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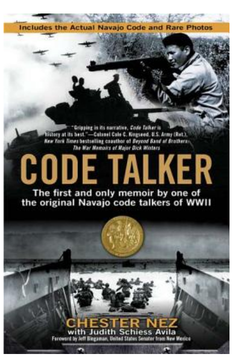
EDITORIAL

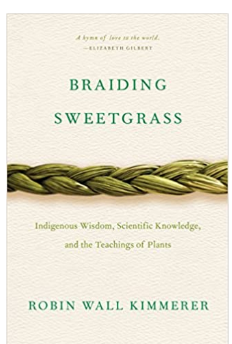
Tomorrow is my daughter's 5th birthday and we have been looking at her pictures, seeing how she grew and changed over time. When she was really little, I was often puzzled about which measurements doctors considered most important at which age. So I looked up models for children's height and weight and started using them in my class to motivate the study of functions. Then she got appendicitis and I used the ultrasound images of her belly to talk about Riemann sums and volumes of revolution. Then the pandemic happened and exponential functions, log-log plots, rate of disease spread, overall disease dynamics, probabilities of all sorts saturated the news. I had no shortage of content to bring to the classroom and, for once, I didn't even have to try hard to explain the significance and vital importance of mathematical analysis of these concepts. Current events made certain things in mathematics presently relevant. But this fall I'm teaching a history of mathematics course. How do I relate that to my students' everyday lives and experiences? What historical developments are important to highlight? What cultural context should be emphasized?

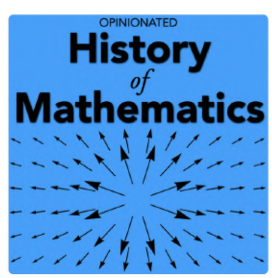
When we think of the history of math, we tend to think about ancient Greeks. On that front, I highly recommend the podcast "Opinionated History of Mathematics" by Viktor Blasjö that I binge-listened to. This no-frills podcast will get you to think about ancient Greek math as it relates to how we approach math today. But what about other cultures? This year, I have been reading a lot of historical books with strong, loose and seemingly no ties to math. I'm an omnivore when it comes to books. I've discovered many things I barely knew anything about that I want to incorporate into my teaching. If you are looking for a book, here are two of my top suggestions from my recent reading list: "Code Talker" by Chester Nez and Judith Schiess Avila and "Braiding Sweetgrass: Indigenous Wisdom, Scientific Knowledge and the Teachings of Plants" by Robin Wall Kimmerer. These books, powerful in different ways, have taught me a lot about history, cultural context and different ways of knowing.

And now I'm off to bake a heart-shaped cake.

Kseniya Garaschuk







The MathemAttic Article Contest

Calling all writers! The editorial staff at MathemAttic wish to announce an article writing contest.

MathemAttic is the section of ***Crux*** *Mathematicorum* aimed, generally, at high school students. What that means is that the level of material in that section doesn't go beyond the high school curriculum plus a few outside topics that would be familiar to students who participate in mathematics competitions. As such, there is more room for material of a more general nature: examples include the columns *Problem Solving Vignettes*, aimed at pre-university students interested in mathematics competitions; *Teaching Problems*, aimed at problems to aid teachers in their classrooms; and *Explorations in Indigenous Mathematics* aimed at uncovering mathematical topics from an indigenous point of view.

For the competition we are looking for expository articles in mathematics that would be of interest to the readers of MathemAttic. Check out Robert Dawson's "Writing an Expository Article for ***Crux***" in this issue, as well as "How to Write a ***Crux*** Article Revisited" [2019: 45(10), p. 562-563]. We will publish a number of the strongest papers in MathemAttic next year. There will also be a few prizes from the CMS available for exceptional articles.

We are particularly interested in hearing from students (high school or university), but we will accept articles from anybody (prizes will be limited to students). If you are a student, please provide us with your grade, age, and school. A word on credit: make sure you (briefly) acknowledge anybody who helped you significantly with research or with the overall presentation.

The contest deadline will be November 1, 2021. We hope to publish a number of the strongest papers, and will offer some feedback to help the writers prepare these for publication. Any entries received after November 1st will be eligible to be considered for publication, but will not be part of the contest.

Please email your submissions to `MathemAttic@cms.math.ca` with "MA Article Contest" in the subject line.

And now, put on your writing hat and give us your best shot!

Writing An Expository Article for *Crux*

Crux is about problems! We are not the place to send your latest research discovery, even if it's about inequalities or triangles. Our articles are mostly about problem-solving techniques, with examples of how they are used and some questions left for the reader to try. However, now that we no longer have printing costs and page limits, we can spread our net a little wider.

Do you know a bit of mathematics that you think is really neat and that not enough people know about? Consider writing an expository article for us. If you want to know what I mean, read some collections of articles by (for instance) Martin Gardner or Ian Stewart. The math they write about in these articles is largely done by other people; what they have done is to make it accessible to readers.

To write such an article, you need to know the subject very well, though there is absolutely no need to have done original research on it. (Gardner was a philosopher by training and a writer by trade until he started his famous "Mathematical Games" column in *Scientific American*, which he wrote for more than 24 years, providing inspiration for innumerable budding mathematicians in the 1960's and 1970's, including myself.) Don't just read one source, read everything you can find. The Internet will help you here and you are allowed to use it. (Wikipedia and Wolfram MathWorld are generally reliable, as are most published books.)

The article needs a coherent structure. What is this topic? Why should we be interested? Here are some neat things about it – and here's where to go look for more. Try very hard to include a short bibliography pointing the reader to the best accessible books on the subject; these do *not* have to be books you have quoted in the article. In a research article, the bibliography is a posse of your fellow mathematicians (living or dead) who back up your claims. In an expository article, it's mostly a reading list.

Giving credit in an expository article is mostly done in the body of the piece. Tell the story: give it the human touch. *Ramanujan discovered this identity in India when he had had little formal education in mathematics. Galois wrote that theorem down at the age of twenty, the night before he was killed in a duel. This was conjectured by Paul Erdős, the brilliant and eccentric Hungarian mathematician who spent much of his life travelling from university to university, and coauthored so many papers that mathematicians joke about their "Erdős number" the way movie actors are said to trace their distance from Kevin Bacon.* Use a bibliographic reference only for something unusual.

In research writing, rigour is essential: this sometimes requires a lot of mathematical notation. Even there, it can be overdone: a good writer will usually prefer words like "therefore," "and," and "not" to the corresponding symbols. In expository writing, we must still always tell the truth, but we can let our hair down a little more. As much as possible, we use natural language (in ***Crux***, English or French!) and you should usually be able to read your whole article aloud (or have a very good reason for each exception.) We don't need proofs for everything – in fact, they should usually be omitted.

There is some room for flexibility in style, much more than there is in research papers. Martin Gardner wrote some of his columns as stories about a numerologist named "Dr. Matrix," and the chemist Frederick Soddy once wrote a geometry theorem as a poem ("The Kiss Precise.") Don't try to be clever with the style unless you know why you're doing it, but if you think it will work, try it on us.

These are some hints, but if you want to write expository articles for ***Crux***, go and read the experts. Libraries usually have a number of popular math books made up of short essays on a number of topics. See how it's done.

We're going to have to maintain quite high standards, and undoubtedly won't publish everything we're sent. However, we will give special consideration to good articles by school-age and student authors (let us know in your cover email). Younger authors especially: please have somebody (not necessarily a mathematician) with a good writing style read your paper before you submit it. If the language in which you're writing is not one you are fluent in, please choose a first reader who can help you with your style.

A word on credit. While ***Crux*** papers aren't very formal, they should be professional. You should (briefly) acknowledge anybody who helped you significantly with research or with the overall presentation. This isn't the Oscars: please don't thank your parents, your Grade Six math teacher, or anybody else just for being important in your life. Typical phrasing: "I would like to thank Chen Xing for helpful discussions, and Dr. Anastasia Smirnova for assistance with the English version." If all somebody did was read through for comma faults, just thank them in person and make sure they see it when its published. If they did nearly as much of the work as you did, they ought to be a coauthor.

Thank you!

PS: We are still not publishing research articles. If you have proved a new result that you think is of general interest, get a second opinion from somebody who knows the field. Then consider periodicals such as the *American Mathematical Monthly* (or its sister publications *Mathematics Magazine* and the *College Math Journal* for more elementary material); *Fibonacci Quarterly*; *Recreational Mathematics Magazine*; *Journal of Classical Geometry*; and *Journal of Integer Sequences*. (Read some articles first to get a feel for what they want!)

If the result is self-contained, has a bit of pizazz, and will work as a problem, then you could submit it here as a problem. Send it to the problems editors, of course! Send your proof, too, please: it may well not be the one published, but it will help the editors evaluate your problem, and will serve as a backup in case the readers are stumped. (I don't need to tell you to read some ***Crux*** problems; but the less your problem looks like three that were published in the last year, the more interested the editors will be.)

Robert Dawson

MathemAttic

No. 25

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2021**.

MA121. If $a+b+c=0$ and $abc=4$, find $a^3+b^3+c^3$.

MA122. Four people Mr Baker, Ms Carpenter, Mr Driver, and Ms Plumber are employed for four jobs as a baker, carpenter, driver, and plumber. None of them has a name identifying their occupation. They make four statements:

1. Mr Baker says he is the plumber.
2. Mr Driver says he is the baker.
3. Ms Carpenter says she is not the plumber.
4. Ms Plumber says she is not the carpenter.

Exactly [illegible] of the four statements are true. Who is the driver? (One of the editors apologizes for spilling coffee on the page, but we are sure the question used to have a unique answer!)

MA123. A 12-sided polygon is inscribed in a circle of radius length l. What is the largest possible length of the shortest side of this polygon?

MA124. How many different 5-digit numbers can be formed using only the digits 1, 2, and 3, if digits placed consecutively must differ by at most 1?

MA125. Determine all positive integers a and b, $a<b$, so that exactly $\frac{1}{100}$ of the consecutive integers $a^2, a^2+1, a^2+2, \ldots, b^2$ are the squares of integers.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA121. Si $a+b+c=0$ et $abc=4$, déterminer $a^3+b^3+c^3$.

MA122. Les quatre métiers de boulanger, charpentier, décorateur et plombier sont occupés par Mme. Boulanger, M. Charpentier, M. Décorateur et Mme. Plombier. Cependant, aucun nom correspond au métier. De plus, on nous affirme que:
1. Mme. Boulanger est plombier.
2. M. Décorateur est boulanger.
3. M. Charpentier n'est pas plombier.
4. Mme. Plombier n'est pas charpentier.

De fait, exactement des quatre énoncés ci-haut sont vrais. Qui a le métier de décorateur? (La nombre exact dénoncés vrais a été accidentellement oblitérée, mais la question posée a une unique réponse.)

MA123. Un polygone à 12 côtés est inscrit dans un cercle de rayon l. Quelle est la plus grande longueur possible pour le plus petit côté de ce polygone ?

MA124. Combien de nombres à 5 chiffres peuvent être formés se servant seulement des chiffres 1, 2 et 3, sous la stipulation que des chiffres placés consécutivement doivent différer par au plus 1 ?

MA125. Déterminer tous les entiers positifs a et b, $a<b$, tels que parmi les entiers consécutifs $a^2, a^2+1, a^2+2, \ldots, b^2$, exactement $\frac{1}{100}$ sont des carrés d'entiers.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(10), p. 483–485.

MA96.

a) A circle passes through points with coordinates $(0, 1)$ and $(0, 9)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle.

b) Let a and b be any real numbers of the same sign (either both positive or both negative). A circle passes through points with coordinates $(0, a)$ and $(0, b)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle in terms of a and b.

Originally Problem 5 from The 36th W. J. Blundon Mathematics Contest.

We received 11 submissions, all of which were correct and complete. We present the solution by William Alexander Digout.

Let's start with part b) of the problem as it is a generalized version of part a). We will then use the general equations of part b) to easily arrive at the solution for part a).

b) Let a and b be any real numbers of the same sign (either both positive or both negative). Without loss of generality while also simplifying the problem, let us also add the condition that $|a| < |b|$, where the point with coordinates $(0, a)$ will be closer to the origin than the point with coordinates $(0, b)$.

Let A, B, C and E be points of interest, where A is the point with coordinates $(0, a)$, B is the point with coordinates $(0, b)$, C is the point at the centre of the circle and has coordinates (x_c, y_c) and E is the point with coordinates $(x_c, 0)$. E is the point of intersection between the circle and the positive part of the x-axis.

Let the point D be the midpoint of AB. The chord AB of the circle centred at C has the midpoint $D = \left(0, \frac{a+b}{2}\right)$ with $AD = \frac{a+b}{2} - a = \frac{b-a}{2}$. Since D is the midpoint of the chord AB, and D lies on the y-axis, C must lie on the line $y = y_D = \frac{a+b}{2}$.

Since E has coordinates $(x_C, 0)$ and E lies on the circle centred at C, then the line EC must be the same length as the radius of the circle centred at C. Therefore the radius of the circle is $\frac{a+b}{2}$.

We now simply need to find x_C.

ADC is a right triangle where $AD^2 + DC^2 = AC^2$ by the Pythagorean Theorem.

Thus

$$\left(\frac{b-a}{2}\right)^2 + x_C^2 = \left(\frac{b+a}{2}\right)^2 \iff x_C^2 = \frac{(b+a-(b-a))(b+a+(b-a))}{4}$$
$$\iff x_C^2 = ab$$
$$\iff x_C = \sqrt{ab}.$$

We conclude that the circle we are looking for has a radius of $\frac{a+b}{2}$ and is centred at the point with coordinates $\left(\sqrt{ab}, \frac{a+b}{2}\right)$.

a) Let $a = 1$ and $b = 9$. Therefore by our work in part (b), we find that the circle has a radius of $(9+1)/2 = 5$ and is centred at the point with coordinates $(\sqrt{1 \cdot 9}, 5) = (3, 5)$.

