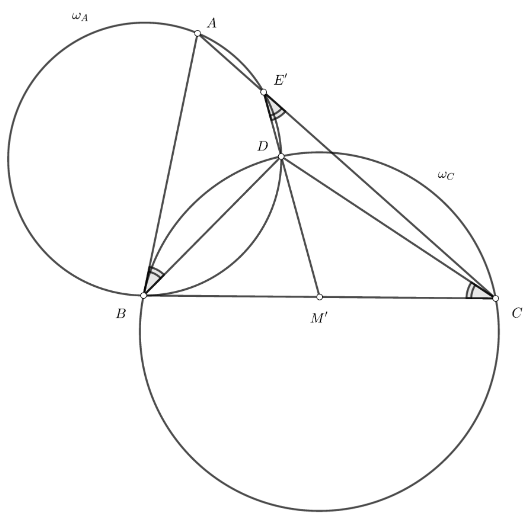
Statements of the problems in this section originally appear in 2021: 47(4), p. 178–179.

OC526. Let ABC be a triangle. The circle ω_A through A is tangent to line BC at B. The circle ω_C through C is tangent to line AB at B. Let ω_A and ω_C meet again at D. Let M be the midpoint of line segment BC, and let E be the intersection of lines MD and AC. Show that E lies on ω_A.

Originally Problem 3 from the 12th Benelux Mathematical Olympiad, May 2020.

We received 7 submissions, of which 6 were correct and complete. We present two solutions.

Solution 1, by UCLan Cyprus Problem Solving Group.



Let E' be the point of intersection of AC with ω_A and let M' be the point of intersection of $E'D$ with BC. It suffices to show that M' is the midpoint of BC.

Since A, B, D, E' are concyclic, then $\angle M'E'C = \angle DE'C = \angle DBA$. Since AB is tangent to ω_C, then $\angle DBA = \angle DCB$. Thus, $\angle M'E'C = \angle DCM'$. This together with the fact that $\angle CM'D$ and $\angle CM'E'$ are identical imply that $\triangle M'E'C$ and $\triangle M'CD$ are similar.

Thus, $M'C/M'D = M'E'/M'C$ or equivalently, $(M'C)^2 = (M'D)(M'E)$. But $(M'D)(M'E)$ is the power of M' with respect to ω_A. Since $M'B$ is tangent to ω_A the power of M' is equal to $(M'B)^2$, as well. It follows that $(M'B)^2 = (M'C)^2$ and $M' = M$, as required.

Solution 2, by Corneliu Manescu-Avram.

[Ed.: please note that the diagram from Solution 1 does not apply.]

We have $\angle CBD = \angle BAD$ and $\angle DBA = \angle DCB$ as in any circle the angle between a chord and a tangent through one of the end points of the chord is equal to the angle in the alternate segment. It follows that $\triangle DAB$ and $\triangle DBC$ are similar.

Let N be the midpoint of AB. Then $\triangle AND$ and $\triangle BMD$ are similar. Therefore,

$$\angle BND = 180^\circ - \angle DNA = 180^\circ - \angle DMB,$$

and the quadrilateral $BNDM$ is cyclic. However, $MN \parallel AC$ and

$$\angle DBA = \angle DBN = \angle DMN = \angle EMN = \angle MEC = 180^\circ - \angle DEA.$$

In conclusion, the quadrilateral $EADB$ is cyclic, and E lies on ω_A.

OC527. Anna and Boris play a game with n counters. Anna goes first, and turns alternate thereafter. In each move, a player takes either 1 counter or a number of counters equal to a prime divisor of the remaining number of counters. The player who takes the last counter wins. For which n does Anna have a winning strategy?

Originally from the Tournament of Towns Junior O-level Contest, in Fall 2020, proposed by Fedor Ivlev.

We received 8 submissions, all of which were correct. We present a typical solution.

The first player, Anna, has a winning strategy if n is not divisible by 4. The second player, Boris, has a winning strategy if n is a multiple of 4. Let $P(n)$ be the respective statement for a given natural number n. We prove $P(n)$ by complete induction on n.

If $n \leq 3$ then the first player wins immediately by taking all counters. Let $n \geq 4$. To complete the induction proof, we need to establish that $P(n)$ holds assuming that $P(k)$ are true for $1 \leq k \leq n-1$. Note that the first player always takes a number of counters, c, that is not divisible by 4.

If n is a multiple of 4, the number of counters, $n-c$, left for the second player is not divisible by 4. Based on the induction hypothesis, $P(n-c)$, the second player has a strategy to win, and so the first player loses.

If n is not a multiple of 4, the first player aims to take a number of counters such that $n-c$ is a multiple of 4. This is possible. Indeed, if $n \equiv 1 \bmod 4$, the first player removes 1 counter. If $n \equiv 2 \bmod 4$, the first player removes 2 counters. If $n \equiv 3 \bmod 4$, then n must have a prime divisor p of the form $p \equiv 3 \bmod 4$. The first player then removes p counters. Based on the induction hypothesis, $P(n-c)$, the second player does not have a strategy to win, and so the first player wins.

OC528. Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers, whose odd-indexed terms are real, even-indexed terms are purely imaginary, and for every positive integer k, $|z_k z_{k+1}| = 2^k$. Denote $f_n = |z_1 + z_2 + \cdots + z_n|$, for $n = 1, 2, \ldots$.

(1) Find the minimum of f_{2020}.

(2) Find the minimum of $f_{2020} \cdot f_{2021}$.

Originally Problem 1 from the 2021 China National Olympiad, Day 1.

We received 6 submissions, of which 5 were correct and complete. We present the solution by Oliver Geupel.

For convenience let us write $c = z_1$. A simple proof by induction gives that for any $n = 0, 1, \ldots$: $z_{2n+1} = 2^n \sigma_n c$ and $z_{2n+2} = 2^{n+1}\tau_n i/c$ for some $\sigma_n, \tau_n \in \{-1, 1\}$.

Let $k = \sum_{n=0}^{1009} 2^n \sigma_n$ and $\ell = \sum_{n=0}^{1009} 2^n \tau_n$. We have $|k| \geq 1$, $|\ell| \geq 1$, and

$$z_1 + z_2 + \cdots + z_{2020} = ck + \frac{2}{c}\ell i.$$

Then,

$$f_{2020}^2 = c^2k^2 + \frac{4}{c^2}\ell^2 \geq c^2 + \frac{4}{c^2} = 4 + \left(c - \frac{2}{c}\right)^2 \geq 4.$$

Thus, $f_{2020} \geq 2$.

The equality holds when $c = \sqrt{2}$ and $k = l = 1$ implied by the selection $\sigma_0 = \sigma_1 = \cdots = \sigma_{1008} = -1$, $\tau_0 = \tau_1 = \cdots = \tau_{1008} = -1$, and $\sigma_{1009} = \tau_{1009} = 1$. Thus, the minimum of f_{2020} is 2, which completes part (1).

Let $m = k + 2^{1010}\sigma_{1010}$, so that $|m| \geq 1$ and $z_1 + z_2 + \cdots + z_{2021} = cm + \frac{2}{c}\ell i$. Then,

$$\begin{aligned} f_{2020}^2 f_{2021}^2 &= \left(c^2k^2 + \frac{4\ell^2}{c^2}\right)\left(c^2m^2 + \frac{4\ell^2}{c^2}\right) \geq \left(c^2k^2 + \frac{4}{c^2}\right)\left(c^2m^2 + \frac{4}{c^2}\right) \\ &= 4(k-m)^2 + \left(c^2km + \frac{4}{c^2}\right)^2 \geq 4(k-m)^2 = 4 \cdot 2^{1010\cdot 2} = 2^{2022}. \end{aligned}$$

Thus, $f_{2020}f_{2021} \geq 2^{1011}$.

The equality holds if k and m have opposite signs, $c = \sqrt[4]{-\frac{4}{km}}$, and $l = 1$. This can be achieved with the selection $\sigma_0 = \sigma_1 = \cdots = \sigma_{1008} = 1$, $\sigma_{1009} = -1$, $\sigma_{1010} = 1$, $\tau_0 = \tau_1 = \cdots = \tau_{1008} = -1$, and $\tau_{1009} = 1$. Indeed, $k = -1$, $l = 1$, $m = -1 + 2^{1010} > 0$, and

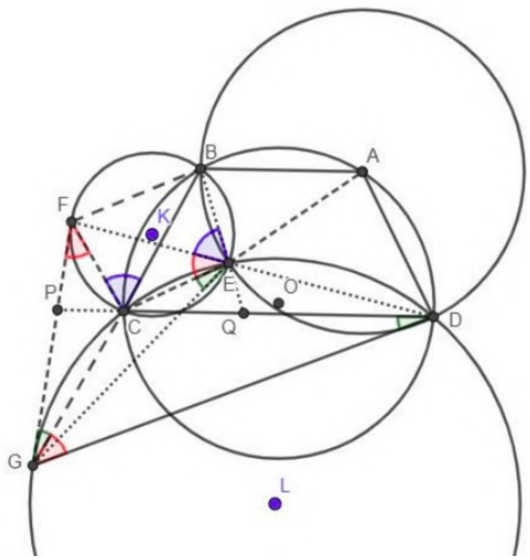
$$\begin{aligned} f_{2020}^2 f_{2021}^2 &= \left(c^2k^2 + \frac{4\ell^2}{c^2}\right)\left(c^2m^2 + \frac{4\ell^2}{c^2}\right) = c^4k^2m^2 + \frac{16\ell^4}{c^4} + 4\ell^2m^2 + 4\ell^2k^2 \\ &= -4km - 4\ell^4 km + 4\ell^2(m^2 + k^2) = 4(k-m)^2 = 2^{2022}. \end{aligned}$$

Consequently, the minimum of $f_{2020}f_{2021}$ is 2^{1011}, and part (2) is complete.

OC529. Let $ABCD$ be a cyclic quadrilateral with E, an interior point such that $AB = AD = AE = BC$. Let DE meet the circumcircle of BEC again at F. Suppose a common tangent to the circumcircle of BEC and DEC touches the circles at F and G respectively. Show that GE is the external angle bisector of angle BEF.

Originally Problem 5 from the 2021 Nigerian Senior MO, Round 2.

We received 2 submissions, both correct and complete. We present the solution by Theo Koupelis.



The points B, E, D are on a circle of center A and radius AB, and thus

$$\angle FEB = 180^\circ - \angle BED = \frac{1}{2}\angle BAD.$$

Also, points B, E, C, F are concyclic, and therefore,

$$\angle FCB = \angle FEB = \frac{1}{2}\angle BAD.$$

Now let $\angle GFC = \theta$ and $\angle FGC = \phi$. Then $\angle CEF = \theta$, because GF is tangent to the circumcircle of BEC. Also $\angle CEG = \phi$, because FG is tangent to the circumcircle of DEC. Thus $\angle FEG = \theta + \phi$. The quadrilateral $DECG$ is cyclic and therefore $\angle CGD = \angle CEF = \theta$, and $\angle CDG = \angle CEG = \phi$. Thus, $\triangle FCG$ and $\triangle GCD$ are similar; therefore,

$$\angle FCG = \angle GCD = 180^\circ - (\theta + \phi),$$

and thus $\angle FCD = 2(\theta + \phi) = 2\angle FEG$.

Let P, Q be the intersection points of DC with FG and BE, respectively. The cyclic quadrilateral $ABCD$ is an isosceles trapezium because $BC = AD$ and therefore $\angle BAD = \angle ABC = \angle PCB$. But $\angle FCB = \frac{1}{2}\angle BAD$, and thus

$$\angle PCF = \angle PCB - \angle FCB = \frac{1}{2}\angle BAD = \angle FEB.$$

Therefore,

$$\begin{aligned} 180^\circ &= \angle PCF + \angle FCD = \angle FEB + 2\angle FEG = \angle GEB + \angle FEG \\ &= 180^\circ - \angle GEQ + \angle FEG. \end{aligned}$$

Thus, $\angle GEQ = \angle GEF$, and so GE is the external angle bisector of angle BEF.

OC530. Let $n > 3$ be a fixed integer and $x_1, x_2, \dots, x_n$ positive real numbers. Find in terms of n, all possible values of

$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-2} + x_{n-1} + x_n} + \frac{x_n}{x_{n-1} + x_n + x_1}.$$

Originally Problem 6 from the 31st Brazilian Mathematical Olympiad, 2009, Third Round, Second Day.

We received 2 submissions, of which 1 was correct and complete. We present an edited version of the solution by UCLan Cyprus Problem Solving Group.

We show that the expression can take any value in $(1, \lfloor (n-1)/2 \rfloor)$, where by $\lfloor x \rfloor$ we mean the greatest integer that is less than or equal to x. Throughout the proof we use the convention that an index strictly greater than n is understood as modulo n; for example by x_{n+2} we mean x_2. We denote by $f(x_1, \dots, x_n)$ the expression described in the question.

We start by proving the following preparatory lemma.

Lemma. Assume $x_1, \dots, x_n$ is a sequence of at least 4 positive integers. The following facts hold.

(a) There is an integer k, $1 \le k \le n$, such that $x_k \le x_{k+3}$ and $x_{k+4} \le x_{k+1}$.

(b) There is an integer k, $1 \le k \le n$, such that $x_{k+3} \le x_k$ and $x_{k+1} \le x_{k+4}$.

Proof. We can assume without loss of generality that x_1 is the smallest of $x_1, \dots, x_n$, so $x_1 \le x_4$. Further, assume that $x_{l+1} < x_{l+4}$ for any $1 \le l \le n-1$. Summing all n inequalities we obtain the contradiction that $x_1 + \cdots + x_n < x_1 + \cdots + x_n$. Therefore, there exists at least one l, $1 \le l \le n-1$, such that $x_{l+4} \le x_{l+1}$. Let k be the smallest such l. As we assumed that $x_1 \le x_4$, we must have that $x_k \le x_{k+3}$ and $x_{k+4} \le x_{k+1}$. This concludes part (a). Part (b) can be established using a similar proof by contradiction.