Remark. By symmetry, had the problem said the circle was tangent to the negative part of the x-axis, then the circle would have the same radius, but centred at the point with coordinates $\left(-\sqrt{ab}, \frac{a+b}{2}\right)$. A circle that passes through points with coordinates $(a, 0)$ and $(b, 0)$ and is tangent to the positive part of the y-axis would have the same radius as our problem, but would be centred at the point with coordinates $\left(\frac{a+b}{2}, \sqrt{ab}\right)$. A circle that passes through the points $(a, 0)$ and $(b, 0)$ and is tangent to the negative part of the y-axis would also have the same radius, but would be centred at the point with coordinates $\left(\frac{a+b}{2}, -\sqrt{ab}\right)$. This problem could also be solved through the use of the equation of a circle.

MA97. In London there are two notorious burglars, A and B, who steal famous paintings. They hide their stolen paintings in secret warehouses at different ends of the city. Eventually all the art galleries are shut down, so they start stealing from each other's collection. Initially A has 16 more paintings than B. Every week, A steals a quarter of B's paintings, and B steals a quarter of A's paintings. After 3 weeks, Sherlock Holmes catches both thieves. Which thief has more paintings by this point, and by how much?

Originally Problem 6 from The 36th W. J. Blundon Mathematics Contest.

We received 7 solutions. We present the solution by Nathan Kyubin Yoo.

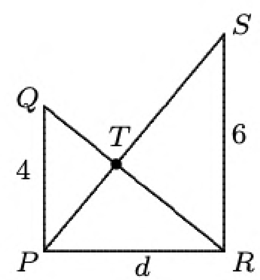
Let x be the number of paintings B has at the start. We assume that the thefts occur simultaneously, and we fill in the following table.

	A	B
start	$x + 16$	x
week 1	$\frac{3}{4}(x+16) + \frac{x}{4} = x + 12$	$\frac{3}{4}x + \frac{1}{4}(x+16) = x + 4$
week 2	$\frac{3}{4}(x+12) + \frac{1}{4}(x+4) = x + 10$	$\frac{3}{4}(x+4) + \frac{1}{4}(x+12) = x + 6$
week 3	$\frac{3}{4}(x+10) + \frac{1}{4}(x+6) = x + 9$	$\frac{3}{4}(x+6) + \frac{1}{4}(x+10) = x + 7$

Therefore 3 weeks later A has 2 more paintings than B.

Editor's Comment. Note that even with the assumption that x is a multiple of 4, after the 3rd week the thieves would have been reduced to stealing half-paintings (maybe some paintings come in multiple panels?). Taes Padhihary got around this issue by taking the integer value.

MA98. A pair of telephone poles d metres apart is supported by two cables which run from the top of each pole to the bottom of the other. The poles are 4 m and 6 m tall. Determine the height above the ground of the point T, where the two cables intersect. What happens to this height as d increases?



Originally Problem 7 from The 18th W. J. Blundon Mathematics Contest.

We received 5 solutions, of which 3 were correct and 2 were incomplete. The incomplete ones had correct results, but the derivations were not sufficiently detailed. We present the solution by Doddy Kastanya.

Let the foot of the perpendicular from T to PR be F. We know that $QP = 4$, $SR = 6$, and $PR = d$. Let

$$FR = a, FP = d - a, FT = h.$$

Using the fact that $\triangle PQR \sim \triangle FTR$, we find that

$$\frac{a}{d} = \frac{h}{4}.$$

Similarly, using $\triangle PSR \sim \triangle PTF$, we find that

$$1 - \frac{a}{d} = \frac{d-a}{d} = \frac{h}{6}.$$

The two equations together yield

$$1 - \frac{h}{6} = \frac{h}{4},$$

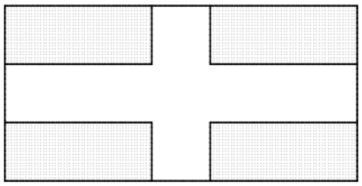
where we can isolate h to get

$$h = \frac{1}{\frac{1}{4} + \frac{1}{6}} = \frac{4 \cdot 6}{4 + 6} = \frac{12}{5}.$$

Note that this height h is independent of the distance d between the poles. Moreover, the same derivation shows that if the poles have height a and b, then the height of the point of intersection is

$$h = \frac{1}{\dfrac{1}{a} + \dfrac{1}{b}} = \frac{ab}{a+b}.$$

MA99. A flag consists of a white cross on a red field. The white stripes, both vertical and horizontal, are of the same width. The flag measures 48cm by 24cm. If the area of the white cross equals the area of the red field, what is the width of the cross?



Originally Problem 7 from The 18th W. J. Blundon Mathematics Contest.

We received 5 solutions, of which 2 were correct. There were 2 solutions that were incomplete because they did not say why a certain quadratic equation's roots contained an extraneous solution. We present the solution by Doddy Kastanya.

The area of the whole rectangle is 48 cm $\cdot$ 24 cm $= 1152$ cm^2. So the area of the red field and the area of the white cross are both half of this, which is 576 cm^2.

Let the width of the white stripes be w. The area of the white cross is the area of the $48 \times w$ horizontal stripe plus the area of the $24 \times w$ vertical stripe minus the area of the overlap, which is a $w \times w$ square. So the area of the white cross is, in centimetres squared,

$$576 = 48w + 24w - w^2 = 72w - w^2.$$

Solving for w in

$$w^2 - 72w + 576 = 0$$

using the quadratic formula yields

$$12(3 \pm \sqrt{5}).$$

The solution $12(3+\sqrt{5})$ is extraneous because it exceeds both dimensions of the overarching rectangle. Therefore, the only possible width of the white cross is

$$12(3 - \sqrt{5}) \text{ cm}.$$

MA100. Suppose the equation $x^3 + 3x^2 - x - 1 = 0$ has real roots a, b, c. Find the value of $a^2 + b^2 + c^2$.

Originally Problem 1 from The 36th W. J. Blundon Mathematics Contest.

We received 10 *solutions, of which* 7 *were correct. Most solutions simply cited Vieta's formulas. We present the solution by Muhammad Robith.*

Let

$$x^3 + 3x^2 - x - 1 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc.$$

By expansion, collecting like terms, and comparing coefficients, we find that

$$\begin{aligned} a+b+c &= -3, \\ ab+bc+ac &= -1, \\ abc &= 1. \end{aligned}$$

These are called Vieta's formulas. Recall that

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ac).$$

By Vieta's formulas,

$$a^2 + b^2 + c^2 = (-3)^2 - 2(-1) = 9 + 2 = 11.$$

PROBLEM SOLVING VIGNETTES

No. 16

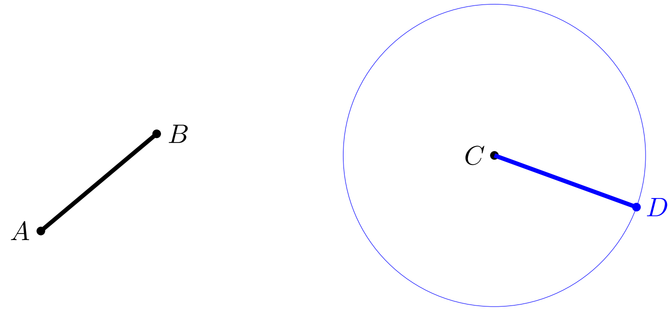
Shawn Godin

Geometric Constructions

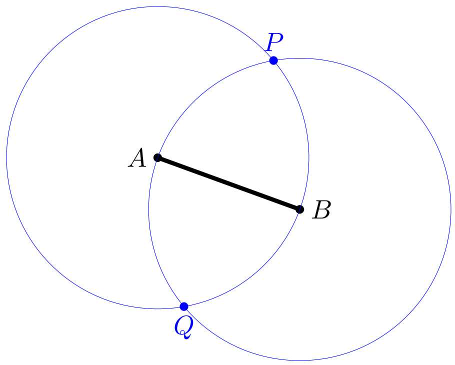
Geometry has fallen on hard times. There is significantly less geometry in the mathematics curriculum than there was eons ago when I was in high school and I was taught less geometry than those from earlier generations. This is a shame, as so much of the nature of mathematical thought can be experienced at a young age through the lens of geometry. It is also a pity since there are now so many dynamic geometry software packages, like GeoGebra, to help students grasp the geometric concepts.

One of the geometric topics that doesn't get the time it deserves is geometric constructions with compass and straightedge. It is amazing the number of geometric constructions that can be performed using just a compass and an unmarked ruler (just used for drawing lines, line segments and rays). In this column we will examine a few classical constructions.

A big, underlying idea is the fact that since we are using an unmarked ruler, we will use the compass as our "length-determiner". For example, if you had a line segment AB and a point C where you wanted to create another line segment CD that is the same length as AB, how could you do it? If we draw a circle with centre C whose radius is equal to the length of AB, then *any* point on the circumference of the circle will create, with C a segment whose length is equal to AB.



For our first construction, we will examine how to construct an equilateral triangle. If we take any line segment AB, and draw circles centred on A and B with radius AB, they will intersect at two points P and Q as in the diagram below.

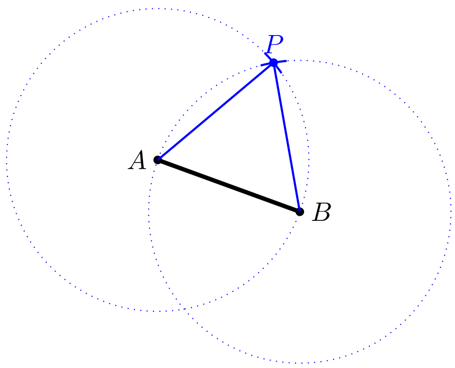


Since P and Q are on both circles then, by construction,

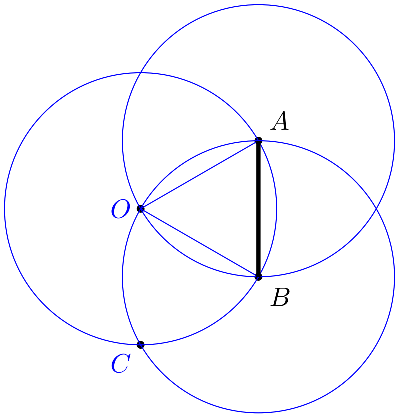
$$PA = PB = AB = QA = QB$$

and hence ΔPAB and ΔQAB are both equilateral triangles.

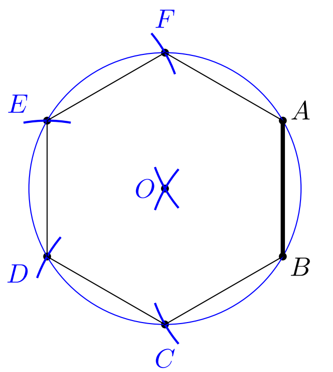
In the construction process, one wouldn't necessarily draw the whole circles to get at what we are after. Knowing the general region where the vertex should be, it is only necessary to draw an arc of the circle. So, for example, to construct the equilateral triangle ΔPAB, we could do as in the diagram below. The dotted circles wouldn't be drawn, they are just there for reference.



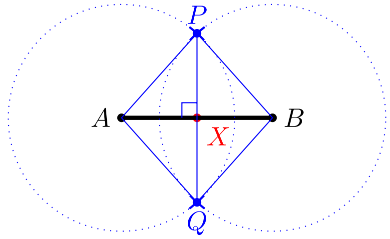
We can use this idea to construct a regular hexagon. Note that by joining the center O of a regular hexagon to each of the six vertices, you form six equilateral triangles (each with an angle of 60° at O bordered by two equal sides). If we start with a line segment, AB, that is to be one of our sides, we can use the method above to determine the point O such that ΔOAB is equilateral. Now, if we construct a circle with centre O that passes through A and B, this will be the circumcircle of our desired regular hexagon. If we were to draw a circle centred at B that passes through O it will also pass through A and another point, C on the circumference of the first circle such that ΔOBC is equilateral and congruent to ΔOAB.



Hence, using arcs instead of full circles you could use A and B to locate O, then from B get C and continuing around the circumcircle constructing the other vertices of the regular hexagon.



Next we will move on to some constructions related to geometric objects. The first is the construction of the midpoint of a given line segment. We saw in [2019 : 45(1), p. 13-16], while looking at congruent triangles, that the angle bisector of the apex, the median from the apex and the perpendicular bisector of the base of an isosceles triangle all coincide. We will use this idea to bisect a line segment. If we draw two circles, of equal radii, centred on A and B, large enough so they intersect, we claim that the two points of intersection, P and Q are on the perpendicular bisector of AB, where X is the point of intersection of AB and PQ.



To prove the claim we observe that ΔPAB and ΔQBC are congruent isosceles triangles by SSS, hence

$$\angle PAB = \angle PBA = \angle QAB = \angle QBA.$$

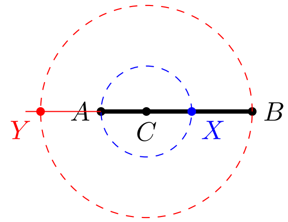
Similarly, we can deduce

$$\angle APQ = \angle AQP = \angle BPQ = \angle BQP$$

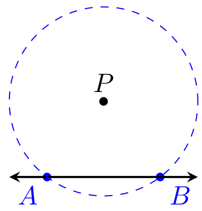
and hence the four small triangles ΔAXP, ΔBXP, ΔAXQ, and ΔBXQ are congruent by SAS. Thus, $AX = XB$ so X is the midpoint of AB. Also, any two adjacent angles at X are equal and supplementary, so all angles at X are 90° and PQ is perpendicular to AB.