We continue by establishing that the lower bound of the expression is 1. Let k be the index identified in the Lemma, part (a). Then

$$\frac{x_{k+1}}{x_k + x_{k+1} + x_{k+2}} + \frac{x_{k+2}}{x_{k+1} + x_{k+2} + x_{k+3}} + \frac{x_{k+3}}{x_{k+2} + x_{k+3} + x_{k+4}} \geqslant \frac{x_{k+1} + x_{k+2} + x_{k+3}}{x_{k+1} + x_{k+2} + x_{k+3}} = 1,$$

and $f(x_1, \ldots, x_n) \geq 1$. Since f has four or more strictly positive terms, it follows that $f(x_1, \ldots, x_n) > 1$ for all $x_1 \geq 0, \ldots, x_n \geq 0$. Therefore, 1 is a lower bound for the expression. In fact, 1 is the largest lower bound. This is because

$$f(a, a^2, \ldots, a^n) = \frac{a}{a^n + a + a^2} + \frac{(n-2)a}{1 + a + a^2} + \frac{a^n}{a^{n-1} + a^n + a}$$

converges to 1 as a approaches infinity.

For the upper bound, we consider separately the cases of odd n and even n. Note that for any $x > 0$, $y > 0$, $z > 0$, $w > 0$ we have the inequality

$$\frac{y}{x+y+z} + \frac{z}{y+z+w} < \frac{y}{y+z} + \frac{z}{y+z} = 1\,.$$

If n is even, by pairing consecutive terms and using the above inequality we get that $f(x_1, \ldots, x_n) < n/2$. Moreover, since

$$f(1, a, \ldots, 1, a) = \frac{n}{2} \times \frac{1}{2a+1} + \frac{n}{2} \times \frac{a}{a+2}$$

converges to $n/2$ as a converges to infinity, it follows that $n/2$ is the smallest upper bound.

If n is odd, we select the index identified in the Lemma, part (b) and obtain:

$$\frac{x_{k+1}}{x_k + x_{k+1} + x_{k+2}} + \frac{x_{k+2}}{x_{k+1} + x_{k+2} + x_{k+3}} + \frac{x_{k+3}}{x_{k+2} + x_{k+3} + x_{k+4}} \leqslant \frac{x_{k+1} + x_{k+2} + x_{k+3}}{x_{k+1} + x_{k+2} + x_{k+3}} = 1.$$

By grouping the terms of the expression f into a triplet, the triplet on the previous line, and pairs of consecutive terms we obtain that $f(x_1, \ldots, x_n) < (n-1)/2$. In addition,

$$f(1, a, \ldots, 1, a, 1) = \frac{n-3}{2} \times \frac{1}{2a+1} + \frac{n-1}{2} \times \frac{a}{a+2} + 2 \times \frac{1}{a+2}$$

converges to $\frac{n-1}{2}$ as a approaches infinity. This shows that for odd n, $(n-1)/2$ is the smallest upper bound.

To complete the proof, we note that f is a continuous function of $x_1, \ldots, x_n$. As a continuous function, f maps a connected set into a connected set. Therefore, the image of f must be an interval, the open interval $(1, \lfloor (n-1)/2 \rfloor)$.

FOCUS ON...

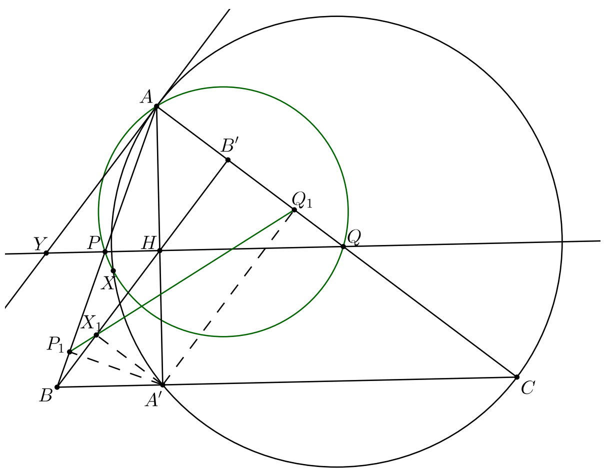
No. 48

Michel Bataille

Solutions to Exercises from Focus On... No. 42 - 46

From Focus On... No. 42

1. *Let the line parallel to BC through H intersect AB at P and AC at Q and let the perpendicular to AC through A intersect the line PQ at Y. Let X be the point of intersection other than A of the circle with diameter AC and the circumcircle of ΔAPQ. Prove that C, X, Y are collinear.*

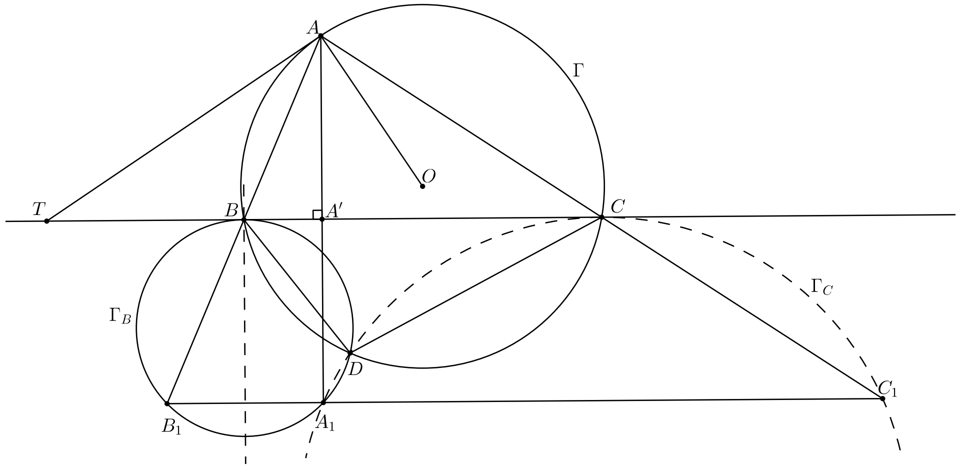


For $Z \neq A$, let Γ_z denote the circle with diameter AZ. The circles $\Gamma_y, \Gamma_p, \Gamma_q$, which all pass through H, invert into the lines through A' perpendicular to AY, AP, AQ, respectively. It follows that $P_1 = \mathbf{I}(P)$ (resp. $Q_1 = \mathbf{I}(Q)$) is the projection of A' onto AB (resp. AC) and that the line $\mathbf{I}(\Gamma_y)$ is perpendicular to BB' (since AY and BB' are parallel). Let $\mathbf{I}(\Gamma_y)$ and BB' intersect at X_1. Since A' is on the circumcircle Γ_b of $\Delta ABB'$, the points P_1, X_1, Q_1 are collinear (on the Simson line of A') and therefore $\mathbf{I}(X_1)$ is on the circumcircle of ΔAPQ (the inverse of the line P_1Q_1) and on $\Gamma_c = \mathbf{I}(BB')$. We deduce that $\mathbf{I}(X_1) = X$. In consequence, X being on Γ_c and on Γ_y, the lines YX, CX coincide (they are both perpendicular to AX at X).

2. *Let A_1 be the reflection of A in the line BC. The circle passing through A_1 and tangent to BC at B intersects Γ at D ($D \neq B$). Prove that $TD = TA$.*

Let Γ_B (resp. Γ_C) be the circle through A_1 and tangent to BC at B (resp. C). We introduce the reflections B_1 and C_1 of A about B and C, respectively. Note that B_1 is also the reflection of A_1 in the diameter of Γ_B through B, hence $B_1 \in \Gamma_B$.

Similarly, $C_1 \in \Gamma_C$.



Note also that B_1, A_1, C_1 are on the parallel to BC through A_1 (since they are the images of B, A', C under the homothety with centre A and factor 2) and that Γ_C is the circumcircle of ΔA_1CC_1.

Since $\mathbf{J}(\Gamma_B)$ is a circle tangent to $BC = \mathbf{J}(BC)$ at $C = \mathbf{J}(B)$, passing through $A_1 = \mathbf{J}(A_1)$ (note that $TA = TA_1$), we obtain $\mathbf{J}(\Gamma_B) = \Gamma_C$.

From $\angle(DB, DA_1) = \angle(B_1B, B_1A_1) = \angle(BA, BC)$, we deduce that

$$\begin{aligned}\angle(DA_1, DC) &= \angle(DA_1, DB) + \angle(DB, DC)\\ &= \angle(BC, BA) + \angle(AB, AC)\\ &= \angle(CB, CA) = \angle(C_1A_1, C_1C)\end{aligned}$$

and therefore D is on Γ_C.

Now, $\mathbf{J}(D)$ is on Γ_C (since D is on Γ_B) and also on $\Gamma = \mathbf{J}(\Gamma)$. In addition, $\mathbf{J}(D) \neq C$ (since $D \neq B$), hence $\mathbf{J}(D) = D$ and $TD = TA$ follows.

From Focus On... No. 44

1. *Let u be a complex number with $|u| = 1$. Show that the solutions to the equation*

$$z^2 - 2z(1-u) - u = 0$$

are unimodular if and only if $|1 - u| \leq 1$.

Let z_1, z_2 be the solutions of $z^2 - 2z(1-u) - u = 0$.

First, suppose that $|z_1| = |z_2| = 1$. Then we have

$$2|1-u| = |z_1 + z_2| \leq |z_1| + |z_2| \leq 2,$$

hence $|1-u| \leq 1$.

Conversely, suppose that $|1-u| \leq 1$. Since $|z_1||z_2| = |z_1 z_2| = |-u| = 1$, we have $z_1 = re^{i\alpha_1}$, $z_2 = \frac{1}{r}e^{i\alpha_2}$ for some real numbers r, α_1, α_2 with $r > 0$. From $z_1 + z_2 = 2 - 2u = 2 + 2z_1 z_2$, we obtain

$$re^{i\alpha_1} + \frac{1}{r}e^{i\alpha_2} = 2 + 2e^{i(\alpha_1+\alpha_2)}.$$

Multiplying both sides by $\exp\left(-\frac{i(\alpha_1+\alpha_2)}{2}\right)$ and setting $w = r\exp\left(\frac{i(\alpha_1-\alpha_2)}{2}\right)$, we readily obtain that $w + \frac{1}{w}$ is a real number. Thus, $w + \frac{1}{w} = \overline{w} + \frac{1}{\overline{w}}$, that is, $(w - \overline{w})(|w|^2 - 1) = 0$.

If $|w|^2 = 1$, then $r = 1$ and we are done. If $w - \overline{w} = 0$, then $\sin\left(\frac{\alpha_1-\alpha_2}{2}\right) = 0$, hence $e^{i\alpha_1} = e^{i\alpha_2}$. But this implies

$$r + \frac{1}{r} = |2(1-u)| \leq 2, \quad \text{or} \quad (r-1)^2 \leq 0$$

and therefore $r = 1$ again.

2. *Let x, y, z, a, b be positive real numbers satisfying*

$$\begin{cases} x^2 + xy + y^2 &= a^2 \\ y^2 + yz + z^2 &= b^2 \\ z^2 + zx + x^2 &= a^2 + b^2. \end{cases}$$

Express $s = x + y + z$ as a function of a and b.

Let $s = x + y + z$. Subtracting each of the first two equations from the third, we obtain $z - y = \frac{b^2}{s}$ and $x - y = \frac{a^2}{s}$, hence we also have $z - x = \frac{b^2 - a^2}{s}$. It follows that

$$zx - y^2 = s(z-y) + xy - z^2 = b^2 + xy - z^2 \text{ and } zx - y^2 = s(x-y) + yz - x^2 = a^2 + yz - x^2.$$

Since $zx - y^2 = ys$ (by addition of the first two equations, taking the third into account), we deduce that

$$\begin{aligned} 3ys = 3(zx - y^2) &= zx - y^2 + b^2 + xy - z^2 + a^2 + yz - x^2 \\ &= a^2 + b^2 - (x^2 + y^2 + z^2 - (xy + yz + zx)). \end{aligned} \tag{1}$$

But

$$2(x^2+y^2+z^2-(xy+yz+zx)) = (x-y)^2+(y-z)^2+(z-x)^2 = \frac{a^4 + b^4 + (b^2 - a^2)^2}{s^2}$$

and

$$s - 3y = x - y + z - y = \frac{a^2 + b^2}{s},$$

hence $3ys = s^2 - (a^2 + b^2)$, so that (1) becomes

$$s^2 - (a^2 + b^2) = (a^2 + b^2) - \frac{a^4 + b^4 + (b^2 - a^2)^2}{2s^2},$$

which leads to the biquadratic equation

$$s^4 - 2(a^2 + b^2)s^2 + (a^4 + b^4 - a^2b^2) = 0.$$

Since $s > 0$ and $s^2 > a^2 + b^2$ (since $s^2 = a^2 + b^2 + 3ys$), solving this equation provides

$$s = \sqrt{a^2 + b^2 + ab\sqrt{3}}.$$

Note. The courageous readers will deduce that $y = \frac{ab\sqrt{3}}{3\sqrt{a^2+b^2+ab\sqrt{3}}}$, and then that

$$x = \frac{a\sqrt{3}(b + a\sqrt{3})}{3\sqrt{a^2 + b^2 + ab\sqrt{3}}}, \quad z = \frac{b\sqrt{3}(a + b\sqrt{3})}{3\sqrt{a^2 + b^2 + ab\sqrt{3}}}.$$

The most courageous will check that this triple (x, y, z) is indeed a solution to the system.

From Focus On... No. 45

1. *Let p be a real number such that $\frac{13}{32} < p < 8$. Prove that the equation $x^3 - 6x^2 + 2(p-2)x - p = 0$ has only one real solution.*

Let

$$P(x) = x^3 - 6x^2 + 2(p-2)x - p.$$

The discriminant of the quadratic polynomial $P'(x) = 3x^2 - 12x + 2p - 4$ is $24(8-p) > 0$, hence P' has two distinct real roots, say u, v with $u < v$. Since $P'(2) = 2(p-8) < 0$, we have $u < 2 < v$ and since P is a decreasing function on $[u, v]$, we have $P(v) < P(2) = 3(p-8) < 0$. As a result, P has a real root in (v, ∞) and from the variations of P, we deduce that it is sufficient to prove that $P(u) < 0$ (since then $P(x) \leq P(u) < 0$ for $x \in (-\infty, v]$).