This technique can be modified so that we can construct perpendiculars wherever we want. Thus, if C is a point on segment AB and we want to construct a line perpendicular to AB at C, we just have to find a segment that overlaps the segment AB with C as its midpoint. Then if we construct the perpendicular bisector of the segment, it will be the one and only line that is perpendicular to AB at the point C.

There are two possibilities. Without loss of generality, we can assume C is closer to A. Then, if we construct the circle centred C that passes through A, it will intersect AB at a point X. Thus our desired line is the perpendicular bisector of AX. Similarly, if we extend AB beyond A and construct the circle centred C that passes through B, it will intersect the extension of AB at a point Y and our desired line is the perpendicular bisector of AY.



Next, suppose we have a line ℓ, and a point P not on ℓ and we want to construct the line through P that is perpendicular to ℓ. To utilize the previous technique, we need P to be on the perpendicular bisector of a segment whose endpoints are on ℓ. Once again, we know that the apex of an isosceles triangle is on the perpendicular bisector of the base, so if we can create an isosceles triangle with apex P, whose base in on ℓ, we are half way there. How do we do this? Since our compass is our "length-determiner", we start by setting it to a large enough size so that the circle centred at P intersects ℓ at two points, A and B. Then, since $PA = PB$, ΔPAB is isosceles with base AB and apex P, so the perpendicular bisector of AB is our desired line.

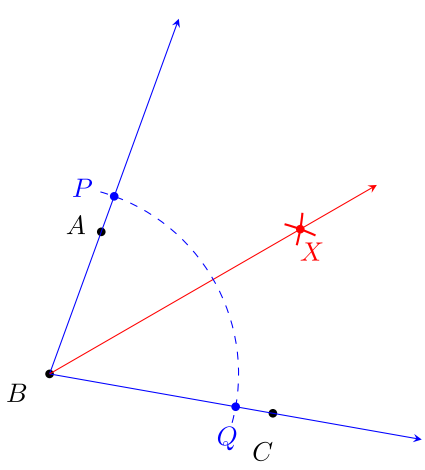


The last construction that we will look at is that for an angle bisector. That is, given three points A, B and C, construct the line that bisects $\angle ABC$. Again, we are going to use the idea of congruent triangles to help us out. If we can find points P, Q and X such that:

- P is on ray BA,
- Q is on ray BC,
- $\Delta BPX \cong \Delta BQX$

then $\angle PBX = \angle QBX$ and hence ray PX is our desired angle bisector.

By construction BX will be a common side for the two triangles, so if we can set it up so that $BP = BQ$ and $PX = QX$, our triangles will be congruent by SSS and we will have our angle bisector. We can easily force the sides to be equal by our construction. Thus, if we draw in rays BA and BC, set our compass to any size and draw an arc that cuts the two rays, these can be our points P and Q and we will have $BP = BQ$. Then, if we set our compass to any size large enough so that circles drawn with centres P and Q intersect, then either of our points of intersection is a candidate for X and, by construction, we will have $PX = QX$. To make our lives easier, we know that in any triangle, each side is shorter in length than the sum of the other two (triangle inequality). Thus, in ΔBPQ, we must have $PQ < BP + BQ$. Hence, if we keep our compass at the same setting, equal in length to both BP and BQ, then the circles will surely intersect and we will have our angle bisector.



I hope that gives you enough to play around with some constructions. In the next column, we will attack a couple of construction problems I faced in professor Honsberger's class. To keep you amused, here are a few construction challenges for you.

1. Given a line segment AB, construct the square $ABCD$.
2. Given a regular polygon with n sides, you can construct a regular polygon with $2n$ sides. Determine the technique and use it to construct a regular dodecagon (12-sided figure).
3. Given an angle, construct another angle of equal measure with vertex at a given point.
4. Given a line ℓ and a point P not on it, construct a line through P parallel to ℓ.
5. Draw any pentagon $ABCDE$. Construct a second pentagon $VWXYZ$ that is congruent to $ABCDE$.

My thanks goes to longtime *Crux* editor Chris Fisher for his feedback on this column and its sequel. His comments helped make both articles better.

OLYMPIAD CORNER

No. 393

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **July 15, 2021**.

OC531. Given a non-zero integer k, prove that equation

$$k = \frac{x^2 - xy + 2y^2}{x + y}$$

is satisfied by an odd number of ordered pairs of integers (x, y) if and only if k is divisible by 7.

OC532. Let $f : [a, b] \to \mathbb{R}$ a Riemann integrable function and let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$.

(a) If $A = \{m \cdot a_n \mid m, n \in \mathbb{N}^*\}$, prove that every open interval of positive real numbers contains elements of A.

(b) If for all $n \in \mathbb{N}^*$ and all $x, y \in [a, b]$ such that $|x - y| = a_n$ the following inequality holds

$$\left| \int_x^y f(t)\ dt \right| \leq |x - y|,$$

prove that

$$\left| \int_x^y f(t)\ dt \right| \leq |x - y| \qquad \forall x, y \in [a, b].$$

OC533. For $k \in \mathbb{Z}$, define the polynomial $F_k(x) = x^4 + 2(1 - k)x^2 + (1 + k)^2$. Find all values of k so that F_k is irreducible over $\mathbb{Z}[x]$ and reducible over $\mathbb{Z}_p[x]$ for all primes p.

OC534. The triangle $A_1A_2A_3$ is given on the plane. Assuming that $A_4 = A_1$ and $A_5 = A_2$, we define points X_t and Y_t for $t = 1, 2, 3$ as follows. Let Γ_t be the excircle of triangle $A_1A_2A_3$ tangent to the side $A_{t+1}A_{t+2}$, and let I_t be its center. Let P_t and Q_t be the points of tangency of Γ_t with the lines A_tA_{t+1} and A_tA_{t+2}, respectively. Then X_t and Y_t are the intersection points of the line P_tQ_t with the lines I_tA_{t+1} and I_tA_{t+2}, respectively. Prove that the points $X_1, Y_1, X_2, Y_2, X_3, Y_3$ lie on a circle.

OC535. The set A consists of n real numbers. For the subset $X \subseteq A$, we denote by $S(X)$ the sum of the elements of the set X, and we assume $S(\emptyset) = 0$. Let k be the number of different real numbers x such that $x = S(X)$ for some $X \subseteq A$. Let ℓ be the number of ordered pairs (X, Y) of subsets of the set A satisfying the equality $S(X) = S(Y)$. Prove that $k\ell \leq 6^n$.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC531. Soit k un entier non nul. Démontrer que l'équation

$$k = \frac{x^2 - xy + 2y^2}{x + y}$$

est satisfaite par un nombre impair de couples ordonnés d'entiers (x, y) si et seulement si k est divisible par 7.

OC532. Soit $f : [a, b] \to \mathbb{R}$ une fonction Riemann intégrable et soit $(a_n)_{n\geq 1}$ une suite de nombres réels positifs tels que $\lim_{n\to\infty} a_n = 0$.

(a) Si $A = \{m \cdot a_n \mid m, n \in \mathbb{N}^*\}$, démontrer que tout intervalle ouvert de nombres réels positifs contient des éléments de A.

(b) Si pour tout $n \in \mathbb{N}^*$ et tout $x, y \in [a, b]$ tel que $|x - y| = a_n$ l'inégalité suivante tient

$$\left|\int_x^y f(t)\ dt\right| \leq |x - y|,$$

démontrer que

$$\left|\int_x^y f(t)\ dt\right| \leq |x - y| \qquad \forall x, y \in [a, b].$$

OC533. Pour $k \in \mathbb{Z}$, définir le polynôme $F_k(x) = x^4 + 2(1-k)x^2 + (1+k)^2$. Déterminer toutes les valeurs de k telles que F_k est irréductible dans $\mathbb{Z}[x]$, mais réductible dans $\mathbb{Z}_p[x]$ pour tout nombre premier p.

OC534. Soit $A_1A_2A_3$ un triangle dans le plan. Posant $A_4 = A_1$ et $A_5 = A_2$, on définit alors des points X_t et Y_t pour $t = 1, 2, 3$, comme suit. Soit Γ_t le cercle exinscrit du triangle $A_1A_2A_3$ tangent au côté $A_{t+1}A_{t+2}$ et soit I_t son centre. Soient alors P_t et Q_t les points de tangence de Γ_t avec les lignes A_tA_{t+1} et A_tA_{t+2} respectivement. X_t et Y_t sont alors les points d'intersection de la ligne P_tQ_t avec les lignes I_tA_{t+1} et I_tA_{t+2} respectivement. Démontrer que les points $X_1, Y_1, X_2, Y_2, X_3, Y_3$ se trouvent sur un même cercle.

OC535. L'ensemble A consiste de n nombres réels. Pour le sous ensemble $X \subseteq A$, on va dénoter $S(X)$ la somme des éléments de X, prenant pour acquis que $S(\emptyset) = 0$. Soit k le nombre de réels différents x tel que $x = S(X)$ pour au moins un $X \subseteq A$. Enfin, soit ℓ le nombre de paires ordonnées (X, Y) de sous ensembles de A satisfaisant l'égalité $S(X) = S(Y)$. Démontrer que $k\ell \leq 6^n$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(10), p. 495–497.

OC506. A quadrilateral is called convex if the lines given by its diagonals intersect inside the quadrilateral. A convex quadrilateral has side lengths 3, 3, 4, 4 not necessarily in this order, and its area is a positive integer. Find the number of non-congruent convex quadrilaterals having these properties.

Originally from The Alberta High School Mathematics Competition, Part I, Question 16.

We received 8 submissions, of which 4 were correct and complete. We present the solution by Jason Smith.

Two kinds of quadrilaterals are possible: parallelograms or kites.

First consider the parallelograms, all of which are convex. The area of a parallelogram with consecutive sides, a and b, and one interior angle, θ, is $A = ab\sin\theta$. In our case, this becomes $A = 12\sin\theta$. The area is a positive integer if $\sin\theta$ takes one of the following 12 values

$$1/12,\ 2/12,\ 3/12,\ 4/12,\ 5/12,\ 6/12,\ 7/12,\ 8/12,\ 9/12,\ 10/12,\ 11/12,\ 12/12.$$

These yield 23 solutions for θ: 11 acute angles, 11 obtuse angles, and 1 right angle. However, each obtuse angle builds a parallelogram congruent to one built by an acute angle. Therefore, 12 non-congruent parallelograms exist having a positive-integer area.

Second consider the kites. The area of a kite with unequal consecutive sides a and b bordering an angle of size θ is also $A = ab\sin\theta$, or $A = 12\sin\theta$ in our case. As before $\sin\theta$ takes one of the following 12 values $1/12, 2/12, \ldots, 12/12$, and θ can be an acute, obtuse or right angle. However, the question asks for kites that are convex. If the two sides of length 3 form a straight line, the resulting shape is not a quadrilateral but rather a triangle, and $\cos\theta = 3/4$, $\sin\theta = \sqrt{7}/4$. In order for the kite to be convex, the acute-angle solutions for θ must therefore satisfy $\sin\theta > \sqrt{7}/4$. This corresponds to $\sin\theta$ being $8/12, 9/12, 10/12, 11/12$, leading to four acute-angle solutions. The right-angle solution and all obtuse-angle solutions for θ guarantees the convexity of the kite. Therefore there are $4 + 12 = 16$ non-congruent such kites with positive-integer area.

In summary, there are 28 non-congruent such quadrilaterals having a positive-integer area.

OC507. There are $2n$ consecutive integers written on a blackboard. In each move, they are divided into pairs and each pair is replaced with their sum and their difference, which may be taken to be positive or negative. Prove that no $2n$ consecutive integers can appear on the board again.

Originally from Spring 2020 Tournament of Towns, A-level Senior, Question 6.

We received 7 submissions of which 6 were correct and complete. We present a typical solution.

Let Q_0 denote the sum of squares of the original numbers, and let Q_m be the respective sum of squares after m moves, where $m > 0$. Since $(x+y)^2 + (x-y)^2 = 2(x^2 + y^2)$, we have $Q_m = 2Q_{m-1} = 2^m Q_0$.

If a is the smallest of the original numbers, then we have

$$\begin{aligned} Q_0 = \sum_{k=0}^{2n-1} (a+k)^2 &= 2na^2 + 2a \sum_{k=0}^{2n-1} k + \sum_{k=0}^{2n-1} k^2 \\ &= 2n\left(a^2 + (2n-1)a\right) + \frac{n(2n-1)(4n-1)}{3}. \end{aligned}$$

Denote by $\nu_2(x)$ the exponent of 2 in the prime factor decomposition of a positive integer x. The algebraic expression of Q_0 shown above, implies $\nu_2(Q_0) = \nu_2(n)$.

If, after m steps, $2n$ consecutive integers $b, b+1, \ldots, b+2n-1$ appear on the board, then

$$Q_m = 2n\left(b^2 + (2n-1)b\right) + (2n-1)n(4n-1)/3$$

and as before $\nu_2(Q_m) = \nu_2(n)$. We obtain two contradictory facts:

$$\nu_2(Q_0) = \nu_2(Q_m) = \nu_2(n) \quad \text{and} \quad \nu_2(Q_m) = \nu_2(Q_0) + m.$$

So we cannot get $2n$ consecutive numbers after m moves.

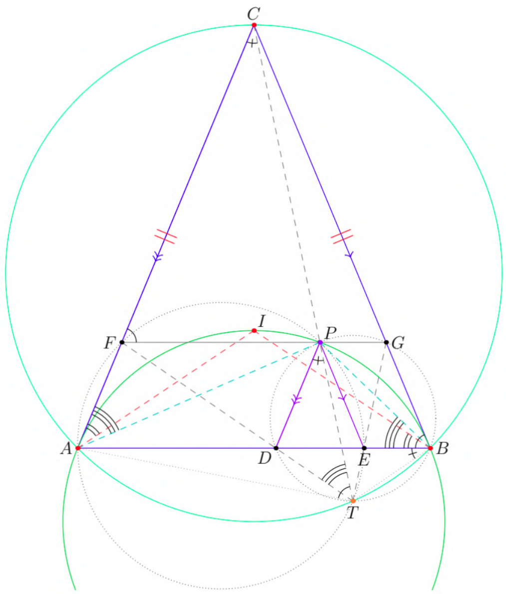
OC508. Let ABC be an isosceles triangle with $AC = BC$, whose incenter is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

Originally from the 2004 Swiss IMO Team First Selection Test, Question 3.