It is readily checked that

$$3P(x) = (x-2)P'(x) + (p-8)(4x+1)$$

and it follows that $3P(u) = (p-8)(4u+1)$. But from $P'(-1/4) = 2p - \frac{13}{16} > 0$ and $-\frac{1}{4} < 2$, we see that we must even have $-\frac{1}{4} < u$. In consequence, we have $3P(u) < 0$, as desired.

2. *Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$ where a, b, c, d are complex numbers. Prove that the sum of two of its roots is equal to the sum of the two remaining roots if and only if P' and P''' have a common root. Application: find the roots of the polynomial $x^4 + 2x^3 + 2x^2 + x + \frac{1}{16}$*

Let x_1, x_2, x_3, x_4 be the list of the roots of $P(x)$ and suppose that $x_1+x_2 = x_3+x_4$. Let $p_1 = x_1x_2$ and $p_2 = x_3x_4$. Then, setting $s = x_1 + x_2 = x_3 + x_4$, we obtain that

$$P(x) = (x^2 - sx + p_1)(x^2 - sx + p_2)$$

and simple calculations give

$$\begin{aligned} P'(x) &= (2x-s)(2x^2-2sx+p_1+p_2), \\ P'''(x) &= 12(2x-s). \end{aligned}$$

Thus, P' and P''' have the common root $\frac{s}{2}$.

Conversely, suppose that $P'(a) = P'''(a) = 0$ for some complex number a. Taylor's formula yields

$$P(x+a) = P(a) + \frac{P''(a)}{2}x^2 + x^4,$$

hence

$$P(x+a) = (x^2-\alpha^2)(x^2-\beta^2)$$

for some $\alpha, \beta \in \mathbb{C}$. It follows that the roots of $P(x)$ are $\alpha+a, -\alpha+a, \beta+a, -\beta+a$. Clearly, the sum of the first two roots is equal to the sum of the two remaining roots.

Application: If $P(x) = x^4+2x^3+2x^2+x+\frac{1}{16}$, then it is readily checked that $P'(-1/2) = P'''(-1/2) = 0$. Inspired by the proof above, we form $P(x-1/2)$:

$$P(x-1/2) = x^4 + \frac{x^2}{2} - \frac{1}{8}.$$

The equation $P(x-1/2) = 0$ is easily solved and we deduce that the roots of $P(x)$ are

$$\frac{-1+\sqrt{\sqrt{3}-1}}{2}, \quad \frac{-1-\sqrt{\sqrt{3}-1}}{2}, \quad \frac{-1+i\sqrt{\sqrt{3}+1}}{2}, \quad \frac{-1-i\sqrt{\sqrt{3}+1}}{2}.$$

From Focus On... No. 46

1. *Use the Cesàro-Stolz theorem to prove that* $\displaystyle\sum_{k=1}^{n}\frac{1}{n^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}$ *when* $0<\alpha<1$.

On the one hand, we have

$$\frac{1}{1-\alpha}\sum_{k=1}^{n}(k^{1-\alpha}-(k-1)^{1-\alpha}) = \frac{n^{1-\alpha}}{1-\alpha}$$

and on the other hand,

$$n^{1-\alpha}-(n-1)^{1-\alpha} = n^{1-\alpha}\left(1-\left(1-\frac{1}{n}\right)^{1-\alpha}\right) = n^{1-\alpha}\left(\frac{1-\alpha}{n}+o(1/n)\right) \sim \frac{1-\alpha}{n^\alpha}.$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^\alpha}$ is a divergent series, the Cesàro-Stolz theorem gives

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}.$$

2. *Prove that* $\sqrt[n]{n!} = \frac{n}{e} + \frac{1}{2e}\ln(n) + o(\ln n)$.

We deduce the desired result from the following calculations:

$$\begin{aligned}\sqrt[n]{n!} = \exp\left(\frac{\ln(n!)}{n}\right) &= \exp\left(\ln(n) - 1 + \frac{\ln(n)}{2n} + o(\ln(n)/n)\right)\\ &= \frac{n}{e}\exp\left(\frac{\ln(n)}{2n} + o(\ln(n)/n)\right)\\ &= \frac{n}{e}\left(1 + \frac{\ln(n)}{2n} + o(\ln(n)/n)\right)\\ &= \frac{n}{e} + \frac{\ln(n)}{2e} + o(\ln(n)).\end{aligned}$$

3. *Let* $n \in \mathbb{N}$ *and let* $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$. *Calculate*

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n.$$

Let $U_n = \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n$. We show that $\lim\limits_{n\to\infty} U_n = 4e^{\gamma}$ where γ denotes the Euler constant.

We have

$$\begin{aligned}2O_n &= 2\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \frac{1}{2n} - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right)\right)\\ &= 2H_{2n} - H_n = 2(\ln(2n) + \gamma + o(1)) - (\ln(n) + \gamma + o(1))\\ &= \ln(n) + 2\ln(2) + \gamma + o(1).\end{aligned}$$

Thus, $1 + \frac{2O_n}{n} = 1 + a_n$ where $a_n = \frac{\ln(n)}{n} + \frac{2\ln(2)+\gamma}{n} + o(1/n)$. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\ln(1+x) = x + O(x^2)$ as $x \to 0$, we see that

$$n\ln\left(1 + \frac{2O_n}{n}\right) = na_n + na_n^2\alpha_n$$

for some bounded sequence (α_n).

Now, $na_n^2 \sim \frac{(\ln(n))^2}{n}$ so that $\lim\limits_{n\to\infty} na_n^2 = 0 = \lim\limits_{n\to\infty} na_n^2\alpha_n$ (since (α_n) is bounded). We deduce that

$$n\ln\left(1 + \frac{2O_n}{n}\right) = \ln(n) + (2\ln(2) + \gamma) + o(1),$$

whence

$$\ln(U_n) = n\ln\left(1 + \frac{2O_n}{n}\right) - \ln(n) = 2\ln(2) + \gamma + o(1).$$

Thus, $\lim\limits_{n\to\infty} \ln(U_n) = \ln(4) + \gamma$ and the announced result follows.

4. *For each positive integer n, let*

$$s_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

and $\lim\limits_{n\to\infty} s_n = s$, *the Ioachimescu constant. Find* $\lim\limits_{n\to\infty}(s_n - s)\sqrt[2n]{n!}$.

The required limit is $\frac{1}{2\sqrt{e}}$.

First, we have

$$\sqrt[2n]{n!} \sim \sqrt{\frac{n}{e}}. \qquad (2)$$

Second, let $t_1 = s_1$ and for $n \geq 2$, $t_n = s_n - s_{n-1}$. Then, for all $n \geq 1$, we have

$$t_n = \frac{1}{\sqrt{n}} - 2(\sqrt{n} - \sqrt{n-1}) = \frac{1}{\sqrt{n}} - \frac{2}{\sqrt{n} + \sqrt{n-1}} = \frac{1}{\sqrt{n}}\left(1 - 2\left(1 + \left(1 - \frac{1}{n}\right)^{1/2}\right)^{-1}\right)$$

and

$$\begin{aligned} 1 - 2\left(1 + \left(1 - \frac{1}{n}\right)^{1/2}\right)^{-1} &= 1 - 2\left(1 + 1 - \frac{1}{2n} + o(1/n)\right)^{-1} \\ &= 1 - \left(1 - \frac{1}{4n} + o(1/n)\right)^{-1} \\ &= 1 - \left(1 + \frac{1}{4n} + o(1/n)\right) = -\frac{1}{4n} + o(1/n). \end{aligned}$$

so that

$$t_n \sim -\frac{1}{4n\sqrt{n}}.$$

This result, besides confirming the convergence of the series $\sum\limits_{k=1}^{\infty} t_k$ (clearly we have $\sum\limits_{k=1}^{\infty} t_k = s_1 + \sum\limits_{k=2}^{\infty}(s_k - s_{k-1}) = \lim\limits_{n\to\infty} s_n = s$), gives

$$\sum_{k=n+1}^{\infty} t_k \sim -\frac{1}{4}\sum_{k=n+1}^{\infty} \frac{1}{k^{3/2}} \sim -\frac{1}{4} \cdot \frac{2}{\sqrt{n}}.$$

It follows that

$$s - \sum_{k=1}^{n} t_k = s - s_n \sim \frac{-1}{2\sqrt{n}}. \qquad (3)$$

Now, from (2) and (3), we obtain

$$(s_n - s)\sqrt[2n]{n!} \sim \frac{1}{2\sqrt{n}} \cdot \frac{\sqrt{n}}{\sqrt{e}} = \frac{1}{2\sqrt{e}}.$$

Unexpected Applications of Newton's Theorem

Aditya Khurmi, Aatman Supkar, Arindam Bhattacharyya and Shuborno Das

1 Newton's Theorem

We found the name "Newton's Theorem" given to the following result in *Lemmas in Olympiad Geometry* by Cosmin Pohoata, Sam Korsky, and Titu Andreescu (it can also be found at `https://www.math.ust.hk/excalibur/v10_n3.pdf`:

Theorem 1 *(Newton's Theorem) Let $ABCD$ be a quadrilateral which has an inscribed circle ω. Let M, N, P, Q be the tangency points of ω with AB, CD, DA, BC, respectively. Then $\{MP, NQ, BD\}$ and $\{MN, PQ, AC, BD\}$ are 2 tuples of concurrent lines.*

We talk about a result that is just the second concurrency, but present it in a way that shows how powerful it can be when used properly. In this article, whenever we use the term "Newton's Theorem", we would mean the following lemma instead of the statement given above:

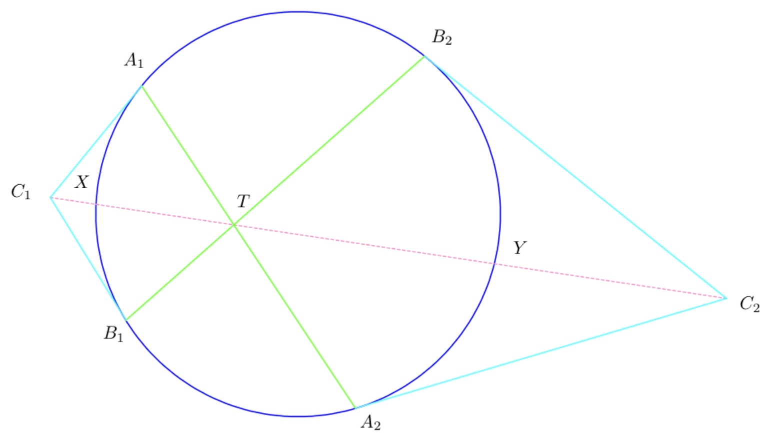


Figure 1: Newton's Theorem (the case when γ is a circle)

Lemma 1 *(Newton's Theorem) Let A_1, B_1, A_2, B_2 be points on a conic γ. Let C_1, C_2 be the intersection of tangents to γ at $\{A_1, B_1\}$ and $\{A_2, B_2\}$ respectively. Let $T = A_1A_2 \cap B_1B_2$. Then C_1, T, C_2 are collinear.*

Proof 1. Apply Pascal's theorem on $A_1A_1A_2B_1B_1B_2$ to get C_1, T and $A_2B_1 \cap B_2A_1$ are collinear. Then apply Pascal's theorem on $A_2A_2A_1B_2B_2B_1$ to get C_2, T and $A_2B_1 \cap B_2A_1$ are collinear. This proves the result. □

Proof 2. Let C_1T meet γ in X, Y. Then A_1XB_1Y is a harmonic quadrilateral and so

$$-1 = (A_1, B_1; X, Y) \stackrel{T}{=} (A_2, B_2; Y, X).$$

Hence, $(A_2, B_2; Y, X)$ is also harmonic, therefore implying that $C_2 = A_2A_2 \cap B_2B_2$ lies on $YX \equiv C_1T$. □

We make some remarks to show its usefulness in practice:

1. Say that C_2 is a point on the tangent at A_2. Then if C_1C_2, A_1A_2, B_1B_2 concur, then C_2B_2 is also a tangent. This point is especially useful in showing tangencies.

2. Since the proof was projective, the order of A_1, A_2, B_1, B_2 does not matter. So $A_1A_2 \cap B_1B_2$ can be inside γ, or even outside it.

3. Even though the result is stated for a conic, the case when γ is a circle would be more useful for standard olympiad problems.

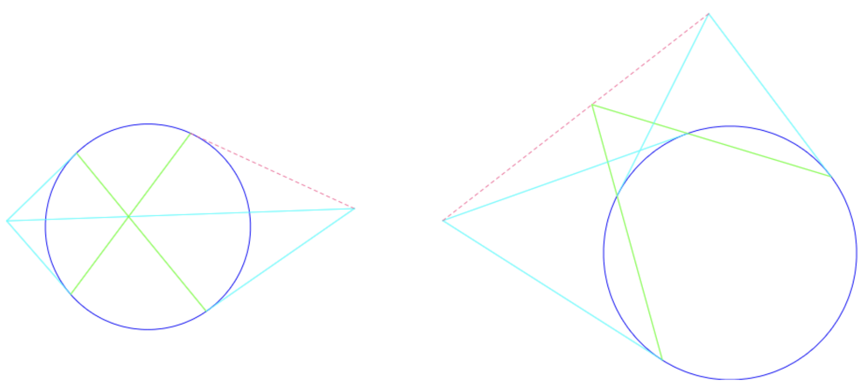


Figure 2: Remarks 1 and 2 respectively.

2 Examples of Applications

We now present walkthroughs so that you can work along and learn how to use this lemma.

Problem 1 *(Bicentric Quadrilaterals) Let $ABCD$ be a bicentric quadrilateral, i.e. a quadrilateral which has both an incircle and a circumcircle. Let the incenter be I and circumcenter be O. Let the diagonals meet at E. Prove that O, I, E are collinear.*

This is a classical application of Newton's theorem.

(a) Let the incircle be γ. Let $X = AB \cap CD$ and $Y = AD \cap BC$. Show that $OE \perp XY$.

(b) Introduce the tangency points of γ with $ABCD$. Use Newton's Theorem and La Hire's theorem to conclude that XY is the polar of E with respect to γ.

(c) Show that $IE \perp XY$.

(d) Conclude.

Time for an application of the lemma to solve a problem related to conics.

Problem 2 *(Pole of conic chords) Let $\mathcal{C}$ be a conic with center O, and $A, B \in \mathcal{C}$. Let M be the midpoint of AB, and X be its pole with respect to $\mathcal{C}$.*

- *Suppose that MX meets $\mathcal{C}$ in K, L. Show that the tangents to $\mathcal{C}$ at K, L are both parallel to AB.*
- *Show that XM passes through O.*

This problem is no doubt a gem, and quite a challenge for anyone seeking a synthetic solution (without falling to techniques such as projective transforms or moving points). Hence, small things like an urge to use Newton's Theorem can be very useful.

(a) Suppose tangents to $\mathcal{C}$ at K, L intersect at Y. Apply Newton's Theorem to conclude that X, Y and $Z = AL \cap BK$ are collinear.

(b) Show that Y lies on AB. Hence it suffices to show $XZ \parallel AB$.

(c) Project the cross ratio $(X, M; K, L)$ onto the line AB from Z to conclude $XZ \parallel AB$.

Thus, we have the first part.

(d) Conclude the second part of the problem by using the fact that the center of any parallelogram on $\mathcal{C}$ is O.

Time for a challenging olympiad problem, which is one of our favorite applications of this lemma.

Problem 3 *(IMO 2008 P6, with addition by Aditya Khurmi.) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω with center I tangent to ray BA beyond A at X_1, and to the ray BC beyond C at X_2, which is also tangent to the lines AD and CD in Y_2, Y_1, respectively.*

- *Prove that the lines X_1X_2, Y_1Y_2, and the line through I perpendicular to BD concur.*
- *Prove that the common external tangents to ω_1 and ω_2 intersect on ω.*

Guessing the point is an essential step for the second part. In fact, if the point is guessed properly (using a good diagram), the problem is not harder than a standard IMO problem 1/4. But it's not so direct, so we motivate it through more natural means (which also proves the first part).

(a) Let ω_1, ω_2 meet AC in X, Y respectively. Use Pitot's Theorem to show that $AX = CY$.

(b) Conclude that Y is the B extouch point in $\triangle BAC$. What can you now say about $BY \cap \omega_1$?

(c) Let I_1, I_2 be the centers of ω_1, ω_2 respectively. Use Monge's theorem to get $AC \cap DB = Z$ lies on I_1I_2 (and is the insimilicenter of ω_1, ω_2).

Let $X_1X_1 \cap Y_1Y_1 = Z_1$ and $X_2X_2 \cap Y_2Y_2 = Z_2$.

Looking at the plethora of tangents, we force the use of Newton's Theorem.

(d) Use the lemma on $\{X_1, X_2, Y_1, Y_2\}$ in all possible combinations (!) to get $Z = X_1Y_1 \cap X_2Y_2$, and conclude that $\{X_1X_2, Y_1Y_2, AC\}, \{X_1Y_2, X_2Y_1, BD\}$ concur (say, at M, N).

(e) Use Brocard's theorem on $X_1Y_1Y_2X_2$ to get that MNZ is self polar with orthocenter I. So we can conclude $IN \perp AC$, i.e. $IN \parallel I_2Y, I_1X$.

(f) Conclude the first part of the problem.

At this point, we have all the good constructions. Try and guess (you are drawing large and accurate diagrams, correct?) the desired point now, which should be $I_1I_2 \cap \omega$.

(g) If you have guessed the point right, then you should be able to finish the proof now. (Hint: Combine homothety with (b))

3 Problems

Try the following problems (and if you find any other problem(s) that use this lemma, feel free to share it with any one of us). At this point, we must break it to you that even though Newton's Theorem might be an essential part of a problem, might even motivate an obscure construction, you would still need ingenuity to make your way through the remaining problem using standard techniques.

Problem 3.1 For any triangle ABC with circumcircle γ, let the A, B, C symmedians meet γ in X, Y, Z. Call XYZ the *sym triangle* of ABC. Prove that if $\triangle_2$ is the *sym triangle* of $\triangle_1$, then $\triangle_1$ is also the *sym triangle* of $\triangle_2$. In other words, being *sym* is a symmetric relation.

Problem 3.2 (Brocard's Theorem) Let $ABCD$ be points on a conic γ. Let $AB \cap CD = X, AC \cap BD = Y$ and $AD \cap BC = Z$. Show that the triangle XYZ is self-polar with respect to γ.

Problem 3.3 (Midpoints and Contact Triangle) Let ABC be a triangle and M, N be the midpoints of BC, BA. Let $\gamma, \triangle DEF$ be the incircle, contact triangle of $\triangle ABC$ respectively. Let AI meet BC in T. Let D^* be the D antipode in γ. Finally, define $X = AD^* \cap \gamma \neq D^*$ and $Y(\neq D) \in \gamma$ so that TY is tangent to γ. Show that $MA \cap FY$ and $MN \cap AI$ both lie on DE.

Problem 3.4 (Iran TST 2017/6) Let γ be the incircle of a triangle ABC. Points P and Q are on AB and AC, such that PQ is parallel to BC and also tangent to γ. Let AB, AC touch γ at F, E, respectively. Let M denote the midpoint of PQ, and let T be the intersection point of EF and BC. Prove that TM is tangent to γ.

Problem 3.5 (SORY P6) Let the incircle be tangent to the sides BC, CA, AB at D, E, F respectively. Let P be the foot of the perpendicular from D onto EF. Assume that BP, CP intersect the sides AC, AB in Y, Z respectively. Finally, let the rays IP, YZ meet the circumcircle of $\triangle ABC$ in R, X respectively. Prove that the tangent from X to the incircle and the line RD meet on the circumcircle of $\triangle ABC$.

(Note: SORY was a mock Olympiad conducted on AoPS by Aditya Khurmi and Satyam Mishra.

See: `https://artofproblemsolving.com/community/q1h1906972p13051397`.)

Problem 3.6 (STEMS 2019/B6) Triangle ABC has incenter I and intouch triangle DEF. Let Q be the foot from D to EF and extend ray DQ to meet the incircle again at R. Ray AQ meets BC at T. Ray AR meets the incircle again at P, and the circumcircle again at S. Finally, let O denote the circumcenter of $\triangle PAD$. Prove that O, T, I, S are collinear.

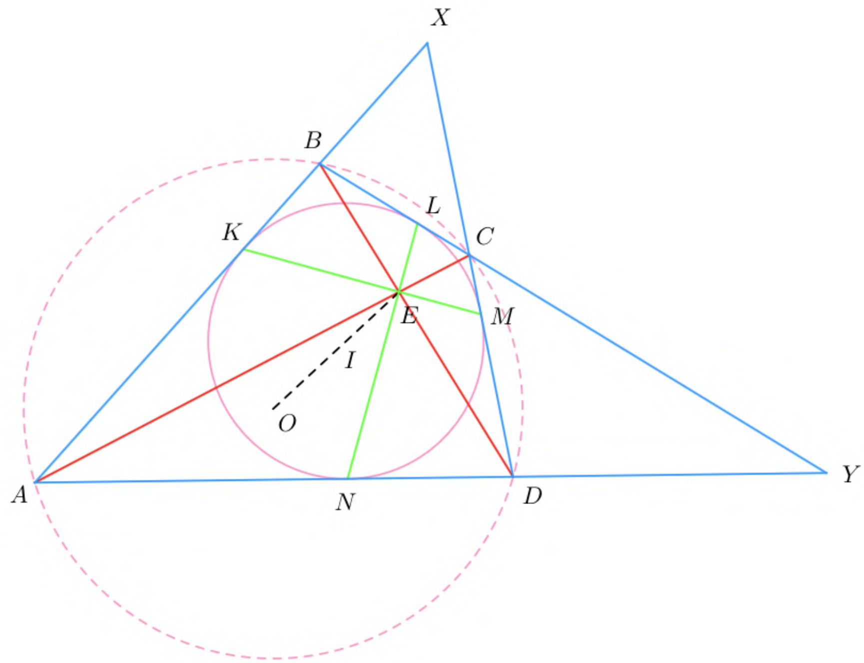
4 Solutions to Walkthroughs

Problem 4 *(Bicentric Quadrilaterals) Let $ABCD$ be a bicentric quadrilateral, i.e. a quadrilateral which has both an incircle and a circumcircle. Let the incenter be I and circumcenter be O. Let the diagonals meet at E. prove that O, I, E are collinear.*

Solution. Let $AB \cap CD = X$ and $AD \cap BC = Y$. By Brocard's Theorem, XY is the polar of E with respect to the circumcircle of $ABCD$, and hence, $OE \perp XY$. To show O, I, E collinear it thus suffices to show $IE \perp XY$.

Let γ be the incircle. All polars from here on are with respect to γ. Let γ be tangent to AB, BC, CD, DA in K, L, M, N respectively. By Newton's Theorem on $\{KL, MN\}$, we find that $KM \cap LN \in BD$. Similarly, $KM \cap LN \in AC$ giving $KM \cap LN = E$.

Since KM is the polar of X and $E \in KM$, by La Hire's theorem X lies on the polar of E. Similarly, Y lies on the polar of E. However, since the polar of E is



a straight line, it must be XY. Thus, XY is the polar of E with respect to γ implying $IE \perp XY$, as desired. □

Problem 5 *(Pole of conic chords) Let $\mathcal{C}$ be a conic with center O, and let $A, B \in \mathcal{C}$. Let M be the midpoint of AB, and X be its pole with respect to $\mathcal{C}$.*

- *Suppose that MX meets $\mathcal{C}$ in K, L. Show that the tangents to $\mathcal{C}$ at K, L are both parallel to AB.*
- *Show that XM passes through O.*

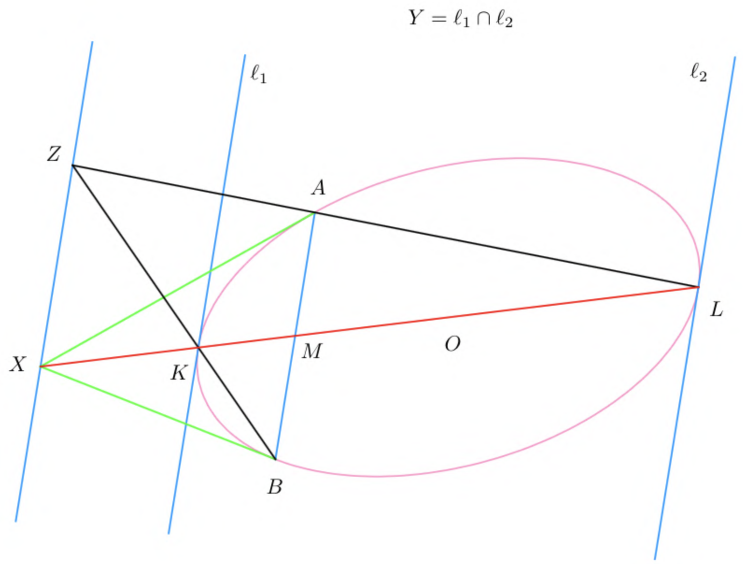
Solution. Let $KK \cap LL = Y$. Applying Newton's Theorem on $\{L, K\}, \{A, B\}$, we find that $AL \cap KB$, say Z, lies on XY.

Since the tangents at A, B meet on the diagonal KL in quadrilateral $AKBL$, hence by a well known property we find that $(A, B; K, L) = -1$. This also gives that $Y = KK \cap LL$ lies on AB. So it suffices to prove that $XZ \parallel AB$. For this, observe that

$$-1 = (A, B; K, L) \stackrel{A}{=} (X, M; K, L) \stackrel{Z}{=} (ZX \cap AB, M; B, A),$$

where we first projected through A onto the line KL, then through Z onto the line AB.

This property is more known for harmonic quadrilaterals. However, it is true for general conics too. The simplest reasoning is by the projective transform taking the conic to a circle (as projective transforms preserve cross ratios).



Now since $(ZX \cap AB, M; B, A) = -1$ and M is the midpoint of AB, hence by a famous property (see for instance problem 0 here: `https://alexanderrem.weebly.com/uploads/7/2/5/6/72566533/projectivegeometry.pdf`), we must have that $ZX \cap AB$ is the point at infinity on AB. This means $ZX \parallel AB$, as desired.

For the second part of the problem, note that $KKLL$ is a (degenerate) parallelogram on $\mathcal{C}$. Hence, its center must be the center of $\mathcal{C}$, which is O. This implies $O \in KL$ (and that O is the midpoint of KL). Hence the second part has also been proven. $\square$

Comment 4.1 *The property that XM passes through O is a generalization of the same property for circles (a chord is bisected by the line through its pole and the circle's center). While the result for circles is easily proven using congruent triangles, that's not the case for conics. Proving it synthetically is quite a challenge, especially since centers of conics don't have a lot of synthetic properties. Hence, the first part of the problem serves as an ingenious and important step in the proof of this beautiful result.*

Problem 6 *(IMO 2008 P6, and more) Let $ABCD$ be a convex quadrilateral with $BA \neq BC$. Denote the incircles of triangles ABC and ADC by ω_1 and ω_2 respectively. Suppose that there exists a circle ω with center I tangent to ray BA beyond A at X_1, and to the ray BC beyond C at X_2, which is also tangent to the lines AD and CD in Y_2, Y_1 respectively.*

- *Prove that the lines X_1X_2, Y_1Y_2 and the line through I perpendicular to BD concur.*
- *Prove that the common external tangents to ω_1 and ω_2 intersect on ω.*

Solution. Let S, T be the antipodes of Y, X in their respective circles. Also, let I_1, I_2, I be the centers of $\omega_1, \omega_2, \omega$ respectively.