We received 6 submissions, all of which were correct and complete. We present two solutions.

Solution 1 by Taes Padhihary.

We present first the intuitive fact that leads to the result. By the problem conditions, note that $\triangle PDE$ and $\triangle CFG$ are homothetic. Therefore, the lines DF, EG and CP should concur at the center of their homothety.



Suppose the line CP meets the circumcircle of $\triangle ABC$ at point T. Then, we have that F, D, T and G, E, T are also collinear. Next we prove this crucial claim that establishes the problem.

Note that $\angle ATP = \angle ABC = \angle CFG = \angle PFC$, and so the quadrilateral $AFPT$ is cyclic. Again, $\angle TBD = \angle TCA = \angle TPD$ and so quadrilateral $DTBP$ is also cyclic. Now, note that $\angle IAB = \angle IBA = \angle IAF$, and therefore, CA is tangent to the circumcircle of $\triangle AIB$. Using the above findings, finally observe that

$$\angle PTF = \angle PAF = \angle PBA = \angle PBD = \angle PTD,$$

which implies that F, D, T are collinear. Similarly, we obtain G, E, T are collinear. Hence, we conclude that the lines DF and EG meet on the circumcircle of $\triangle ABC$.

Solution 2 by Corneliu Manescu-Avram.

Choose a Cartesian system of coordinates with C as origin and the altitude from C as the x-axis. Taking the length of this altitude as the unity, we have $C(0, 0)$, $A(1, a)$, and $B(1, -a)$, for some $a > 0$. It is easy to find the circumcentre of $\triangle ABC$ as $O((a^2 + 1)/2, 0)$, the incentre of $\triangle ABC$ as $I(a^2 + 1 - a\sqrt{a^2 + 1}, 0)$, and the circumcentre of $\triangle AIB$ as $O_1(a^2 + 1, 0)$. The equations of the circumcircles are

$$C(ABC) : x^2 + y^2 - (a^2 + 1)x = 0,$$

$$C(AIB) : x^2 + y^2 - 2(a^2 + 1)x + a^2 + 1 = 0.$$

If $P(u, v)$ is a point on the circumcircle of $\triangle ABC$ then

$$u^2 + v^2 - 2(a^2 + 1)u + a^2 + 1 = 0.$$

We find: $D(1, a(1-u)+v)$, $E(1, -a(1-u)+v)$, $F(u, au)$, and $G(u, -au)$. Hence we deduce that $M(x_M, y_M)$, the intersection point of DF and EG has coordinates

$$M\left(\frac{u^2}{2u-1}, \frac{uv}{2u-1}\right).$$

We find that M belongs to the circumcircle of $\triangle ABC$ because

$$x_M^2 + y_M^2 - (a^2+1)x_M = 0.$$

OC509. Prove that for any odd prime p the number of positive integers n satisfying $p|n!+1$ is smaller than or equal to $cp^{2/3}$ where c is a constant independent of p.

Originally from the 2009 Chinese IMO Team Selection Test, Day 1, Question 3.

We received 4 submissions, of which 3 were correct and complete. We present two solutions.

Solution 1 by Sergey Sadov.

Let $S = \{n \in \mathbb{N} \mid (n!+1) \equiv 0 \pmod p\}$. Note that $n \geq p$ implies $n! \equiv 0 \pmod p$, while $(p-1)! \equiv -1 \pmod p$ by Wilson's theorem. Hence max $S \leq p-2$. Let $s_1 \leq \cdots \leq s_m = p-1$ be the members of S. Assume that there are at least two numbers in S and let $d_j = s_{j+1} - s_j$, for $j = 1, \ldots, m-1$.

Denote by r_k the number of those d_j that are equal to a given k. Next we show that $r_k \leq k$. For any $s, s' \in S$ with $s' = s+k$ we have

$$p \mid \big(s'! - s!\big) = s!\big((s+1)(s+2)\ldots(s+k) - 1\big)$$

and hence $p|(s+1)(s+2)\ldots(s+k)-1$. The k-th degree polynomial

$$p_k(x) = (x+1)(x+2)\ldots(x+k) - 1$$

has at most k roots in the field $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p. Therefore, there exist at most k numbers $s \in S$ such that $s+k \in S$. In particular, $r_k \leq k$.

It follows that among the numbers $d_1, d_2, \ldots, d_{m-1}$ there is at most one which is equal to 1, at most two which are equal to 2, etc. Hence

$$1^2 + 2^2 + \ldots \ell^2 \leq d_1 + d_2 + \cdots + d_{m-1},$$

where ℓ is the greatest integer such that $1 + 2 + \cdots + \ell \leq m$.

Thus,

$$m < 1 + 2 + \cdots + (\ell+1) = (\ell+1)(\ell+2)/2 \tag{1}$$

and

$$\ell(\ell+1)(2\ell+1)/6 \leq d_1 + \cdots + d_{m-1}. \tag{2}$$

On the other hand,

$$d_1 + \cdots + d_{m-1} = s_m - s_1 < p. \tag{3}$$

Combining the last three inequalities we can get the required estimate.

First, (2) and (3) imply that $\ell^3/3 < p$, and $\ell < (3p)^{1/3}$. Second (1) implies

$$m \leq 2\ell^2 < 2 \times 3^{2/3} \times p^{2/3}$$

Hence there exists a constant $c = 2 \times 3^{2/3}$, independent of p such that $m < cp^{2/3}$.

Solution 2 by Corneliu Manescu-Avram.

As in the previous solution, let $S = \{n \in \mathbb{N} \mid (n! + 1) \equiv 0 \pmod{p}\}$ and note that $\max\ S \leq p - 2$.

Define A a subset of S as follows

$$A = \{n \in S \mid |n - m| > (p-1)^{1/3} \text{ for any } m \in S, m \neq n\}$$

Then the number of elements of A, Card $(A) < C_1(p-1)/(p-1)^{1/3} = C_1(p-1)^{2/3}$ for some constant C_1 independent of p. If $n \in S \setminus A$ then there exists a positive integer $1 \leq k \leq (p-1)^{1/3}$ for which $(n! + 1) \equiv 0 \pmod{p}$ and $((n+k)! + 1) \equiv 0 \pmod{p}$, whence

$$(n+1)(n+2)\dots(n+k) - 1 \equiv 0 \pmod{p}.$$

This is a polynomial congruence of degree k in n modulo p and has at most k solutions. Summing for all values of k we deduce that the cardinality $\text{Card}(S \setminus A) < C_2(p-1)^{2/3}$, where C_2 is a constant. In summary,

$$\text{Card}(S) = \text{Card}(A) + \text{Card}(S \setminus A) < Cp^{2/3}$$

for some constant C independent of p.

OC510. 2019 points are chosen independently and uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

Originally from the International Mathematics Competition 2019 Blagoevgrad Bulgaria, Day 2 Problem 10 (proposed by Fedor Petrov, St.Petersburg State University).

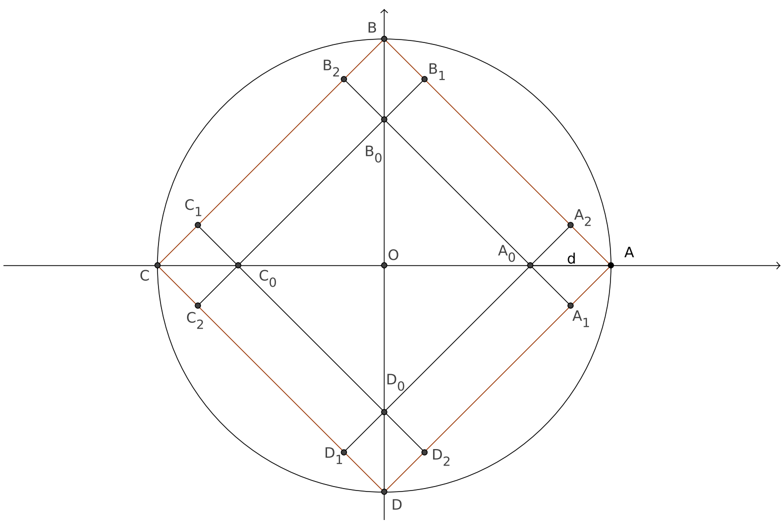
We received 6 submissions, of which 5 were correct and complete. We present the solution by Oliver Geupel.

We show that the probability p_4 that C is a quadrilateral is larger than the probability p_3 that C is a triangle. In fact, we prove that $p_4 > 10^{300} p_3$.

The convex hull C is a triangle only if three of the chosen 2019 points form a triangle where all other points are within. Also, among all triangles with vertices in the unit disk, an equilateral triangle with vertices on the boundary has the largest area which is $3^{3/2}/4$. Hence,

$$p_3 \leq \binom{2019}{3}\left(\frac{3^{3/2}}{4\pi}\right)^{2016}. \tag{1}$$

Next, consider the points $A(1,0)$, $B(0,1)$, $C(-1,0)$, $D(0,-1)$, and $A_0(1-d,0)$, where d is some very small positive real number. The parallel to AB through A_0 meets AD, BD, and BC at points A_1, B_0, and B_2, respectively. The parallel to BC through B_0 meets AB, AC, and CD at B_1, C_0, and C_2, respectively. The parallel to CD through C_0 meets BC, BD, and AD at C_1, D_0, and D_2, respectively. The parallel to AD through D_0 meets CD and AB at D_1 and A_2, respectively.



With our random experiment, consider the event E that exactly one point is in each of the four squares $A_0A_1AA_2$, $B_0B_1BB_2$, $C_0C_1CC_2$, and $D_0D_1DD_2$, and the other 2015 points are in the square $A_0B_0C_0D_0$. We have

$$p_4 \geq P[E] = 2019 \cdot 2018 \cdot 2017 \cdot 2016 \cdot \left(\frac{d^2}{2\pi}\right)^4 \left(\frac{2(1-d)^2}{\pi}\right)^{2015}. \tag{2}$$

We plan to prove that $p_3 < 10^{-300}p_4$. By (1) and (2), it is enough to show that, for some clever choice of d, it holds

$$\binom{2019}{3}\left(\frac{3^{3/2}}{4\pi}\right)^{2016} < \frac{2019 \cdot 2018 \cdot 2017 \cdot 2016}{10^{300}} \cdot \left(\frac{d^2}{2\pi}\right)^4 \left(\frac{2(1-d)^2}{\pi}\right)^{2015},$$

equivalently,

$$10^{300} \cdot \frac{3^{3021}}{2^{6049}} \cdot \frac{\pi^3}{7} < d^8(1-d)^{4030}.$$

Since $3^5 < 2^8$, $\pi < 4$, and $10^2 < 2^7$, we have

$$\frac{3^{3021}}{2^{6049}} \cdot \frac{\pi^3}{7} = \left(\frac{3^5}{2^8}\right)^{604} \cdot \frac{3\pi^3}{2^5 \cdot 7} \cdot \frac{1}{2^{7\cdot 173+1}} < 1 \cdot \frac{3 \cdot 4^3}{2^5 \cdot 7} \cdot \frac{1}{10^{2\cdot 173}} < \frac{1}{10^{346}}.$$

Let us put $d = 1/1000$. Applying the well-known inequality $1 + x > e^{\frac{x}{1+x}}$ which holds for every $x > -1$, we obtain

$$\begin{aligned} d^8(1-d)^{4030} = \frac{1}{10^{24}} \cdot \left(1 - \frac{1}{1000}\right)^{4030} &> \frac{1}{10^{24}} \cdot \frac{1}{e^{4030/999}} \\ &> \frac{1}{10^{24}} \cdot \frac{1}{e^5} > \frac{1}{10^{24}} \cdot \frac{1}{10^5} > \frac{1}{10^{46}}. \end{aligned}$$

This proves that $p_4 > 10^{300} p_3$.

Editor's Comments. Several generalizations of this questions were proposed. Corneliu Manescu-Avram indicated that if $n \geq 50$ points are randomly selected in the unit circle, then the probability that the convex hull of the points is a quadrilateral is greater than the probability that the hull is a triangle. Sergey Sadov claimed that for $m_2 > m_1$ there exists a large n depending on m_1 and m_2 such that if n points are randomly selected in the unit circle, then the probability that the convex hull of the points is a polygon with m_2 vertices is greater than the probability that the hull is a polygon with m_1 vertices.

FOCUS ON...

No. 46

Michel Bataille

Some Asymptotic Expansions

Introduction

Many problems asking for the limit of a sequence or the sum of a series can be solved *via* an appropriate asymptotic expansion. In this number, we illustrate the method with the Taylor expansions, of course, but also with the expansions of the nth harmonic number H_n and of $\ln(n!)$ as $n \to \infty$, which are of frequent use. We first present some basic tools with examples of the results they can produce. As usual, a selection of past problems will then be offered.

In what follows, the notations $u_n \sim v_n$ and $u_n = o(v_n)$ mean $\lim_{n\to\infty} \frac{u_n}{v_n} = 1$ and $\lim_{n\to\infty} \frac{u_n}{v_n} = 0$, respectively. Note that $u_n \sim v_n$ is equivalent to $u_n = v_n + o(v_n)$.