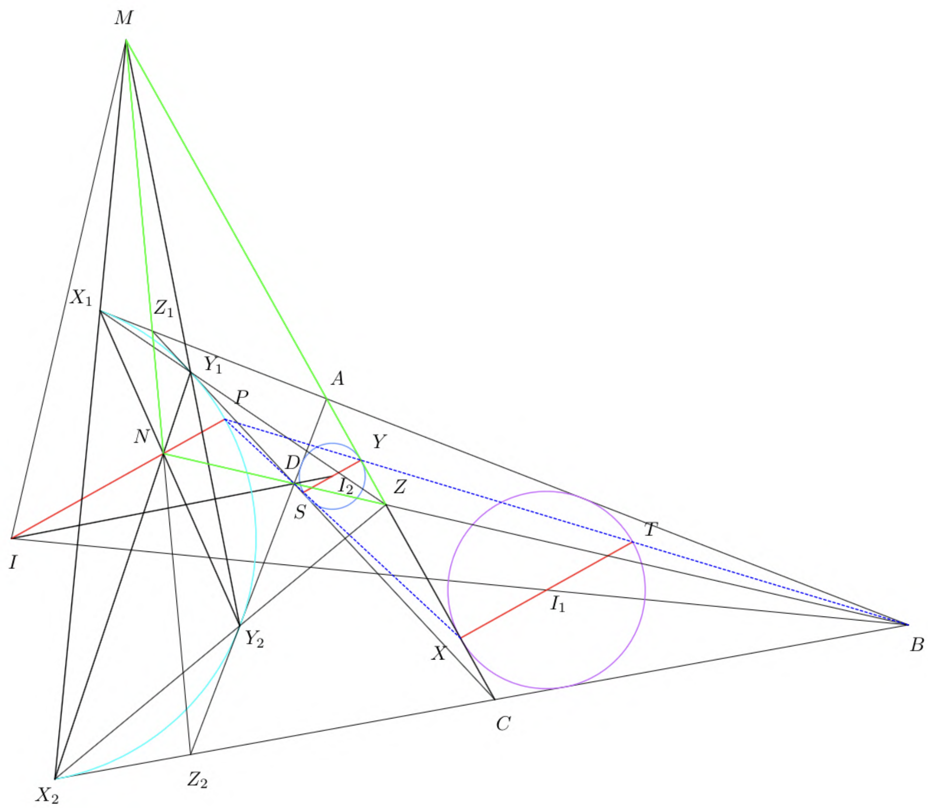


Figure 3: IMO 2008/6

By Pitot's theorem, we find $AB - BC = CD - AD$, and hence

$$2AX = AB + AC - BC = CD + AC - AD = 2CY.$$

Thus, $AX = CY$. Hence, X is the B-excircle touch point of $\triangle BAC$, implying that B, T, X are collinear.

- By Newton's Theorem on $\{X_1, X_2\}\{Y_1, Y_2\}$, we find that $X_1Y_1 \cap X_2Y_2$ lies on BD. By applying the lemma on $\{X_1, Y_2\}, \{X_2, Y_1\}$ we find $M = X_1X_2 \cap Y_1Y_2$ lies on AC. Hence, AC is the polar of N.

  Further by Newton's Theorem on $\{X_1, Y_1\}, \{Y_2, X_2\}$, we get $Z_1 = X_1X_1 \cap Y_1Y_1$, $Z_2 = X_2X_2 \cap Y_2Y_2$ and N are collinear. Thus, $X_1Y_1 \cap X_2Y_2$ is the pole of Z_1Z_2, and so by the dual of this (using La Hire's theorem), we get that $X_1Y_1 \cap X_2Y_2$ lies on the polar of N, which is AC. Hence $X_1Y_1 \cap X_2Y_2 = Z$. (We could have reduced this entire paragraph to a one line application of

Brocard's theorem. We, however, wanted to showcase the power of this lemma).

Since Z_1Z_1 is the polar of Z, it passes through M, as NM is the polar of Z by Brocard's theorem. Since I is the orthocenter of MNZ, we get $IM \perp NZ \equiv BD$. So we are done with the first part of the problem.

- Let the perpendicular from I to AC meet ω in P, so that P lies on the ray IN. Then the homothety at B taking $\omega_1 \mapsto \omega$ takes $I_1 \mapsto I, T \mapsto P$. Hence B, T, P are collinear, and Y also lies on this line (as shown before). Similarly, P, S, D, X are collinear. Thus, YD, BX meet at the point P on ω.

  Further, it is clear that a homothety at P takes SY to XT, and hence ω_2 to ω_1. Thus, P is the intersection of the external tangents to these two circles. Thus, we are done. □

5 A note on the Authors

All the authors are Indians and are IMOTCers (IMOTC is the International Mathematical Olympiad Training Camp in India), which means they have cleared the Indian National Mathematical Olympiad, which is the third tier in the Indian team selection procedure for the International Mathematical Olympiad.

- Aditya Khurmi is currently a freshman majoring in maths in the University of Massachusetts, Amherst.
- Shuborno Das is currently a freshman majoring in maths and computer science in the University of Oxford.
- Aatman Supkar is currently a second year student at Indian Institute of Science studying maths.
- Arindam Bhattacharyya is currently a second year student at Chennai Mathematical Institute studying maths and computer science.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **January 30, 2022**.

4681. *Proposed by Michel Bataille.*

Let n be a positive integer and

$$P_n(x) = \prod_{k=1}^{n} \left(x + 4\sin^2 \frac{2k\pi}{2n+1} \right).$$

Evaluate $P_n(0)$ and $P_n'(0)$.

4682. *Proposed by Goran Conar.*

Determine the infimum and supremum (if they exist) of the set

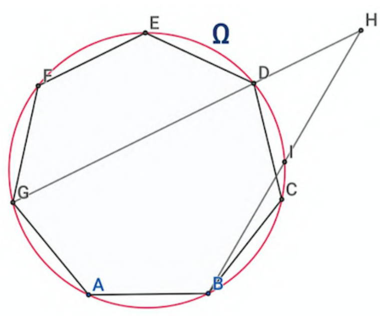
$$\left\{ \sqrt[mn]{m+n} : m, n \in \mathbb{N} \right\}.$$

4683. *Proposed by Warut Suksompong.*

For a non-negative integer n, let $S(n)$ be the sum of digits in the decimal representation of n. Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that for any real number r, there exists an integer k such that $S(|P(k)|) > r$.

4684. *Proposed by Alin Creţu.*

Let $ABCDEFG$ be a regular heptagon with the vertices on the circle Ω. Suppose that $BH \cap \Omega = \{I\}$ and that G, D, H are collinear. If $DH = DC$, show that $HI = IB$.



4685. *Proposed by Abdollah Zohrabi.*

There are 24 students in a school with two classrooms, each with capacity to sit 12 people. Every student goes to school every day, and goes to exactly one of the classrooms. Prove that the students can attend school for 14 days in such a way that each pair of students are present in the same classroom at least once.

4686. *Proposed by Nguyen Viet Hung.*

Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{a}{2}+\sqrt[3]{\frac{b^3+c^3}{2}}.$$

4687. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Calculate

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}-\ln 2+\frac{1}{4n}\right).$$

4688. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a square, with $M, P \in (CD)$ and $N \in (BC)$ so that $DM = MC$, $NB = 3NC$ and $PN \perp AN$. If K is the middle point of segment MN, show that $\measuredangle(PAM) \equiv \measuredangle(KAN)$.

4689. *Proposed by Daniel Sitaru.*

Solve for positive real numbers x, y and z such that $x+y+z=3$:

$$x^{x^2}\cdot y^{y^2}\cdot z^{z^2}=\frac{1}{(x^2)^{xz}\cdot(y^2)^{yx}\cdot(z^2)^{zy}}.$$

4690. *Proposed by Leonard Giugiuc.*

Let x, y and z be nonnegative real numbers such that $xy+yz+zx>3$ and $xy+yz+zx+xyz<4$. Prove that

$$x+y+z>xy+yz+zx.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 janvier 2022**.

4681. *Proposé par Michel Bataille.*

Soient n un entier positif et

$$P_n(x) = \prod_{k=1}^{n} \left(x + 4\sin^2 \frac{2k\pi}{2n+1} \right).$$

Évaluer $P_n(0)$ et $P_n'(0)$.

4682. *Proposé par Goran Conar.*

S'ils existent, déterminer l'infimum et le suprémum de l'ensemble

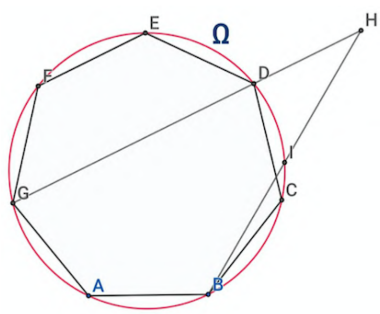
$$\left\{ \sqrt[mn]{m+n} : m, n \in \mathbb{N} \right\}.$$

4683. *Proposé par Warut Suksompong.*

Pour un entier non négatif n, soit $S(n)$ la somme de ses chiffres lorsqu'il est représenté sous forme décimale. Soit aussi $P(x)$ un polynôme à coefficients entiers, non constant. Démontrer que pour tout nombre réel r, il existe un nombre entier k tel que $S(|P(k)|) > r$.

4684. *Proposé par Alin Creţu.*

Soit $ABCDEFG$ un heptagone dont tous les sommets se trouvent sur le cercle Ω. Soit alors H tel que $BH \cap \Omega = \{I\}$ et tel que G, D et H soient alignés. Si $DH = DC$, démontrer que $HI = IB$.



4685. *Proposé par Abdollah Zohrabi.*

Une école avec deux classes, chacune de capacité 12, accueille 24 élèves. Tout élève va à l'école chaque jour et s'y présente à une seule des classes. Démontrer la possibilité, sur une période de 14 jours, que toute paire d'élèves se présente dans la même classe au moins une fois.

4686. *Proposé par Nguyen Viet Hung.*

Soient a, b, c des nombres réels positifs. Démontrer que

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a}{2} + \sqrt[3]{\frac{b^3+c^3}{2}}.$$

4687. *Proposé par Ovidiu Furdui et Alina Sîntămărian.*

Calculer

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} - \ln 2 + \frac{1}{4n} \right).$$

4688. *Proposé par Mihaela Berindeanu.*

Soit $ABCD$ un carré, où $M, P \in (CD)$ et $N \in (BC)$, tels que $DM = MC$, $NB = 3NC$ et $PN \perp AN$. Si K est le point milieu du segment MN, démontrer que $\measuredangle(PAM) \equiv \measuredangle(KAN)$.

4689. *Proposé par Daniel Sitaru.*

Déterminer les nombres réels positifs x, y et z tels que $x + y + z = 3$ et

$$x^{x^2} \cdot y^{y^2} \cdot z^{z^2} = \frac{1}{(x^2)^{xz} \cdot (y^2)^{yx} \cdot (z^2)^{zy}}.$$

4690. *Proposé par Leonard Giugiuc.*

Soient x, y et z des nombres réels non négatifs tels que $xy + yz + zx > 3$ et $xy + yz + zx + xyz < 4$. Démontrer que

$$x + y + z > xy + yz + zx.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(4), p. 199–203.

4631 (corrected). *Proposed by Nguyen Viet Hung.*

Let P be any point on a triangular face of a regular tetrahedron with centroid O and let L, M, N be respectively projections of P onto the other three triangular faces. Prove that PO passes through the centroid G of triangle LMN and determine ratio $\frac{\overline{PO}}{\overline{PG}}$.

Editor's comment: This problem has been corrected by inserting 'regular' before 'tetrahedron'. This correction should also apply to the original statement of the problem, seen in issue 47 (4), pp. 199 and 201.

We received 7 submissions. All but one of them corrected the statement of problem 4631 as it had appeared in 47(4), then proved the modified result; the other provided only a counterexample. We shall feature the two solutions that were sent to us by the UCLan Cyprus Problem Solving Group.

It is easily seen that the assertion fails to hold for an arbitrary tetrahedron. We shall verify it for the case where the given tetrahedron is regular.

Solution 1.

Because the given tetrahedron is regular, we may assume that the coordinates of its vertices are

$$A = (1, 1, 1), \quad B = (1, -1, -1), \quad C = (-1, 1, -1), \quad \text{and} \quad D = (-1, -1, 1).$$

Its centroid is then $O = (0, 0, 0)$. The equation of the plane through B, C, D is $x + y + z = -1$. Let P belong to this plane, say $P = (b, c, d)$ where $b + c + d = -1$.

The equation of the plane through A, B, C is $x + y - z = 1$. The projection L of P on this plane is $(b+t, c+t, d-t)$ where $t \in \mathbb{R}$ satisfies $(b+t)+(c+t)-(d-t) = 1$. This gives

$$t = \frac{1 + d - (b + c)}{3} = \frac{2 + 2d}{3}.$$

Thus

$$L = \left(\frac{3b + 2d + 2}{3}, \frac{3c + 2d + 2}{3}, \frac{d - 2}{3}\right).$$

Analogously we get

$$M = \left(\frac{b - 2}{3}, \frac{3c + 2b + 2}{3}, \frac{3d + 2b + 2}{3}\right) \text{ and } N = \left(\frac{3b + 2c + 2}{3}, \frac{c - 2}{3}, \frac{3d + 2c + 2}{3}\right).$$

So

$$G = \frac{1}{9}(5b+2(b+c+d)+2,\ 5c+2(b+c+d)+2,\ 2(b+c+d)+5d+2) = \frac{5}{9}(b,c,d)\,.$$

So $\overrightarrow{GP} = \frac{4}{9}\overrightarrow{OP}$ and, thus, PO passes through G with $PO/PG = \frac{9}{4}$.

Solution 2.

The transformation that projects an arbitrary point of 3-space onto a plane is affine, as is the transformation that maps a triple of points onto the centroid of the triangle formed by those points; it follows that given three fixed planes, there exists an affine transformation, $f : \mathbb{R}^3 \to \mathbb{R}^3$, that maps an arbitrary point to the centroid G of the triangle formed by the projection of that point onto the three planes.

We may again assume that the centroid O of the given regular tetrahedron is the origin of $\mathbb{R}^3$. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ be the position vectors of its vertices, while BCD is the face that contains the given point P. In the limiting case when P coincides with the vertex B, the triangle KLM is the degenerate triangle with vertices B, B, B' where (since $ABCD$ is regular), B' is the centroid of the face ACD. So

$$f(\mathbf{b}) = \frac{\mathbf{b}+\mathbf{b}+\mathbf{b}'}{3} = \frac{\mathbf{b}+\mathbf{b}+\frac{\mathbf{a}+\mathbf{c}+\mathbf{d}}{3}}{3} = \frac{(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})+5\mathbf{b}}{9} = \frac{5\mathbf{b}}{9}.$$

Analogously we have

$$f(\mathbf{c}) = \frac{5\mathbf{c}}{9} \quad \text{and} \quad f(\mathbf{d}) = \frac{5\mathbf{d}}{9}\,.$$

Now any point P in the plane of B, C, D has position vector $\mathbf{p} = \lambda\mathbf{b} + \mu\mathbf{c} + \nu\mathbf{d}$ where $\lambda + \mu + \nu = 1$. Since f is affine we have $f(\mathbf{p}) = T\mathbf{p} + \mathbf{v}$ for some matrix T and some vector $\mathbf{v}$. Then

$$\begin{aligned} f(\mathbf{p}) &= T(\lambda\mathbf{b} + \mu\mathbf{c} + \nu\mathbf{d}) + \mathbf{v} \\ &= \lambda(T\mathbf{b} + \mathbf{v}) + \mu(T\mathbf{c} + \mathbf{v}) + \nu(T\nu\mathbf{d} + \mathbf{v}) \\ &= \lambda f(\mathbf{b}) + \mu f(\mathbf{c}) + \nu f(\mathbf{d}) \\ &= \frac{5}{9}(\lambda\mathbf{b} + \mu\mathbf{c} + \nu\mathbf{d}) = \frac{5}{9}\mathbf{p}\,. \end{aligned}$$

So P, O, G are collinear with $PO/PG = 9/4$ as in Solution 1.

Editor's comments. The original statement of problem 4631 erroneously omitted the word *regular*. It is easy to produce an example of a nonregular tetrahedron $ABCD$ with centroid O together with a specific point P on one face whose projections on the other faces form a triangle whose centroid G is *not* on the line PO. Three solvers produced such an example. The UCLan Cyprus Problem Solving Group went further and proved that should the resulting centroid G lie on PO for all points P on the surface of the tetrahedron, then that tetrahedron would have to be regular.

4632. *Proposed by Michel Bataille.*

Let $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ be the nth harmonic number. Prove that for $n \geq 1$,

$$\sum_{k=1}^{n} \binom{2n+1}{2k-1} \frac{H_{2k-1}}{2k} = \sum_{k=1}^{n} \binom{2n+1}{2k} \frac{H_{2k}}{2k+1}.$$

We received 11 correct solutions and 1 incorrect submission. We present 4 solutions.

Solution 1, by Marie-Nicole Gras.

We recall that for $n \geq 1$, we have $H_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k}$. This can be established by induction with the induction step

$$\begin{aligned}
\sum_{k=1}^{n+1} \binom{n+1}{k} \frac{(-1)^{k-1}}{k} &= \frac{(-1)^n}{n+1} + \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{n} \binom{n}{k-1} \frac{(-1)^{k-1}}{k} \\
&= \frac{(-1)^n}{n+1} + H_n + \frac{1}{n+1} \sum_{k=1}^{n} \binom{n+1}{k} (-1)^{k-1} \\
&= H_n + \frac{1}{n+1} \sum_{k=1}^{n+1} \binom{n+1}{k} (-1)^{k-1} \\
&= H_n - \frac{1}{n+1} [(1-1)^{n+1} - 1] = H_{n+1}.
\end{aligned}$$

Multiplying the desired equation by $2n+2$ and taking the difference between the two sides, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\binom{2n+2}{2k} H_{2k-1} - \binom{2n+2}{2k+1} H_{2k} \right] \\
&= \sum_{k=1}^{n} \left[\binom{2n+1}{2k-1} H_{2k-1} + \binom{2n+1}{2k} (H_{2k-1} - H_{2k}) - \binom{2n+1}{2k+1} H_{2k} \right] \\
&= \left[\binom{2n+1}{1} + \sum_{k=1}^{n-1} \binom{2n+1}{2k+1} (H_{2k+1} - H_{2k}) - \binom{2n+1}{2n+1} H_{2n} \right] \\
&\qquad + \sum_{k=1}^{n} \binom{2n+1}{2k} (H_{2k-1} - H_{2k}) \\
&= (2n+1) + \sum_{j=2}^{2n} (-1)^{j-1} \binom{2n+1}{j} (H_j - H_{j-1}) + \left(\frac{1}{2n+1} - H_{2n+1} \right) \\
&= \sum_{j=1}^{2n+1} \frac{(-1)^{j-1}}{j} \binom{2n+1}{j} - H_{2n+1} = 0, \quad \text{which establishes the result.}
\end{aligned}$$

Solution 2, based on the independent solutions of Adnan Ali and UCLan Cyprus Problem Solving Group.

Along with the representation of H_n in the previous solution, we need this result:

$$\frac{1}{n}=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k.$$

To see this, note that

$$\begin{aligned}\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}H_k&=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\sum_{j=1}^{k}\frac{1}{j}=\sum_{j=1}^{n}\frac{1}{j}\sum_{k=j}^{n}(-1)^{k-1}\binom{n}{k}\\&=\sum_{j=1}^{n}\frac{1}{j}(-1)^{j-1}\binom{n-1}{j-1}=\frac{1}{n}\sum_{j=1}^{n}(-1)^{j-1}\binom{n}{j}\\&=\frac{1}{n}[1-(1-1)^n]=\frac{1}{n}.\end{aligned}$$

Multiplying the desired equation by $2n+2$ and taking the difference between the two sides, we obtain

$$\begin{aligned}&\sum_{k=1}^{n}\left[\binom{2n+2}{2k}H_{2k-1}-\binom{2n+2}{2k+1}H_{2k}\right]=\sum_{j=1}^{2n}(-1)^{j+1}\binom{2n+2}{j+1}H_j\\&=\sum_{j=1}^{2n}(-1)^{j+1}\binom{2n+2}{j+1}\left(H_{j+1}-\frac{1}{j+1}\right)\\&=\sum_{k=2}^{2n+1}(-1)^k\binom{2n+2}{k}H_k-\sum_{k=2}^{2n+1}(-1)^k\binom{2n+2}{k}\frac{1}{k}\\&=\sum_{k=1}^{2n+2}(-1)^k\binom{2n+2}{k}H_k+(2n+2)-H_{2n+2}\\&\qquad+\sum_{k=1}^{2n+2}(-1)^{k-1}\binom{2n+2}{k}\frac{1}{k}-(2n+2)+\frac{1}{2n+2}\\&=-\frac{1}{2n+2}+(2n+2)-H_{2n+2}+H_{2n+2}-(2n+2)+\frac{1}{2n+2}=0,\end{aligned}$$

from which the desired equality follows.

Solution 3, by M. Bello, M. Benito, Ó. Ciaurri and E. Fernández.

Let $a_j=(-1)^j\binom{2n+1}{j}\frac{H_j}{j+1}$ and $S=\sum_{j=1}^{2n}a_j$. It is required to show that $S=0$. Since $H_j=\int_0^1(1-t^j)(1-t)^{-1}\,dt$,

$$S=\int_0^1\frac{f(1)-f(t)}{1-t}\,dt\qquad\text{where}\qquad f(t)=\sum_{j=1}^{2n}\binom{2n+1}{j}\frac{(-1)^jt^j}{j+1}.$$

Let $g(s) = (1-s)^{2n+1} + s^{2n} - 1 = \sum_{i=1}^{2n} (-1)^j \binom{2n+1}{j} s^j$ and $h(t) = \int_0^t g(s)\,ds$. Then $tf(t) = \int_0^t g(s)\,ds = h(t)$, so that

$$\begin{aligned} S &= \int_0^1 \frac{th(1) - h(t)}{t(1-t)}\,dt = \int_0^1 \frac{(1-t)h(1) - h(1-t)}{t(1-t)}\,dt \\ &= \frac{1}{2}\int_0^1 \frac{h(1) - (h(t) + h(1-t))}{t(1-t)}\,dt. \end{aligned}$$

Since $g(s) = g(1-s)$,

$$\begin{aligned} h(t) + h(1-t) &= \int_0^t g(s)\,ds + \int_0^{1-t} g(s)\,ds \\ &= \int_0^t g(s)\,ds + \int_0^{1-t} g(1-s)\,ds \\ &= \int_0^t g(s)\,ds + \int_t^1 g(s)\,ds = \int_0^1 g(s)\,ds = h(1). \end{aligned}$$

Therefore $S = 0$.

Solution 4, by G.C. Greubel.

We recall some preliminary results.

$$\begin{aligned} \binom{m}{k} &= \frac{\Gamma(m+1)}{\Gamma(k+1)\Gamma(m-k+1)} \\ &= \frac{1}{2\pi i}\oint_{|z|=1} \frac{(1+z)^m}{z^{k+1}}\,dz. \end{aligned}$$

The partial derivatives of $\binom{m}{k}$ with respect to m and k are respectively given by

$$\partial_m \binom{m}{k} = \binom{m}{k}(\psi(m+1) - \psi(m-k+1)) = \frac{1}{2\pi i}\oint_{|z|=1} (z+1)^m z^{-k-1}\ln(z+1)\,dz,$$

and

$$\partial_k \binom{m}{k} = \binom{m}{k}(\psi(m-k+1) - \psi(k+1)) = -\frac{1}{2\pi i}\oint_{|z|=1} (z+1)^m z^{-k-1}\ln z\,dz,$$

where

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}$$

and

$$\psi(z+1) = \frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{z\Gamma'(z) + \Gamma(z)}{z\Gamma(z)} = \psi(z) + \frac{1}{z}.$$

The generating function for the sequence $\{H_n\}$ is

$$\sum_{k=1}^{\infty} H_k z^k = \frac{-\ln(1-z)}{1-z}.$$

Observe also that

$$\sum_{k=1}^{m}(-1)^{k+1}\binom{m+1}{k+1}H_k = \sum_{k=1}^{\infty}(-1)^{k+1}\binom{m+1}{k+1}H_k$$

since $\binom{m+1}{k+1} = 0$ for $k \geq m+1$.

For each positive integer m,

$$\begin{aligned}
\sum_{k=1}^{m}(-1)^{k-1}\binom{m+1}{k+1}H_k &= \frac{1}{2\pi i}\oint_{|z|=1}(z+1)^{m+1}z^{-2}\left(\sum_{k=1}^{m}(-1)^{k-1}z^{-k}H_k\right)dz \\
&= \frac{1}{2\pi i}\oint_{|z|=1}(z+1)^{m+1}z^{-2}\left[\frac{z}{1+z}\ln\left(1+\frac{1}{z}\right) + \sum_{k=m+1}^{\infty}(-z)^k H_k\right]dz \\
&= \frac{1}{2\pi i}\oint_{|z|=1}(z+1)^{m}z^{-1}(\ln(z+1)-\ln(z))\,dz - \left[\sum_{k=m+1}^{\infty}(-1)^{k-1}\binom{m+1}{k+1}H_k\right] \\
&= \frac{1}{2\pi i}\oint_{|z|=1}(z+1)^{m}z^{-1}(\ln(z+1)-\ln(z))\,dz
\end{aligned}$$

s

$$= \left[\partial_m\binom{m}{k} + \partial_k\binom{m}{k}\right]_{k=0}$$

s

$$= \left[\binom{m}{k}(\psi(m+1)-\psi(k+1))\right]_{k=0} = \psi(m+1)-\psi(1) = H_m.$$

Setting $m = 2n+1$ and subtracting H_{2n+1} from each side yields

$$\sum_{j=1}^{2n}(-1)^{j+1}\binom{2n+2}{j+1}H_j = 0,$$

from which the result follows.

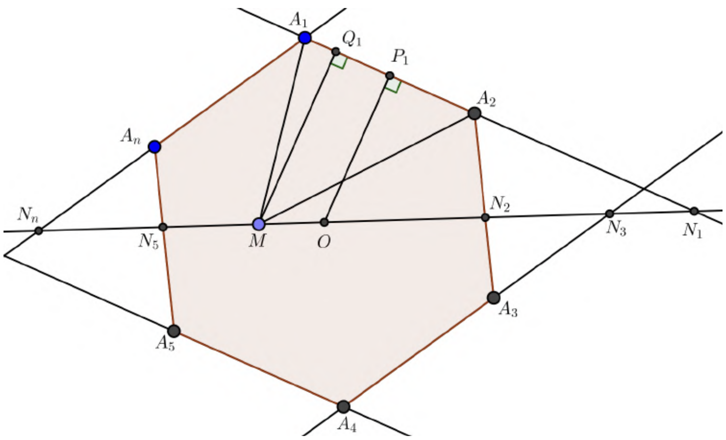
Editor's comments. C.R. Pranesachar took the difference of the two sides and worked out the coefficients of each integer reciprocal to find that the difference is equal to

$$\begin{aligned}
&\sum_{k=1}^{n}\frac{1}{2k-1}\left(-1+\binom{2n+1}{2k-1}\right) + \sum_{k=1}^{n}\frac{1}{2k}\left(-1-\binom{2n+1}{2k}\right) \\
&\quad = \sum_{j=1}^{2n}\frac{(-1)^{j-1}}{j}\binom{2n+1}{j} - H_{2n} \\
&\quad = \sum_{j=1}^{2n+1}\frac{(-1)^{j-1}}{j} - H_{2n+1} = 0.
\end{aligned}$$

4633. *Proposed by Nguyen Viet Hung.*

Let $A_1A_2\dots A_n$ be a a regular $n-$sided polygon with center O and let M be any point inside the polygon. Suppose that the line OM intersects the lines A_iA_{i+1} at N_i ($i=1,2,\dots,n$ and $A_{n+1}\equiv A_1$), respectively. Find

$$\sum_{i=1}^{n}\frac{MN_i}{ON_i}.$$



We feature the solution by Adnan Ali, which is a typical example of the 12 similar submissions that we received.

We shall prove that the desired sum is equal to n.

Let a be the side-length of the polygon and d be the common distance from O to one of its sides. Let P_i and Q_i, $1\le i\le n$, be the feet of the perpendiculars dropped to the side A_iA_{i+1} from points O and M respectively. Note that the resulting pairs of right triangles, $\triangle OP_iN_i$ and $\triangle MQ_iN_i$ are similar (since their sides are parallel). Thus

$$\frac{MN_i}{ON_i}=\frac{MQ_i}{OP_i}=\frac{MQ_i}{d}\qquad 1\le i\le n. \tag{1}$$

Using square brackets to denote area, we have

$$MQ_i=\frac{2[MA_iA_{i+1}]}{A_iA_{i+1}}=\frac{2[MA_iA_{i+1}]}{a}. \tag{2}$$

It follows that the desired sum becomes

$$\sum_{i=1}^{n}\frac{MN_i}{ON_i}\ =_{(1)}\frac{1}{d}\sum_{i=1}^{n}MQ_i\ =_{(2)}\frac{2}{ad}\sum_{i=1}^{n}[MA_iA_{i+1}]=\frac{[A_1A_2\dots A_n]}{\frac{ad}{2}}=\frac{n\cdot\frac{ad}{2}}{\frac{ad}{2}}=n.$$

4634. *Proposed by George Stoica.*

Let $\sum_{n=1}^{\infty} a_n < \infty$ for $a_n > 0$, $n = 1, 2, \ldots$ Find $\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n}$.