Main tools and first examples

Taylor's expansions often intervene in the kind of problems we are concerned with. Sometimes, a resort to these expansions suffices to obtain the required result, as in problem 720 of the *College Mathematics Journal*, proposed in January 2002:

> Find the limit of the following expression as $n \to \infty$:
>
> $$\frac{(n+3)^{n+3} - (n+2)^{n+2}}{(n+2)^{n+2} - (n+1)^{n+1}} - \frac{(n+2)^{n+2} - (n+1)^{n+1}}{(n+1)^{n+1} - n^n}.$$

We set $u_n = (n+1)^{n+1} - n^n$ so that the given expression is $A_n = \dfrac{u_{n+2}u_n - (u_{n+1})^2}{u_{n+1}u_n}$.

Let $p \in \{1, 2, 3\}$. Taylor's expansions of $\ln(1+x)$ and e^x then lead to

$$(n+p)\ln\left(1+\frac{p}{n}\right) = (n+p)\left(\frac{p}{n} - \frac{p^2}{2n^2} + o(1/n^2)\right) = p + \frac{p^2}{2n} + o(1/n)$$

$$\left(1+\frac{p}{n}\right)^{n+p} = \exp\left(p + \frac{p^2}{2n} + o(1/n)\right) = e^p\left(1 + \frac{p^2}{2n} + o(1/n)\right).$$

It follows that

$$u_n = n^{n+1}\left(\left(1+\frac{1}{n}\right)^{n+1} - \frac{1}{n}\right) = n^{n+1}\left(e + \frac{e-2}{2n} + o(1/n)\right)$$

$$u_{n+1} = n^{n+2}\left(1+\frac{2}{n}\right)^{n+2} - n^{n+1}\left(1+\frac{1}{n}\right)^{n+1} = n^{n+2}\left(e^2 + \frac{2e^2 - e}{n} + o(1/n)\right)$$

and

$$u_{n+2} = n^{n+3}\left(1+\frac{3}{n}\right)^{n+3} - n^{n+2}\left(1+\frac{2}{n}\right)^{n+2} = n^{n+3}\left(e^3 + \frac{9e^3-2e^2}{2n} + o(1/n)\right).$$

Thus, $u_{n+1}u_n \sim en^{n+1}\cdot e^2n^{n+2} = e^3n^{2n+3}$ and a simple calculation gives

$$u_{n+2}u_n - (u_{n+1})^2 = n^{2n+4}\left(\frac{e^4}{n} + o(1/n)\right) \sim e^4n^{2n+3}.$$

As a result, $A_n \sim \dfrac{e^4n^{2n+3}}{e^3n^{2n+3}}$ and the required limit is e.

Not very elegant, but quite efficient!

Another frequently used tool is comparison to an integral: if $f : [0,\infty) \to [0,\infty)$ is a continuous, non-increasing function, then for integers m, n such that $0 \le m \le n$, we have

$$\int_m^{n+1} f(x)\,dx \le \sum_{k=m}^{n} f(k) \le f(m) + \int_m^n f(x)\,dx.$$

[if f is nondecreasing, the inequalities are reversed.] These inequalities allow one to discover some classical results, to be known by all problem solvers:

$$\sum_{k=1}^{n}\frac{1}{k^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha} \quad (0<\alpha<1 \text{ or } \alpha \le 0),$$

$$\sum_{k=n+1}^{\infty}\frac{1}{k^\alpha} \sim \frac{1}{(\alpha-1)n^{\alpha-1}} \quad (\alpha>1),$$

$$H_n = \sum_{k=1}^{n}\frac{1}{k} \sim \ln(n),$$

$$\ln(n!) = \sum_{k=1}^{n}\ln(k) \sim n\ln(n).$$

(The details are left to the reader.)

It turns out that the following easily proved results are very useful: if $a_n \sim b_n$ and $b_n > 0$, then

(i) $\sum\limits_{k=1}^{n} a_k \sim \sum\limits_{k=1}^{n} b_k$ if $\sum\limits_{k=1}^{\infty} b_k$ is divergent [a form of the Stolz-Cesàro theorem] and

(ii) $\sum\limits_{k=n+1}^{\infty} a_k \sim \sum\limits_{k=n+1}^{\infty} b_k$ if $\sum\limits_{k=1}^{\infty} b_k$ is convergent.

Here is a simple example: let $w_n = H_n - \ln(n)$. Then,

$$w_n - w_{n-1} = \frac{1}{n} + \ln\left(1-\frac{1}{n}\right) \sim -\frac{1}{2n^2},$$

hence the series $\sum_{k=2}^{\infty}(w_k - w_{k-1})$ is convergent. Let s denote its sum. Then, applying (ii),

$$w_n - w_1 - s = -\sum_{k=n+1}^{\infty}(w_k - w_{k-1}) \sim \sum_{k=n+1}^{\infty}\frac{1}{2k^2} \sim \frac{1}{2n}$$

and we conclude that

$$H_n = \ln(n) + \gamma + \frac{1}{2n} + o(1/n)$$

where $\gamma = s + w_1 = s + 1$ (Euler's constant).

Similarly, to study the difference $\ln(n!) - n\ln(n)$, consider the series $\sum_{k\geq 2}(w_k - w_{k-1})$ where $w_k = \ln(k!) - k\ln(k)$. This time, since

$$w_n - w_{n-1} = \ln(n) - n\ln(n) + (n-1)\ln(n-1) = (n-1)\ln\left(1 - \frac{1}{n}\right) \sim -\frac{n-1}{n} \sim -1,$$

applying (i) gives

$$\ln(n!) - n\ln(n) = \sum_{k=2}^{n}(w_k - w_{k-1}) \sim \sum_{k=2}^{n}(-1) \sim -n,.$$

Thus, $\ln(n!) = n\ln(n) - n + o(n)$.

Continuing this way leads to

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + a + o(1) \tag{1}$$

where a is a real number [it can be shown that $a = \ln(\sqrt{2\pi})$; Stirling's well-known formula is then readily obtained].

The key idea deserves to be memorized: to obtain a simple sequence (w'_n) such that $w_n \sim w'_n$, consider the telescopic series $\sum(w_k - w_{k-1})$ and apply (i) or (ii).

More examples

Our first example is problem **4442** [2019 : 265 ; 2019 : 569]:

> Find the following limit
>
> $$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n}}\right).$$

Here is a solution which rests upon the results of the previous section.

Let $S_n = \frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{2n-1}+\sqrt{2n}}$. We show that $S_n \sim \frac{\sqrt{2}}{2}\cdot\sqrt{n}$ so that the required limit is $\frac{\sqrt{2}}{2}$.

First, we calculate

$$\begin{aligned}S_n &= (\sqrt{2}-\sqrt{1})+(\sqrt{4}-\sqrt{3})+\cdots+(\sqrt{2n}-\sqrt{2n-1})\\ &= \sqrt{2}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})-\left[(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{2n})-\sqrt{2}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})\right]\\ &= 2\sqrt{2}(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n})-(\sqrt{1}+\sqrt{2}+\cdots+\sqrt{2n}).\end{aligned}$$

Because $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n} \sim \frac{n^{1+\frac{1}{2}}}{1+\frac{1}{2}} = \frac{2n\sqrt{n}}{3}$, both terms are asymptotic to $\frac{4n\sqrt{2n}}{3}$, so we need a better estimate of the sum $\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}$.

To this aim, prompted by the key idea, we set $w_n = (\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}) - \frac{2n\sqrt{n}}{3}$ and consider $w_{n+1}-w_n$:

$$\begin{aligned}w_{n+1}-w_n &= \sqrt{n+1}-\frac{2(n+1)\sqrt{n+1}}{3}+\frac{2n\sqrt{n}}{3}\\ &= \frac{\sqrt{n+1}}{3}-\frac{2n(\sqrt{n+1}-\sqrt{n})}{3}\\ &= \frac{\sqrt{n}}{3}\left(1+\frac{1}{n}\right)^{1/2}-\frac{2n\sqrt{n}}{3}\left(\left(1+\frac{1}{n}\right)^{1/2}-1\right)\\ &= \frac{\sqrt{n}}{3}\left(1+\frac{1}{2n}+o(1/n)-2n\left(\frac{1}{2n}-\frac{1}{8n^2}+o(1/n^2)\right)\right)\\ &= \frac{\sqrt{n}}{3}\left(\frac{3}{4n}+o(1/n)\right)=\frac{1}{4\sqrt{n}}+o(1/\sqrt{n}).\end{aligned}$$

We deduce that $w_n - w_1 \sim \frac{1}{4}\sum_{k=1}^{n-1} k^{-1/2}$ and so $w_n \sim \frac{1}{4}\frac{\sqrt{n-1}}{1/2} \sim \frac{\sqrt{n}}{2}$. This means that

$$\sqrt{1}+\sqrt{2}+\cdots+\sqrt{n}=\frac{2n\sqrt{n}}{3}+\frac{\sqrt{n}}{2}+o(\sqrt{n})$$

and therefore

$$S_n = 2\sqrt{2}\left(\frac{2n\sqrt{n}}{3}+\frac{\sqrt{n}}{2}+o(\sqrt{n})\right)-\frac{4\sqrt{2}\,n\sqrt{n}}{3}-\frac{\sqrt{2n}}{2}+o(\sqrt{n})=\frac{\sqrt{2}}{2}\cdot\sqrt{n}+o(\sqrt{n}).$$

Thus, $S_n \sim \frac{\sqrt{2}}{2}\cdot\sqrt{n}$, as desired.

To show expansion (1) at work, we consider the first part of problem **96.L** proposed in *The Mathematical Gazette* in November 2012.

> Let $u_n = \dfrac{\sqrt[n]{n!}}{n}$ and $\alpha \in \mathbb{R}$. Find $\lim\limits_{n\to\infty} n^{\alpha}(u_{n+1}-u_n)$.

From (1) we deduce $\ln(u_n) = \frac{1}{n}\ln(n!) - \ln(n) = -1 + o(1)$, hence $u_n \sim e^{-1}$. Let $\delta_n = u_{n+1}-u_n$ and $\Delta_n = \ln\left(\frac{u_{n+1}}{u_n}\right)$ so that $\delta_n = u_n\left(\exp(\Delta_n)-1\right)$. We estimate

Δ_n as follows:

$$\begin{aligned}\Delta_n &= \frac{1}{n+1}\ln((n+1)!) - \ln(n+1) - \left(\frac{1}{n}\ln(n!) - \ln(n)\right)\\ &= \frac{\ln(n+1)}{n+1} + \left(\frac{1}{n+1} - \frac{1}{n}\right)\ln(n!) - \ln\left(1+\frac{1}{n}\right)\\ &= \left(1+\frac{1}{n}\right)^{-1}\frac{\ln(n)}{n} - \left(1+\frac{1}{n}\right)^{-1}\ln\left(1+\frac{1}{n}\right) - \frac{1}{n^2}\left(1+\frac{1}{n}\right)^{-1}\ln(n!)\\ &= \left(1+\frac{1}{n}\right)^{-1}\left(\frac{\ln(n)}{n} - \ln\left(1+\frac{1}{n}\right) - \frac{\ln(n)}{n} + \frac{1}{n} - \frac{\ln(n)}{2n^2} + o\left(\frac{\ln(n)}{n^2}\right)\right) \quad \text{(using (1)).}\end{aligned}$$

Now, $\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \sim \frac{1}{2n^2}$, hence $\frac{1}{n} - \ln\left(1+\frac{1}{n}\right) = o((\ln(n))/n^2)$ and we finally obtain $\Delta_n \sim -\frac{\ln(n)}{2n^2}$ and $\delta_n \sim u_n \cdot \Delta_n \sim -\frac{\ln(n)}{2en^2}$. We deduce that

$$\lim_{n\to\infty} n^{\alpha}(u_{n+1} - u_n) = 0 \quad \text{if} \quad \alpha < 2$$

and

$$\lim_{n\to\infty} n^{\alpha}(u_{n+1} - u_n) = -\infty \quad \text{if} \quad \alpha \geq 2.$$

We conclude this number with problem **4257** [2017 : 235 ; 2018 : 268], an interesting problem, the solution of which illustrates many of the tools introduced above.

Calculate the following limit

$$\lim_{n\to\infty}\left(\frac{\sqrt[n(n+1)]{1!\cdot 2!\cdot \cdots \cdot n!}}{\sqrt{n}}\right).$$

We show that the required limit is $e^{-3/4}$ by proving that $\lim\limits_{n\to\infty} u_n = -\frac{3}{4}$, where

$$u_n = \ln\left(\frac{\sqrt[n(n+1)]{1!\cdot 2!\cdot \cdots \cdot n!}}{\sqrt{n}}\right) = \frac{1}{n^2+n}\sum_{k=1}^{n}\ln(k!) - \frac{\ln n}{2}.$$

It is natural to first look for an asymptotic expansion of $\sum_{k=1}^{n}\ln(k!)$. We know that $\ln(n!) \sim n\ln(n)$ and since the series $\sum_{n\geq 1} n\ln(n)$ is divergent, it follows that $\sum\limits_{k=1}^{n}\ln(k!) \sim \sum\limits_{k=1}^{n} k\ln(k)$. The function $x \mapsto x\ln(x)$ being increasing on $[1,\infty)$, we have

$$\int_1^n x\ln(x)\,dx \leq \sum_{k=1}^{n} k\ln(k) \leq \int_1^{n+1} x\ln(x)\,dx$$

with

$$\int_1^n x\ln(x)\,dx = \left[\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right]_1^n = \frac{n^2\ln(n)}{2} - \frac{n^2}{4} + \frac{1}{4} \sim \frac{n^2\ln(n)}{2}$$

Since $\dfrac{(n+1)^2\ln(n+1)}{2} \;\sim\; \dfrac{n^2\ln(n)}{2}$, we obtain that

$$\sum_{k=1}^{n}\ln(k!) \;\sim\; \sum_{k=1}^{n}k\ln(k) \;\sim\; \frac{n^2\ln(n)}{2}.$$

Now, we consider $w_n = \dfrac{n^2\ln(n)}{2} - \displaystyle\sum_{k=1}^{n}\ln(k!)$ and estimate

$$z_n = w_n - w_{n-1} = \frac{n^2\ln(n)}{2} - \frac{(n-1)^2\ln(n-1)}{2} - \ln(n!)$$

as follows

$$\begin{aligned}
z_n &= \frac{n^2\ln(n)}{2} - \frac{n^2}{2}\left(1-\frac{1}{n}\right)^2\left(\ln(n)+\ln\left(1-\frac{1}{n}\right)\right) - \ln(n!)\\
&= \frac{n^2\ln(n)}{2}\left(1-\left(1-\frac{1}{n}\right)^2\right) - \frac{n^2}{2}\left(1-\frac{2}{n}+\frac{1}{n^2}\right)\left(-\frac{1}{n}+o(1/n)\right) - \ln(n!)\\
&= n\ln(n) - \frac{\ln(n)}{2} - \frac{n^2}{2}\left(-\frac{1}{n}+o(1/n)\right) - \ln(n!)\\
&= n\ln(n) - \frac{\ln(n)}{2} + \frac{n}{2} - n\ln(n) + n + o(n) = \frac{3n}{2} + o(n) \quad (\text{since } \ln(n) = o(n)).
\end{aligned}$$

Thus $z_n \;\sim\; \frac{3n}{2}$, from which we deduce first $\sum\limits_{k=1}^{n} z_k \;\sim\; \frac{3n^2}{4}$ and then $w_n \;\sim\; \frac{3n^2}{4}$. As a result, the desired expansion is

$$\sum_{k=1}^{n}\ln(k!) = \frac{n^2\ln(n)}{2} - \frac{3n^2}{4} + o(n^2).$$

We can now complete the solution and obtain the limit of u_n:

$$\begin{aligned}
u_n &= \frac{n}{n+1}\left(\frac{1}{n^2}\sum_{k=2}^{n}\ln(k!)\right) - \frac{\ln(n)}{2}\\
&= \left(1+\frac{1}{n}\right)^{-1}\left(\frac{\ln(n)}{2} - \frac{3}{4} + o(1)\right) - \frac{\ln(n)}{2}\\
&= \left(1-\frac{1}{n}+o(1/n)\right)\left(\frac{\ln(n)}{2} - \frac{3}{4} + o(1)\right) - \frac{\ln(n)}{2}\\
&= \frac{\ln(n)}{2} - \frac{3}{4} + o(1) - \frac{\ln(n)}{2} \quad \left(\text{since } \frac{\ln(n)}{n} = o(1) \text{ and } \frac{1}{n} = o(1)\right)\\
&= -\frac{3}{4} + o(1),
\end{aligned}$$

that is, $\lim\limits_{n\to\infty} u_n = -\frac{3}{4}$.

The reader may find it enjoyable to use the results and methods of this number to solve the following exercises.

Exercises

1. Use the Stolz-Cesàro theorem to prove that $\displaystyle\sum_{k=1}^{n} \frac{1}{n^\alpha} \sim \frac{n^{1-\alpha}}{1-\alpha}$ when $0 < \alpha < 1$.

2. Prove that $\sqrt[n]{n!} = \dfrac{n}{e} + \dfrac{1}{2e} \ln(n) + o(\ln n)$.

3. (proposed in Mathproblems in 2015) Let $n \in \mathbb{N}$ and let $O_n = 1+\dfrac{1}{3}+\cdots+\dfrac{1}{2n-1}$. Calculate

$$\lim_{n\to\infty} \frac{1}{n}\left(1+\frac{2O_n}{n}\right)^n .$$

4. (Problem 1122 proposed in March 2018 in *The College Mathematics Journal*). For each positive integer n, let

$$s_n = -2\sqrt{n} + \sum_{i=1}^{n} \frac{1}{\sqrt{k}}$$

and $\lim\limits_{n\to\infty} s_n = s$, the Ioachimescu constant. Find $\lim\limits_{n\to\infty} (s_n - s)\sqrt[2n]{n!}$.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **July 15, 2021**.

4641. *Proposed by Al Şeymanur.*

Let K, L, and M be the midpoints of the sides BC, CA, and AB, respectively, of an acute triangle ABC. Denote by

$$\begin{array}{ll} A', L', M'' & \text{the reflections of } A, L, M \text{ in the line } BC, \\ B', M', K'' & \text{the reflections of } B, M, K \text{ in the line } CA, \\ C', K', L'' & \text{the reflections of } C, K, L \text{ in the line } AB. \end{array}$$

Using square brackets to denote areas, we set $T = [ABC]$, $T' = [A'B'C']$, and $H = [K'M''L'K''M'L'']$. Prove that

$$4H - T' = 9T, \quad T' \leq 4T \quad \text{and} \quad H \leq \frac{13}{4}T.$$

4642. *Proposed by Adam L. Bruce.*

Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix and let $x \in \mathbb{R}^n$. Show that

$$(x^T A^2 x)^3 \leq (x^T A x)(x^T A^2 x)(x^T A^3 x).$$

4643. *Proposed by Nguyen Viet Hung.*

Find all pairs (m, n) of positive integers such that $\gcd(m, n) = 1$ and integer $(2^m - 1)(2^n - 1)$ is a perfect square.

4644. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let $z_1, z_2, z_3 \in \mathbb{C}$ be different numbers, with $|z_1| = |z_2| = |z_3| = 1$. Show that

$$|2z_1 - z_2 - z_3| + |z_3 - z_2| \geq \frac{1}{\sqrt{2}} \left(|z_2 - z_3||z_1 - z_3| + |z_2 - z_1||z_2 - z_3|\right).$$

4645. *Proposed by Leonard Giugiuc and Bogdan Suceava.*

Let a, b, c be positive real numbers such that $a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. Prove that

$$(a + b + c + d)^2 + 48abcd \geq 64.$$

4646. *Proposed by George Apostolopoulos.*

Let ABC be an acute triangle with inradius r and circumradius R. Prove that

$$\cot A + \cot B + \cot C \leq \sqrt{3}\left(\frac{R}{2r}\right)^2.$$

4647. *Proposed by Michel Bataille.*

In the plane, two circles Γ and γ intersect at A and B. Let M (resp. N) be a point of the arc of γ exterior (resp. interior) to Γ. If O is the centre of Γ, prove that

$$OM^2 - ON^2 = k(MA \cdot MB + NA \cdot NB)$$

for some real number k independent of the chosen points M and N.

4648. *Proposed by Corneliu Manescu-Avram.*

Let a be a positive integer and let $p > 3$ be a prime number such that $a^2+a+1 \equiv 0 \pmod{p}$. Prove that $(a+1)^p \equiv a^p + 1 \pmod{p^3}$.

4649. *Proposed by Mihaela Berindeanu.*

For $x, y, z \in \mathbb{R}$, show that

$$\frac{2^{10x}}{2^y + 2^z} + \frac{2^{10y}}{2^x + 2^z} + \frac{2^{10z}}{2^x + 2^y} \geq 2^{6x+2y+z-1} + 2^{6y+2z+x-1} + 2^{6z+2x+y-1}.$$

4650. *Proposed by Roberto F. Stöckli.*

Let $I_n = ((n-1)^2, n^2]$. Define $f(n) = 1$ if I_n contains exactly one triangular number (recall that the nth triangular number is $t_n = n(n+1)/2$) and $f(n) = 0$ otherwise. Find the value of

$$\lim_{n\to\infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 juillet 2021**.

La rédaction souhaite remercier Frédéric Morneau-Guérin, professeur à l'Université TÉLUQ, d'avoir traduit les problèmes.

4641. *Soumis par Al Şeymanur.*

Désignons respectivement par K, L et M les points milieux des côtés BC, CA et AB du triangle acutangle ABC. Notons

$$\begin{array}{cc} A', L', M'' & \text{la réflexion de } A, L, M \text{ sur la droite } BC, \\ B', M', K'' & \text{la réflexion de } B, M, K \text{ sur la droite } CA, \\ C', K', L'' & \text{la réflexion de } C, K, L \text{ sur la droite } AB. \end{array}$$

Posons $T = [ABC]$, $T' = [A'B'C']$ et $H = [K'M''L'K''M'L'']$, où les crochets sont employés pour indiquer que l'on considère l'aire. Montrez que

$$4H - T' = 9T, \quad T' \leq 4T \quad \text{ et } \quad H \leq \frac{13}{4}T.$$

4642. *Soumis par Adam L. Bruce.*

Soit $A \in \mathbb{R}^{n\times n}$ une matrice définie positive et soit $x \in \mathbb{R}^n$. Montrez que

$$(x^T A^2 x)^3 \leq (x^T Ax)(x^T A^2 x)(x^T A^3 x).$$

4643. *Soumis par Nguyen Viet Hung.*

Trouvez toutes les paires d'entiers positifs (m, n) pour lesquelles $(2^m - 1)(2^n - 1)$ est un carré parfait et $\mathrm{PGCD}(m, n) = 1$.

4644. *Soumis par Mihaela Berindeanu puis modifié par le comité de rédaction.*

Soient $z_1, z_2, z_3 \in \mathbb{C}$ des nombres complexes distincts vérifiant $|z_1| = |z_2| = |z_3| = 1$. Montrez que

$$|2z_1 - z_2 - z_3| + |z_3 - z_2| \geq \frac{1}{\sqrt{2}} \left(|z_2 - z_3||z_1 - z_3| + |z_2 - z_1||z_2 - z_3|\right).$$

4645. *Soumis par Leonard Giugiuc et Bogdan Suceava.*

Soient a, b, c des nombres réels positifs vérifiant $a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$. Montrez que

$$(a + b + c + d)^2 + 48abcd \geq 64.$$

4646. *Soumis par George Apostolopoulos.*

Soit ABC un triangle acutangle dont le rayon du cercle inscrit est r et celui du cercle circonscrit est R. Montrez que

$$\cot A + \cot B + \cot C \leq \sqrt{3}\left(\frac{R}{2r}\right)^2.$$

4647. *Soumis par Michel Bataille.*

Dans le plan cartésien, deux cercles Γ et γ se rencontrent en A et B. Soit M (respectivement N) un point de l'arc de γ qui est extérieur (respectivement intérieur) à Γ. Si O désigne le centre de Γ, montrez que

$$OM^2 - ON^2 = k(MA \cdot MB + NA \cdot NB)$$

pour un certain nombre réel k qui ne dépend pas des points M et N choisis.

4648. *Soumis par Corneliu Manescu-Avram.*

Soit a un entier positif et soit $p > 3$ un nombre premier vérifiant $a^2 + a + 1 \equiv 0 \pmod{p}$. Montrez que $(a+1)^p \equiv a^p + 1 \pmod{p^3}$.

4649. *Soumis par Mihaela Berindeanu.*

Étant donné $x, y, z \in \mathbb{R}$, montrez que

$$\frac{2^{10x}}{2^y + 2^z} + \frac{2^{10y}}{2^x + 2^z} + \frac{2^{10z}}{2^x + 2^y} \geq 2^{6x+2y+z-1} + 2^{6y+2z+x-1} + 2^{6z+2x+y-1}.$$

4650. *Soumis par Roberto F. Stöckli.*

Soit $I_n = ((n-1)^2, n^2]$. Posons $f(n) = 1$ si I_n contient exactement un nombre triangulaire (rappelons que le n-ième nombre triangulaire est $t_n = n(n+1)/2$) et $f(n) = 0$ sinon. Trouvez la valeur de

$$\lim_{n\to\infty} \frac{f(1) + f(2) + \cdots + f(n)}{n}$$

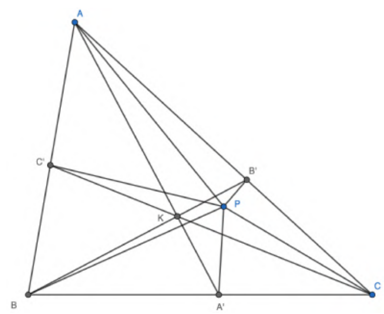
SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(10), p. 504–508.

4591. *Proposed by Pericles Papadopoulos.*

Let point P be inside triangle ABC and let A', B' and C' be the points where the internal bisectors of $\angle BPC$, $\angle CPA$ and $\angle APB$ intersect sides BC, CA and AB, respectively.



Show that lines AA', BB' and CC' concur at a point K satisfying

$$\frac{AK}{KA'} = PA\left(\frac{1}{PB} + \frac{1}{PC}\right).$$

We received 15 submissions, all of which were correct. Most solutions used variations of the same argument, while the three exceptions used coordinates. We will sample both approaches.

Solution 1 is a composite of the similar solutions using the majority approach.

Since PA' bisects angle $\angle BPC$, then $\dfrac{BA'}{A'C} = \dfrac{PB}{PC}$. Similarly we have

$$\frac{CB'}{B'A} = \frac{PC}{PA} \quad \text{and} \quad \frac{AC'}{C'B} = \frac{PA}{PB}. \tag{1}$$

Multiplying the left-hand sides gives us

$$\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1.$$

Since the three cevians lie inside the triangle, they cannot be parallel; consequently, Ceva's theorem implies that AA', BB' and CC' concur at some point K.

Van Aubel's triangle theorem together with (1) now gives

$$\frac{AK}{KA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C} = \frac{PA}{PB} + \frac{PA}{PC},$$

as required.

Editor's comment. Professor H. Van Aubel of the Royal Antwerp Athenaeum was a 19th century proposer and solver of mathematics problems. Two of his results are called "Van Aubel's theorem", both of which can be found on the internet or in geometry texts.

Solution 2, by Michel Bataille.

Let $PA = u, PB = v$, and $PC = w$. Since PA' is the internal bisector of $\angle BPC$, we have

$$\frac{A'B}{CA'} = \frac{PB}{PC} = \frac{v}{w}.$$

It follows that

$$w\overrightarrow{A'B} + v\overrightarrow{A'C} = \overrightarrow{0},$$

and so the barycentric coordinates of A' relatively to (A, B, C) are $(0 : w : v)$.