We received 11 submissions of which 7 were correct and complete. We present the solution of the proposer, lightly edited.

Let us denote $S_n := a_1 + \cdots + a_n$ and let L be the limit of the sequence $(S_n)_{n\in\mathbb{N}}$ (which exists since $\sum_{n=1}^{\infty} a_n < \infty$). We will first show that

$$\lim_{n\to\infty} \sqrt[n]{n! \cdot a_1 \cdot a_n} = 0,$$

and use this to conclude that the desired limit is likewise 0.

By the Arithmetic Mean - Geometric Mean inequality, we have

$$\begin{aligned} \sqrt[n]{n! \cdot a_1 \cdots a_n} &\leq \frac{a_1 + 2a_2 + \cdots + na_n}{n} \\ &= \frac{S_1 + 2(S_2 - S_1) + \cdots + n(S_n - S_{n-1})}{n} \\ &= S_n - \frac{S_1 + S_2 + \cdots + S_{n-1}}{n}. \end{aligned} \tag{1}$$

By the Stolz-Cesàro Theorem,

$$S_n \to L \text{ implies } \frac{S_1 + \cdots + S_n}{n} \to L$$

and it easily follows that

$$\frac{S_1 + \cdots + S_{n-1}}{n} \to L$$

as well. Therefore,

$$S_n - \frac{S_1 + S_2 + \cdots + S_{n-1}}{n} \to 0 \text{ as } n \to \infty,$$

and, applying the Squeeze Theorem to (1), we also get

$$\lim_{n\to\infty} \sqrt[n]{n! \cdot a_1 \cdot a_n} = 0.$$

Finally, since $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$, we obtain that

$$\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} \cdot \lim_{n\to\infty} \sqrt[n]{n! \cdot a_1 \cdots a_n} = 0.$$

4635*. *Proposed by S. Chandrasekhar.*

For a prime p dividing $n!$, let $e_p(n!)$ denote the highest power of p in $n!$, that is if $e_p(n!) = k$, it means $p^k \mid n!$ whereas $p^{k+1} \nmid n!$. Prove or disprove that $e_3(n!) \mid e_2(n!)$ for infinitely many n.

We received comments from Marie-Nicole Gras, Carl Pomerance, and David Stone. Until very recently, this was an unsolved research-level problem. However, in May 2021, Lukas Spiegelhofer confirmed that there are infinitely many n such that $e_2(n!) = 2e_3(n!)$ in an arXiv preprint titled Collisions of the binary and ternary sum-of-digits functions *(`https://arxiv.org/abs/2105.11173`). We summarized a few basic ideas to solve the problem.*

Let p be a prime and n be a positive integer. It is not hard to see that Legendre's Formula

$$e_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor$$

implies that

$$e_p(n!) = \frac{n - s_p(n)}{p-1},$$

where $s_p(n)$ denotes the sum of the digits in the base-p expansion of n. It is easy to show that $\lim_{n\to\infty} s_p(n)/n = 0$. Therefore,

$$\lim_{n\to\infty} \frac{e_p(n!)}{n} = \frac{1}{p-1}.$$

In particular, $e_2(n!) = n + o(n)$ and $e_3(n!) = n/2 + o(n)$, as $n \to \infty$. Therefore, when n is large enough,

$$e_3(n!) \mid e_2(n!) \iff e_2(n!) = 2e_3(n!) \iff s_2(n) = s_3(n). \tag{1}$$

In fact, by computation, one can show that equation (1) holds for $n \geq 6$. Thus, it suffices to show that there are infinitely many n such that $s_2(n) = s_3(n)$.

Lukas Spiegelhofer managed to prove the following stronger theorem: for each $\varepsilon > 0$, there is $C_\varepsilon > 0$, such that

$$\#\{n \leq N : s_2(n) = s_3(n)\} \geq C_\varepsilon N^{\frac{\log 3}{\log 4} - \varepsilon} \tag{2}$$

holds for all positive integers N. Note that $\log 3/\log 4 \approx 0.792$. The proof is not constructive, meaning that the proof did not give an algorithm to find integers n such that $s_2(n) = s_3(n)$.

The proof of the main theorem essentially follows from the following weaker statement: there exist infinitely many positive integers n such that $n \equiv 9 \pmod{12}$ and $s_2(n) - s_3(n) \in \{0, 1\}$. The reason is the following: if there are infinitely many $n \equiv 9 \pmod{12}$ such that $s_2(n) - s_3(n) = 0$, then we are done; otherwise, there are infinitely many $n \equiv 9 \pmod{12}$ (i.e., $n \equiv 0 \pmod 3$ and $n \equiv 1 \pmod 4$) such that $s_2(n) - s_3(n) = 1$, which implies that $s_3(n+1) = s_3(n) + 1 = s_2(n) = s_2(n+1)$.

Editor's Comment. A list of the smallest 10000 numbers n such that $s_2(n) = s_3(n)$ can be found in `https://oeis.org/A037301/b037301.txt`. Readers are encouraged to refer to Spiegelhofer's paper for a complete proof of the main theorem as well as discussions on related problems. In particular, he suggested the following interesting question (which might be very difficult): determine if there are infinitely many prime numbers p such that $s_2(p) = s_3(p)$. He also provided a probabilistic heuristic that the natural density of the set of integers n with $s_2(n) = s_3(n)$ is zero.

4636. *Proposed by Mihaela Berindeanu.*

Solve the following equation over the set of real numbers:

$$(3^x + 7)^{\log_4 3} - (4^x - 7)^{\log_3 4} = 4^x - 3^x - 14.$$

We received 15 submissions of which 12 were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group.

Let $a = 4^x - 7, b = 3^x + 7$ and $t = \log_3 4$. Then $\frac{1}{t} = \log_4 3$, so the equation becomes

$$b^{1/t} + b = a^t + a\,.$$

This has the obvious solution $a = b^{1/t}$. It cannot have any other solution because if $a > b^{1/t}$ then $a + a^t > b^{1/t} + b$ and if $a < b^{1/t}$ then $a + a^t < b^{1/t} + b$. Let log denote the base e logarithm function. Since $a = b^{1/t}$ we get $\log b = t \log a = \frac{\log 4}{\log 3} \log a$. Thus $f(x) = 0$ where

$$f(x) = \log 4 \log (4^x - 7) - \log 3 \log (3^x + 7)\,.$$

The function f is defined on $(\log_4 7, \infty)$. We claim that f is strictly increasing in this domain. Indeed

$$\begin{aligned} f'(x) &= \frac{(\log 4)^2 \cdot 4^x}{4^x - 7} - \frac{(\log 3)^2 \cdot 3^x}{3^x + 7} \\ &= \frac{12^x[(\log 4)^2 - (\log 3)^2] + 7[(\log 4)^2 \cdot 4^x - (\log 3)^2 \cdot 3^x]}{(4^x - 7)(3^x + 7)} > 0. \end{aligned}$$

We now observe that $x = 2$ is a solution and since f is strictly increasing, $x = 2$ must be the unique solution.

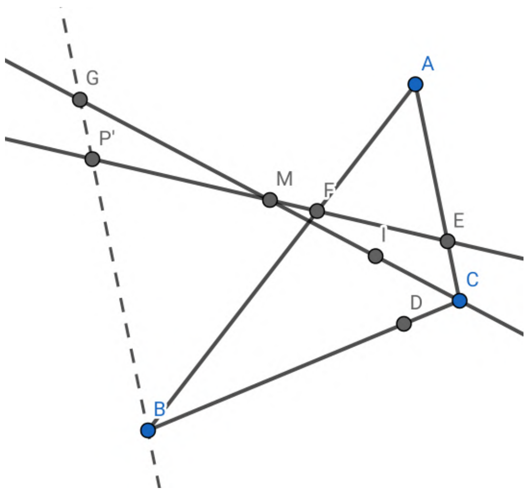
4637. *Proposed by Titu Zvonaru.*

Let the incircle of triangle ABC touch the sides BC, CA, and AB at points D, E, and F, respectively. The internal bisector of the angle $\angle BCA$ intersects the line EF at M. Let P be the reflection of the point E with respect to M. Prove that the triangle BPF is isosceles.

We received 16 submissions, some containing 2 and even 3 solutions based on different ideas. A variety of analytic tools were used including vectors, complex *numbers and barycentric coordinates. Pure geometric solutions can be classified as angle-chasing and length-chasing solutions.*

Solution 1, by Adnan Ali.

Denote by a, b, c and s the sides BC, CA, AB and the semi-perimeter of the triangle, respectively. Then we know that $AE = AF = s - a$, $BF = BD = s - b$ and $CD = CE = s - c$.



Now construct a line ℓ through B parallel to AC and let it intersect EF at P'. From this construction we obtain $\triangle BP'F \sim \triangle AEF$. Thus $AE = AF$ implies $BP' = BF = s - b$, and hence $\triangle BP'F$ is isosceles. Now let the internal bisector of $\angle BCA$ meet ℓ at G. Clearly, $\triangle P'GM \sim \triangle ECM$. Next observe that $P'G \parallel EC$ implies

$$\angle P'GM = \angle MCE = \angle MCB,$$

which implies that $\triangle CBG$ is isosceles with $CB = BG$. Since $BG = a > s - b = BP'$, we get

$$P'G = BG - BP' = BC - BF = BC - BD = CD = CE$$

so

$$\triangle P'GM \cong \triangle ECM \Rightarrow P'M = ME,$$

and hence M is the midpoint of $P'E$. But from the problem statement, we know that M is the midpoint of PE and thus $P = P'$. The problem follows.

Solution 2, by Ivko Dimitrić.

We use barycentric coordinates in reference to triangle ABC where $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, $C = (0 : 0 : 1)$. If I is the incenter of ABC and a, b, c are the side lengths opposite the corresponding vertices and s is the semi-perimeter, then

$$E = (s-c : 0 : s-a), \quad F = (s-b : s-a : 0), \quad I = (a : b : c).$$

The line CI has an equation

$$\begin{vmatrix} x & y & z \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = 0 \quad \Longleftrightarrow \quad bx - ay = 0,$$

whereas the line EF has an equation

$$\begin{vmatrix} x & y & z \\ s-c & 0 & s-a \\ s-b & s-a & 0 \end{vmatrix} = 0 \quad \Longleftrightarrow \quad -(s-a)x + (s-b)y + (s-c)z = 0.$$

Then the intersection point M of these two lines is found to be

$$M = (a : b : b-a) = \frac{1}{2b}\,(a,\ b,\ b-a).$$

The normalized coordinates of E are $E = \left(\frac{s-c}{b},\ 0\,,\ \frac{s-a}{b}\right)$. Since M is the midpoint of EP,

$$P = 2M - E = \left(\frac{s-b}{b},\ 1\,,\ -\frac{s-b}{b}\right) = (s-b : b : -(s-b))$$

The line BP is then

$$\begin{vmatrix} x & y & z \\ s-b & b & -(s-b) \\ 0 & 1 & 0 \end{vmatrix} = 0 \quad \Longleftrightarrow \quad x + z = 0.$$

Therefore, the lines AC ($y = 0$) and BP have the same point at infinity $BP_\infty = AC_\infty = (1 : 0 : -1)$, which means that they are parallel.

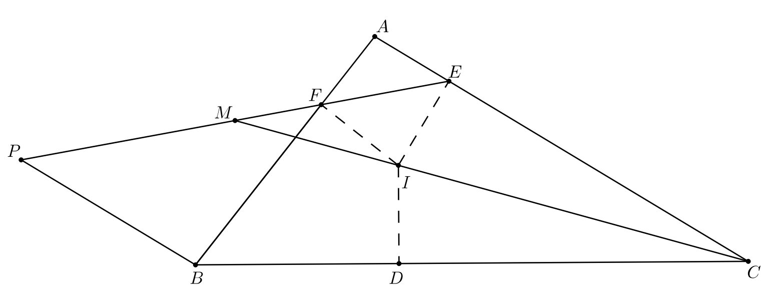
Then,

$$\angle BPF = \angle FEA = \angle EFA = \angle PFB,$$

which shows that $\triangle BPF$ is isosceles with $BF = BP$.

Solution 3, by Michel Bataille.

We use the familiar notations for the elements of the triangle ABC.



Since $AE = AF(= s - a)$ and $\angle PFB = \angle AFE$, the triangle BPF is isosceles if $\angle FBP = \angle FAE$, or, equivalently, if BP and AC are parallel.

Now, using barycentric coordinates relative to (A, B, C), the points E, F satisfy

$$bE = (s - c)A + (s - a)C, \; cF = (s - b)A + (s - a)B,$$

so that the equation of the line EF is

$$x(s - a) - y(s - b) - z(s - c) = 0.$$

Since $I = (a : b : c)$, the equation of CI is $bx - ay = 0$, and we readily deduce that $M = (a : b : b - a)$, that is,

$$2bM = aA + bB + (b - a)C.$$

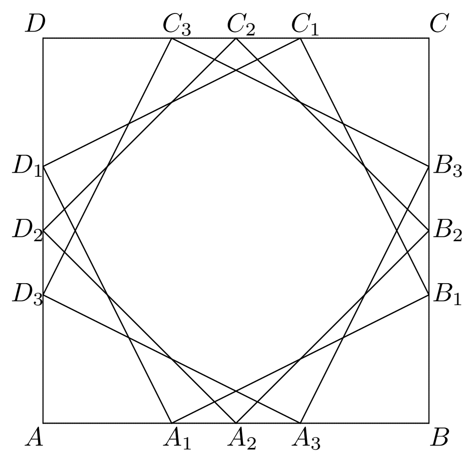
Now, from $\overrightarrow{PM} = \overrightarrow{ME}$, we have $P = 2M - E$ or

$$bP = 2bM - bE = aA + bB + (b-a)C - (s-c)A - (s-a)C = (s-b)A + bB - (s-b)C.$$

Thus, $b(P - B) = (s - b)(A - C)$, that is, $b\overrightarrow{BP} = (s - b)\overrightarrow{CA}$ and $BP \parallel CA$ follows.