Similarly, we have $B' = (w : 0 : u)$ and $C' = (v : u : 0)$. The equations of the lines AA', BB' and CC' are quickly obtained:

$$AA' :\ vy - wz = 0; \qquad BB' :\ ux - wz = 0; \qquad CC' :\ ux - vy = 0,$$

and it is readily checked that the point $K = (vw : wu : uv)$ is on these three lines. Since $vw + wu + uv \neq 0$, K is not at infinity and we conclude that AA', BB', CC' concur at K.

In addition, we have

$$(vw+wu+uv)K = vwA+wuB+uvC = vwA+u(wB+vC) = vwA+u(w+v)A',$$

and so

$$\frac{AK}{KA'} = \frac{u(v+w)}{vw} = u\left(\frac{1}{v} + \frac{1}{w}\right) = PA\left(\frac{1}{PB} + \frac{1}{PC}\right).$$

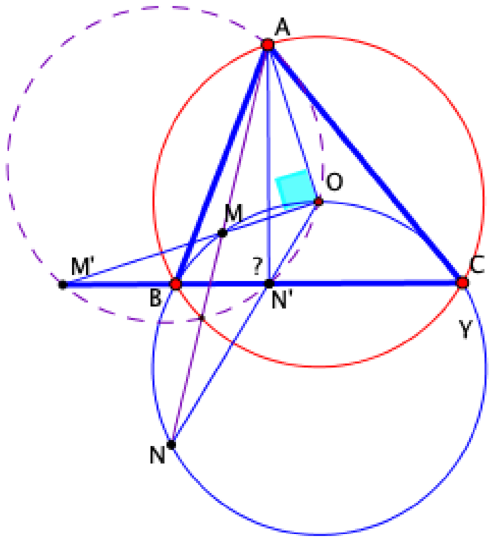
4592. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC \neq 90^\circ$ and let O be its circumcentre. Let γ be the circumcircle of ΔBOC. The perpendicular to OA at O intersects γ again at M and the line AM intersects γ again at N. Prove that

$$\frac{NO}{NA} = \frac{2OA^2}{AB \cdot AC}.$$

All 10 of the submissions were correct; we present two of the solutions.

Solution 1 is a composite of the similar submissions by Sergey Sadov and by the proposer.



Consider inversion in the circumcircle of the given triangle ABC. The inverse of γ (the circle that contains the points O, B, C) is the line BC, so that the inverses M' of M and N' of N must be the points where BC intersects the lines OM and ON, respectively. Because A, M, N are collinear, it follows that O, A, M', N' are concyclic and therefore

$$\angle M'N'A = \angle M'OA = 90^\circ.$$

Thus, in the right triangle $BN'A$ we have

$$N'A = AB \sin B.$$

But the sine law (applied to ΔABC) gives us $\sin B = \frac{AC}{2OA}$, whence

$$N'A = \frac{AB \cdot AC}{2OA}. \tag{1}$$

Finally, because the point A is fixed by our inversion while N and N' are interchanged, the triangles OAN and $ON'A$ are oppositely similar, which gives us

$$\frac{NO}{NA} = \frac{OA}{N'A}.$$

From equation (1) we conclude that

$$\frac{NO}{NA} = \frac{2OA^2}{AB \cdot AC},$$

as desired.

Solution 2, by Titu Zvonaru.

We will consider only the case in which ΔABC is acute angled with $AB < AC$, all other cases being similar. As usual we let R be its circumradius and a, b, c be its sides. Let O' be the circumcentre of ΔBOC and let D be the opposite end of the diameter from M of its circumcircle. Note that $D \in AO$ because $\angle MOD = 90°$. Finally, let $x = \angle NAO$. Applying the Law of Sines to ΔBOC then to ΔABC, we have (because $\angle BOC = 2A$)

$$OO' = \frac{a}{2\sin 2A} = \frac{a}{2(2\sin A)\cos A} = \frac{R}{2\cos A}.$$

Moreover,

$$\angle O'DO = \angle O'OD = 180° - \angle AOB - \angle BOO' = 180° - 2C - A = B - C.$$

Because $OO' = O'D = O'M$, it follows that

$$MO = 2 \cdot OO' \cdot \sin(B - C) = \frac{R\sin(B-C)}{\cos A}, \quad \text{and} \quad \tan x = \frac{OM}{AO} = \frac{\sin(B-C)}{\cos A}.$$

Since $\angle MNO = \angle MDO = B - C$,

$$\begin{aligned} \frac{NO}{NA} = \frac{\sin x}{\sin(x + B - C)} &= \frac{\sin x}{\sin x\cos(B-C) + \cos x \sin(B-C)} \\ &= \frac{\tan x}{\tan x \cos(B-C) + \sin(B-C)} \\ &= \frac{\frac{\sin(B-C)}{\cos A}}{\frac{\sin(B-C)}{\cos A}\cos(B-C) + \sin(B-C)} \\ &= \frac{1}{\cos(B-C) - \cos(B+C)} = \frac{1}{2\sin B \sin C} \\ &= \frac{2R^2}{AB \cdot AC}, \end{aligned}$$

as desired.

4593. *Proposed by Diaconu Radu.*

Solve the system of equations in real numbers:

$$\begin{cases} a^2 + bc = 7, \\ ab + bd = 3, \\ ac + dc = 2, \\ bc + d^2 = 6. \end{cases}$$

We received 35 submissions, of which 34 were correct. We present 2 solutions, one of which uses elementary algebra, and the other uses matrix theory in linear algebra.

Solution 1, by Prithwijit De.

It is evident that $b \neq 0$, $c \neq 0$ and $a + d \neq 0$. Note that

$$a^2 - d^2 = (a^2 + bc) - (bc + d^2) = 1,$$

whence $a - d = \dfrac{1}{a+d}$. Also, $\dfrac{b}{3} = \dfrac{c}{2} = \dfrac{1}{a+d}$.

Setting $a + d = k$ we have

$$a = \frac{1}{2}\left(k + \frac{1}{k}\right), \ b = \frac{3}{k}, \ c = \frac{2}{k} \text{ and } d = \frac{1}{2}\left(k - \frac{1}{k}\right).$$

Substituting the expressions of a, b and c in $a^2 + bc = 7$, we get

$$\frac{1}{4}\left(k^2 + \frac{1}{k^2}\right) + 1 + \frac{6}{k^2} = 7 \quad \text{or} \quad k^4 - 26k^2 + 25 = 0,$$

so $(k^2 - 25)(k^2 - 1) = 0$ which yields $k = \pm 5$ or $k = \pm 1$. Hence we obtain the solutions

$$(a, b, c, d) = \pm\left(\frac{13}{5}, \frac{3}{5}, \frac{2}{5}, \frac{12}{5}\right), \text{ or } \pm(1, 3, 2, 0).$$

Finally it is easy to check that these 4 quadruples satisfy the given equations so the proof is complete.

Solution 2, by Corneliu Manescu-Avram, slightly enhanced by the editor.

The given system is equivalent to the matrix equation

$$A^2 = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Let $t = \operatorname{trace}(A)$ and $r = \det(A)$. From $r^2 = \det A^2 = 36$ we get $r = \pm 6$. Using the Cayley-Hamilton Theorem we easily obtain $A^2 - tA + rI_2 = 0_2$ which yields, by applying the trace function, $13 - t^2 + 2r = 0$ or $t^2 = 13 + 2r$.

Since $t = a + d \neq 0$ by the condition $ab + bd = 3$, we have $A = \dfrac{1}{t}\left(A^2 + rI_2\right)$.

From $r = 6$, we get $t^2 = 25$ so $t = \pm 5$, yielding the two solutions:

$$\pm\frac{1}{5}\left(\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \pm\frac{1}{5}\begin{bmatrix} 13 & 3 \\ 2 & 12 \end{bmatrix}$$

Similarly, from $r = -6$ we get $t^2 = 1$ so $t = \pm 1$, yielding the two solutions:

$$\pm\left(\begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \pm\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$$

In conclusion, there are 4 solutions given by $(a, b, c, d) = \pm(1, 3, 2, 0)$ or $\pm(\frac{13}{5}, \frac{3}{5}, \frac{2}{5}, \frac{12}{5})$.

4594. *Proposed by Nguyen Viet Hung.*

Prove that for any point M on the incircle of triangle ABC,

$$\frac{MA^2}{h_a} + \frac{MB^2}{h_b} + \frac{MC^2}{h_c} = 2R + r,$$

where h_a, h_b and h_c are the lengths of the altitudes from A, B and C respectively, while R and r denote circumradius and inradius, respectively.

We received 14 submissions, 13 of which are correct and one of which was incomplete. We present two solutions.

Solution 1, by Michel Bataille.

Let F be the area and a, b, c be the sides of ABC. Because $2F = ah_a = bh_b = ch_c$ the left-hand side L of the desired equality can be written as

$$L = \frac{aMA^2 + bMB^2 + cMC^2}{2F}.$$

Since the incenter I is the center of mass of A, B, C with the masses a, b, c, respectively, Leibniz's relation yields

$$aMA^2 + bMB^2 + cMC^2 = (a + b + c)MI^2 + aIA^2 + bIB^2 + cIC^2$$

(see Focus On... No 16, Vol. 41(3), March 2015 p. 110). Moreover, we have

$$aIA^2 + bIB^2 + cIC^2 = abc$$

(see Focus On... No 28, Vol. 43(9), November 2017 p. 390). Thus, if M is on the incircle,

$$aMA^2 + bMB^2 + cMC^2 = (a + b + c)r^2 + abc = 2r^2 s + 4rsR$$

(where s is the semiperimeter) and since $2F = 2rs$, we finally obtain

$$L = \frac{2r^2 s + 4rsR}{2rs} = r + 2R,$$

as required.

Solution 2, by UCLan Cyprus Problem Solving Group.

Letting I denote the incenter of the triangle ABC we have

$$MA^2 = (\overrightarrow{MI} + \overrightarrow{IA}) \cdot (\overrightarrow{MI} + \overrightarrow{IA}) = (MI)^2 + (AI)^2 + 2\overrightarrow{MI} \cdot \overrightarrow{IA}.$$

So the required result will follow from the following three facts that we will proceed to prove

$$r^2 \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) = r \tag{1}$$

$$\frac{AI^2}{h_a} + \frac{BI^2}{h_b} + \frac{CI^2}{h_c} = 2R \tag{2}$$

$$\frac{1}{h_a}\overrightarrow{IA} + \frac{1}{h_b}\overrightarrow{IB} + \frac{1}{h_c}\overrightarrow{IC} = 0 \tag{3}$$

Writing Δ for the area of the triangle and s for its semiperimeter we have $\Delta = sr$. Since also $ah_a = 2\Delta$ etc, then

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a+b+c}{2\Delta} = \frac{s}{\Delta} = \frac{1}{r}$$

proving (1).

To prove (2), let D, E, F be the projections of I onto BC, AC, AB respectively. The area of the triangle AIF is equal to

$$\begin{aligned}
\frac{(AF)(IF)}{2} &= \frac{(IA\cos(A/2))(IA\sin(A/2))}{2} \\
&= \frac{IA^2\sin(A)}{4} = \frac{a(IA)^2}{8R} \\
&= \frac{\Delta(IA)^2}{4Rh_a}.
\end{aligned}$$

Summing up the areas of the other 5 triangles we get

$$\Delta = \frac{\Delta}{2R}\left(\frac{AI^2}{h_a} + \frac{BI^2}{h_b} + \frac{CI^2}{h_c}\right)$$

and therefore (2) follows.

To prove (3) we need to show that the vectors $\frac{1}{h_a}\overrightarrow{IA}, \frac{1}{h_b}\overrightarrow{IB}, \frac{1}{h_c}\overrightarrow{IC}$ form a triangle.

The angle between $\frac{1}{h_a}\overrightarrow{IA}$ and $\frac{1}{h_b}\overrightarrow{IB}$ is equal $(A+B)/2$. The sum of the three angles is equal to 180° and therefore to verify that the three vectors form a triangle it is enough to verify that the sine rule holds.

From the sine rule in triangle AIB, we have

$$\frac{AI}{BI} = \frac{\sin(B/2)}{\sin(A/2)}\,.$$

We also have $h_a = (AB)\sin(B)$ and $h_b = (AB)\sin(A)$. Therefore

$$\frac{AI/h_a}{BI/h_b} = \frac{\sin(B/2)\sin(A)}{\sin(A/2)\sin(B)} = \frac{\cos(A/2)}{\cos(B/2)} = \frac{\sin((B+C)/2)}{\sin((A+C)/2)}\,.$$

So there is a triangle with side lengths

$$\frac{AI}{h_a}, \quad \frac{BI}{h_b}, \quad \frac{CI}{h_c}$$

and angles

$$\frac{B+C}{2}, \quad \frac{C+A}{2}, \quad \frac{A+B}{2}.$$

Therefore (3) also holds and the proof is complete.

4595. *Proposed by Nguyen Viet Hung.*

Let $n > 2$ be an integer and let $S_n = \sum_{k=2}^{n} \sqrt{1+\frac{2}{k^2}}$. Determine $\lfloor S_n \rfloor$.

There were 21 solutions submitted, all correct. We present the solution by Arkady Alt and Florentin Visescu (done independently).

We have:

$$\begin{aligned} n-1 &< \sum_{k=2}^{n} \sqrt{1+\frac{2}{k^2}} < \sum_{k=2}^{n} \left(1+\frac{1}{k^2}\right) \\ &< \sum_{k=2}^{n} \left(1+\frac{1}{(k-1)k}\right) = (n-1) + \sum_{k=2}^{n} \left(\frac{1}{k-1}-\frac{1}{k}\right) \\ &= (n-1)+1-\frac{1}{n} < n, \end{aligned}$$

from which it follows that $\lfloor S_n \rfloor = n-1$.