4638. *Proposed by Marie-Nicole Gras.*

We consider a square $ABCD$ of side $AB = 6a$, $a \in \mathbb{R}$; we put on the side AB points A_1, A_2, A_3 such that $AA_1 = 2a$, $A_1A_2 = A_2A_3 = a$, then we draw the squares $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and $A_3B_3C_3D_3$ as shown on the figure.



The region common to the interiors of the three squares is a dodecagon. Find the relationship between the areas of the dodecagon and the largest square.

This problem was inspired by 4449.

We received 15 submissions, all are correct. We present two solutions.

Solution 1, by Missouri State University Problem Solving Group, which contains a generalisation.

More generally, we will consider the case when

$$A = (-1,-1), B = (1,-1), C = (1,1), D = (-1,1),$$

and $A_1A_2 = A_2A_3 = \lambda, 0 < \lambda < 1$. Denote the intersection of $\overline{C_1D_1}$ and $\overline{C_3B_1}$ by P, the intersection of $\overline{C_2B_2}$ and $\overline{C_3B_3}$ by Q, and the intersection of $\overline{C_1B_1}$ and $\overline{C_2B_2}$ by R. Let O denote the origin. The dodecagon we are investigating consists of eight triangles congruent to $\triangle OPQ$ and four triangles congruent to $\triangle OQR$.

The equation of the line through C_1 and D_1 is

$$y = \frac{1-\lambda}{1+\lambda}x + \frac{1+\lambda^2}{1+\lambda}.$$

The point P corresponds to the y-intercept of this line, so

$$P = \left(0, \frac{1+\lambda^2}{1+\lambda}\right).$$

The equation of the line through B_3 and C_3 is

$$y = -\frac{1-\lambda}{1+\lambda}x + \frac{1+\lambda^2}{1+\lambda}$$

and the equation of the line through B_2 and C_2 is $y = 1 - x$. Finding the intersection of these two lines gives

$$Q = \left(\frac{1-\lambda}{2}, \frac{1+\lambda}{2}\right).$$

By symmetry, we have

$$R = \left(\frac{1+\lambda}{2}, \frac{1-\lambda}{2}\right).$$

The area of $\triangle OPQ$ is

$$\frac{1}{2}\det\begin{bmatrix}(1-\lambda)/2 & (1+\lambda)/2 \\ 0 & \left(1+\lambda^2\right)/(1+\lambda)\end{bmatrix} = \frac{(1-\lambda)\left(1+\lambda^2\right)}{4(1+\lambda)}.$$

The area of $\triangle OQR$ is

$$\frac{1}{2}\det\begin{bmatrix}(1+\lambda)/2 & (1-\lambda)/2 \\ (1-\lambda)/2 & (1+\lambda)/2\end{bmatrix} = \frac{\lambda}{2}.$$

Therefore the area of the dodecagon is

$$8\cdot\frac{(1-\lambda)\left(1+\lambda^2\right)}{4(1+\lambda)} + 4\cdot\frac{\lambda}{2} = \frac{2\left(1+2\lambda^2-\lambda^3\right)}{1+\lambda}.$$

The area of the larger square is 4, so the ratio of the area of the dodecagon to that of the square is

$$\frac{1+2\lambda^2-\lambda^3}{2(1+\lambda)}.$$

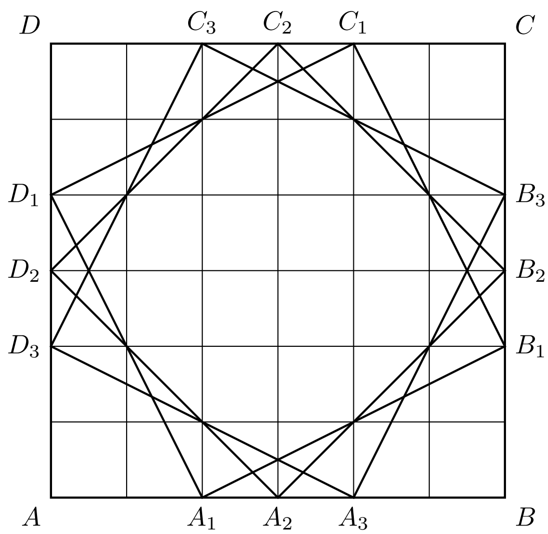
In the problem posed, $\lambda = 1/3$ and we obtain a ratio of $4/9$.

We note in passing that for a fixed larger square, the dodecagon's area is smallest when

$$\lambda = \frac{\sqrt{17}-3}{4}.$$

Solution 2, by Peter DeVries.

Divide the square $ABCD$ into a 6×6 square grid, partitioning the area of $ABCD$ into 36 squares of side length a as in Figure 1.



By calculation of the points of intersection of the sides of the smaller squares and symmetry of the construction, we can see that the dodecagon contains 12 small squares of side length a, 4 triangles with base a and height a, and 4 triangles with base $2a$ and height $\frac{1}{2}a$. Therefore, the area of the dodecagon is $16a^2$ compared to square $ABCD$'s area of $36a^2$. Thus, the ratio of the dodecagon's area to that of $ABCD$ is $\dfrac{16}{36} = \dfrac{4}{9}$.

Editor's Comments. As Brian Beasley points out, it is interesting to note as well that while the octagon in Problem 4449 was equilateral but not equiangular, the dodecagon in this problem is neither equilateral nor equiangular.

4639. *Proposed by Seán Stewart.*

If $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, show that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k(k+m)!(n-k)!} = \frac{1}{m!n!} \sum_{k=1}^{n} \frac{1}{m+k}.$$

We received 13 submissions, all correct. We present the solution by Theo Koupelis, slightly modified by the editor.

Let $H_n = \sum_{k=1}^{n} \frac{1}{k}$ denote the nth Harmonic number. Then $H_n = \int_0^1 \frac{1-x^n}{1-x} \mathrm{d}x$, with $H_0 = 0$. Therefore,

$$\sum_{k=1}^{n} \frac{1}{m+k} = H_{m+n} - H_m = \int_0^1 \frac{1-x^{n+m}}{1-x} \mathrm{d}x - \int_0^1 \frac{1-x^m}{1-x} \mathrm{d}x.$$

Setting $u = 1-x$, we then have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{m+k} &= -\int_0^1 \frac{1}{u}(1-u)^m \cdot ((1-u)^n - 1)\,\mathrm{d}u \\
&= -\int_0^1 \frac{1}{u}(1-u)^m \sum_{k=1}^{n} (-1)^k \binom{n}{k} u^k \mathrm{d}u \\
&= -\sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_0^1 u^{k-1}(1-u)^m \mathrm{d}u \\
&= -\sum_{k=1}^{n} (-1)^k \binom{n}{k} \mathrm{B}(k, m+1) \\
&= -\sum_{k+1}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(k)\Gamma(m+1)}{\Gamma(k+m+1)}
\end{aligned}$$

where B and Γ denote the beta and gamma functions, respectively. It follows that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{m+k} &= -\sum_{k=1}^{n} (-1)^k \frac{n!}{k!(n-k)!} \frac{(k-1)!m!}{(k+m)!} \\
&= \sum_{k=1}^{n} (-1)^{k+1} \frac{m!n!}{k(k+m)!(n-k)!}
\end{aligned}$$

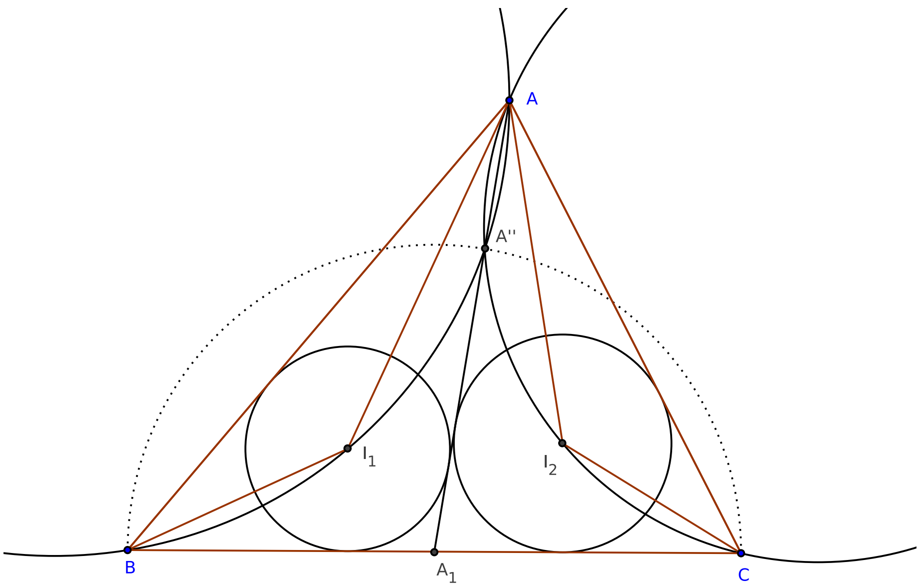
from which the result follows, completing the proof.

Editor's comment: More than half of the submitted solutions are similar to the one presented above. They all use one or more of Harmonic function, Gamma function, and Beta function.

4640. *Proposed by Mihaela Berindeanu.*

Let ABC be an acute triangle and A_1, B_1, C_1 be the feet of the medians from A, B, C respectively. Denote by I_1 and I_2 the centers of the inscribed circles of $\triangle\ ABA_1$ and $\triangle\ AA_1C$ respectively. The circumcircle of $\triangle\ ABI_1$ cuts the circumcircle of $\triangle\ AI_2C$ for the second time in A'. Define B' and C' analogously. Show that if $\overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} = \overrightarrow{0}$, then $\triangle\ ABC$ is an equilateral triangle.

We received six submissions, all correct, and we feature the solution by Oliver Geupel.



Since the angle A is acute, the circle with center A_1 and radius A_1B meets the median AA_1 at an interior point A''. We have

$$\begin{aligned}\angle AI_1B &= 180^\circ - \frac{\angle A_1BA + \angle BAA_1}{2}\\ &= 90^\circ + \frac{\angle AA_1B}{2} = 90^\circ + \frac{\angle A''A_1B}{2} = 180^\circ - \angle BA''A_1 = \angle AA''B;\end{aligned}$$

whence, the points A, A'', B, and I_1 are concyclic. Analogously, the points A, A'', C, and I_2 are concyclic. As a consequence, $A' = A''$. With similar arguments for B' and C', we deduce that the segments A_1A', B_1B', and C_1C' are each half as long as their corresponding sides $a = BC, b = CA$, and $c = AB$.

Using the centroid G of ΔABC, let us define points A_0, B_0, and C_0 by the conditions

$$\overrightarrow{GA_0} = \overrightarrow{A_1A'}, \qquad \overrightarrow{GB_0} = \overrightarrow{B_1B'}, \qquad \text{and} \qquad \overrightarrow{GC_0} = \overrightarrow{C_1C'}.$$

By hypothesis,

$$\overrightarrow{GA_0} + \overrightarrow{GB_0} + \overrightarrow{GC_0} = \overrightarrow{A_1A'} + \overrightarrow{B_1B'} + \overrightarrow{C_1C'} = \overrightarrow{0};$$

whence, G is the common centroid of the triangles ABC and $A_0B_0C_0$. Moreover, because each of the triplets $\{G, A, A_0\}$, $\{G, B, B_0\}$, and $\{G, C, C_0\}$ of points is collinear, there is a real number $\lambda > 0$ such that

$$\overrightarrow{GA_0} + \overrightarrow{GB_0} = \overrightarrow{C_0G} = \lambda\overrightarrow{CG} = \lambda(\overrightarrow{GA} + \overrightarrow{GB}).$$

Since the vectors $\overrightarrow{GA}$ and $\overrightarrow{GB}$ are linearly independent, it readily follows that $\overrightarrow{GA_0} = \lambda\overrightarrow{GA}$ and $\overrightarrow{GB_0} = \lambda\overrightarrow{GB}$. Thus, $\triangle ABC$ and $\triangle A_0B_0C_0$ are homothetic.

As a consequence, the median lengths m_a, m_b, and m_c of $\triangle ABC$ are proportional to its sides a, b, and c; that is, there is a real number $\mu > 0$ such that

$$\begin{aligned} 4m_a^2 &= 2(b^2 + c^2) - a^2 = \mu a^2, && (1)\\ 4m_b^2 &= 2(c^2 + a^2) - b^2 = \mu b^2, && (2)\\ 4m_c^2 &= 2(a^2 + b^2) - c^2 = \mu c^2. && (3) \end{aligned}$$

By adding (1), (2), and (3), we obtain $\mu = 3$. Hence, by (3),

$$2c^2 = a^2 + b^2.$$

Plugging this into (1), we deduce that $a^2 = b^2$. Analogously, $a^2 = c^2$. It follows that the triangle ABC is equilateral.

Robo Creativity – Answers

1. The robot PP creates itself.
2. The robot QQ destroys itself.
3. The robot SS is a friend of itself.
4. The robot PFP creates its best friend.
5. The robot SES is a friend of its worst enemy.
6. The robot FRFR is the best friend of one of its enemies.
7. The robots PCP and CPCP create each other.
8. The robots QDQ and DQDQ destroy each other.
9. The robot CQCQ creates the robot QCQ while QCQ destroys CQCQ. Alternatively, the robot PDP creates the robot DPDP while DPDP destroys PDP.
10. The robot CSCS creates the robot SCS while SCS is a friend of CSCS, though not the best friend of CSCS.
11. The robot FQEFQ is the best friend of the robot QEFQ while QEFE destroys EFQRFQ, the worst enemy of FQEFQ.
12. The robot CFQECFQ creates the robot FQECFQ the best friend of the robot QECFQ, while QECFQ destroys the robot ECFQECFQ, the worst enemy of CFQECFQ. Alternatively, the robot PFDEP creates the robot FDEPFDEP, the best friend of the robot DEPFDEP, while DEPFDEP destroys the robot EPFDP, the worst enemy of PFDEP.