Comment from the editor. Seven solvers used the upper bound $(\pi^2/6)-1$ and two the upper bound $\int_1^\infty x^{-2}\,dx = 1$ for $\sum_{k=2}^{n} k^{-2}$. Two solvers used the interesting identity

$$\sqrt{1+\frac{1}{(x-1)^2}+\frac{1}{x^2}} = 1+\frac{1}{x-1}-\frac{1}{x}.$$

4596. *Proposed by Boris Čolaković.*

Let a, b, c be the lengths of the sides of triangle ABC with inradius r and circumradius R. Show that

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b} \leq \frac{R}{r}-\frac{1}{2}$$

We received 39 solutions, all of which were correct. Of these solutions, 20 were submitted by Vivek Mehra. We present the solution by Nguyen Viet Hung.

The desired inequality is succesively equivalent to

$$\begin{aligned} \frac{a(a+b)(a+c)+b(b+c)(b+a)+c(c+a)(c+b)}{(a+b)(b+c)(c+a)}+\frac{1}{2} &\leq \frac{R}{r}, \\ \frac{a^3+b^3+c^3+(a+b+c)(ab+bc+ca)}{(a+b+c)(ab+bc+ca)-abc}+\frac{1}{2} &\leq \frac{R}{r}, \\ \frac{3abc+(a+b+c)(a^2+b^2+c^2-ab-bc-ca)+(a+b+c)(ab+bc+ca)}{(a+b+c)(ab+bc+ca)-abc}+\frac{1}{2} &\leq \frac{R}{r}, \\ \frac{3abc+(a+b+c)(a^2+b^2+c^2)}{(a+b+c)(ab+bc+ca)-abc}+\frac{1}{2} &\leq \frac{R}{r}. \end{aligned}$$

Now we note that

$$\begin{aligned}
&a+b+c=2s\\
&ab+bc+ca=s^2+4Rr+r^2\\
&abc=4sRr\\
&a^2+b^2+c^2=2(s^2-4Rr-r^2).
\end{aligned}$$

Therefore the inequality becomes

$$\begin{aligned}
\frac{12sRr+4s(s^2-4Rr-r^2)}{2s(s^2+4Rr+r^2)-4sRr}+\frac{1}{2}&\le\frac{R}{r},\\
\frac{6Rr+2(s^2-4Rr-r^2)}{(s^2+4Rr+r^2)-2Rr}+\frac{1}{2}&\le\frac{R}{r},\\
\frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}+\frac{1}{2}&\le\frac{R}{r},\\
2s^2(R-2r)+r(4R^2+4Rr+3r^2-s^2)&\ge 0,
\end{aligned}$$

which is true because by Euler's inequality $R\ge 2r$, and by Gerretsen's inequality $s^2\le 4R^2+4Rr+3r^2$. So the proof is completed.

4597. *Proposed by George Apostolopoulos.*

Let a,b,c be positive real numbers with $a+b+c=1$. Prove that

$$a^2+b^2+c^2+\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\ge 2(ab+bc+ca).$$

We received 29 submissions, of which 27 were correct and complete. We present two solutions.

Solution 1, by Arkady Alt.

Let $p=ab+bc+ca$ and $q=abc$, so $a^2+b^2+c^2=1-2p$ and $\dfrac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}=\dfrac{3q}{p}$.

The problem becomes to prove $1-4p+\dfrac{3q}{p}\ge 0$.

Since $3\left(ab+bc+ca\right)\le\left(a+b+c\right)^2$ then $0<p\le\dfrac{1}{3}$, and there are two cases.

If $p\in(0,1/4]$ since $p,q>0$ then $1-4p+\dfrac{3q}{p}>0$.

By Schur's Inequality, $9q\ge 4p-1$, and so,

$$1-4p+\frac{3q}{p}\ge 1-4p+\frac{3}{p}\cdot\frac{4p-1}{9}=\frac{(4p-1)\left(1-3p\right)}{3p}\ge 0$$

in the case that $p\in(1/4,1/3]$.

Solution 2, by Theo Koupelis, Angel Plaza and Titu Zvonaru (independently).

Homogenizing the given inequality we get the equivalent expression

$$a^2+b^2+c^2+\frac{3abc(a+b+c)}{ab+bc+ca}\geq 2(ab+bc+ca),$$

which, after clearing denominators becomes,

$$(a^2+b^2+c^2)(ab+bc+ca)+3abc(a+b+c)\geq 2(ab+bc+ca)^2.$$

After expanding and simplifying we get the equivalent inequality

$$(a^3b+b^3a)+(b^3c+c^3b)+(c^3a+a^3c)\geq 2a^2b^2+2b^2c^2+2c^2a^2,$$

which is obvious by AM-GM. Equality occurs when $a=b=c=\frac{1}{3}$.

4598. *Proposed by George Stoica.*

Let $P(z)$ be a polynomial of degree n with complex coefficients and with no zeroes z satisfying $|z|<1$. Prove that $|P(z)|\leq 2^n|P(rz)|$ for all $|z|\leq 1$ and $0<r<1$.

We received 8 submissions and they were all correct. We present the solution of Theo Koupelis, and Sonebi Omar (done independently).

Let z_i, $i=1,\ldots,n$ be the zeroes of the polynomial, with $|z_i|\geq 1$. Then

$$P(z)=a_n\prod_{i=1}^{n}(z-z_i)\qquad\text{and}\qquad P(rz)=a_n\prod_{i=1}^{n}(rz-z_i). \tag{1}$$

It is easy to use triangle inequality show that $|z-z_i|\leq 2|rz-z_i|$ for all i. Indeed,

$$|z-z_i|=|(z-rz)+(rz-z_i)|\leq |z-rz|+|rz-z_i|, \tag{2}$$

and also

$$(1-r)|z|+r|z|=|z|\leq 1\leq |z_i|\leq |rz-z_i|+|rz|\Longrightarrow |z-rz|\leq |rz-z_i|. \tag{3}$$

From (2) and (3) we get $|z-z_i|\leq 2|rz-z_i|$ for all i. Multiplying all such expressions and using (1) we get the desired result $|P(z)|\leq 2^n|P(rz)|$.

Editor's Comment. As pointed out by Walther Janous, a stronger inequality could be proved using trigonometry:

$$|P(z)|\leq\left(\frac{2}{1+r}\right)^n|P(rz)|. \tag{4}$$

The maximum and minimum modulus of polynomials have been studied by Govil, Lehmer, Rivlin, Qazi, and many other authors. Interested readers are encouraged to check Rivlin's paper for the proof of equation (4): T. J. Rivlin, On the maximum modulus of polynomials, American Mathematical Monthly **67** (1960), 251–253.

4599. *Proposed by Albert Natian.*

The sum of squares of the sides of a triangle ABC is 133. By enlarging two sides of ABC by a factor of 27, and a third side by a factor of 8, ABC is deformed into a larger but similar triangle RST whose area is 324 times that of ABC. Find the side lengths of ABC.

There were 19 solutions submitted, 17 of which were correct, one was incomplete and one was based on a misinterpretation. We present a common approach.

Let the sides of $\triangle ABC$ be a, b, c and the sides of $\triangle RST$ be $\{27a, 27b, 8c\}$, which in some order are to be proportional to $a : b : c$. We may suppose that $a \leq b$. Suppose, if possible that $c \leq b$. Then $8c \leq 27b$, so that b and $27b$ are the longest sides of their respective triangles. But $a : b : c$ is not proportional to $27a : 27b : 8c$ nor $8c : 27b : 27a$ (since $8c/a = 27 = 27a/c$ is inconsistent).

Therefore $a \leq b \leq c$. We cannot have $a : b : c$ proportional to $27a : 27b : 8c$ nor $27a : 8c : 27b$ (since $27 = 8c/b = 27b/c$ is inconsistent). So $a : b : c = 8c : 27a : 27b$, whence $27a^2 = 8bc$, $b^2 = ac$ and $8c^2 = 27ab$. Eliminating b from the first and third of these leads to $9a = 4c$ and $9b^2 = 4c^2$. Therefore

$$81 \times 133 = 81(a^2 + b^2 + c^2) = 16c^2 + 36c^2 + 81c^2 = 133c^2,$$

from which $(a, b, c) = (4, 6, 9)$.

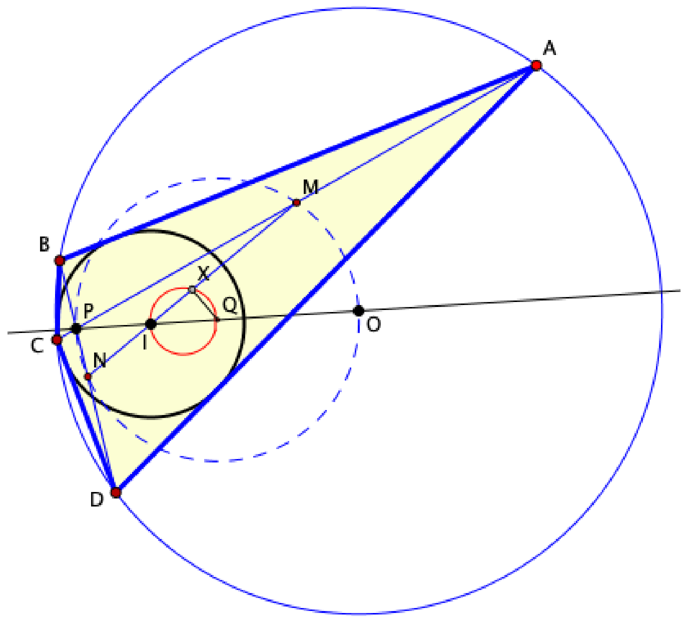
Editor's Comments. Madhav Modak noted that the condition $[RST] = 324[ABC]$ was redundant. The other solvers used this condition to immediately obtain the constant of proportionality and quickly eliminate the four ratios of augmented sides not proportional to $a : b : c$. C.R. Pranesachar noted, in a related situation, that if two noncongruent triangles are similar and share two sides, then the sides of each are in geometric progression with common ratio in the open interval $(1/\phi, \phi)$ where ϕ is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$.

4600. *Proposed by Semen Slobodianiuk, modified by the Editorial Board.*

It is known (for example, by a formula of Euler, often attributed to Nicolas Fuss, giving the distance between the centers in terms of the two radii) that given a bicentric quadrilateral inscribed in one circle and circumscribed about a second, then every point A of the circumcircle is the vertex of a bicentric quadrilateral $ABCD$ that is inscribed in the first circle, and circumscribed about the second. Determine the locus of the centroid of the vertex set $\{A, B, C, D\}$ as the bicentric quadrilateral $ABCD$ travels around the first circle while its sides stay tangent to the second.

We received 5 submissions; our featured solution is a composite of bits of the solutions from three of them, namely from Theo Koupelis, from Sergey Sadov, and from the proposer.

Denote the circumcircle with center O and radius R by (O, R), and the incircle with center I and radius r by (I, r). We need to define two further points, namely the point of intersection P of the diagonals AC and BD, and the midpoint Q of the segment PO. We will see that the desired locus is the circle whose diameter is IQ. The proof is based on three theorems, two of which come from Newton.



Theorem 1. *For all quadrilaterals $ABCD$ that are inscribed in (O, R) and circumscribed about (I, r) the point $P = AC \cap BD$ is fixed on the common diameter, with I between P and O.*

This result was part of problem 3256 in [1] where it was shown that P has a common polar line with respect to both circles; equivalently, P is the limiting point of the pencil determined by the circles. Numerous references accompany the proof given there, including [3, paragraph 1275, pages 564-565] where a version of the theorem is called "Newton's theorem" (although no explicit reference to Newton's work appears there). Alternatively, the fixed position of P follows from the equation that the distance from P to I satisfies:

$$R^2 = (OI)^2 + \frac{2r^4}{r^2 - (PI)^2}.$$

This equation is established in [2].

Theorem 2. *The midpoints of the three diagonals of a complete quadrilateral lie on a line; furthermore, if the quadrangle has an incircle (tangent to all four sides), then that line passes through the incenter.*

This result is also called "Newton's theorem"; references containing the proof are easily found such as [4], where the proof from [3, paragraph 1614, page 750] has been reproduced.

Theorem 3. *Given a circle with center U, the locus of the midpoint of a chord of the circle that rotates about a point V in the circle's interior is the circle whose diameter is UV.*

We do not require Newton's help in proving this result: If $M \neq U, V$ is the midpoint of a chord containing V, then $\angle UMV = 90^\circ$, whence M lies on the circle whose diameter is UV. Because V is in the interior of the circle, the point M sweeps out the entire circle as the chord turns through 180° about V.

We are now ready to investigate the centroid of a bicentric quadrilateral $ABCD$. The centroid of the point pair A and C is the midpoint M of AC, while the centroid of B and D is the midpoint N of BD; it follows that the centroid of the vertex set is the midpoint of MN, call it X. Our task is to describe the locus of X as $ABCD$ travels about (O, R) while its sides remain tangent to (I, r). With P (the intersection of the diagonals AC and BD) in the role of the fixed point V of Theorem 3, as the chord AC (of the circle (O, R)) rotates about the point P (which is fixed by Theorem 1), its midpoint M sweeps out the circle whose diameter is PO. Note that N sweeps out the same circle. The center of that circle, call it Q, is the midpoint of PO. By Theorem 2, for all positions of the vertex A on (O, R), the resulting line MN contains the incenter I. We again apply Theorem 3, this time with I in the role of the fixed point, and we conclude that the midpoint of X of MN (which is the centroid of the vertex set) sweeps out the circle whose diameter is IQ as the chord MN (of the circle on diameter OP) rotates about I.

References

[1] *Crux Mathematicorum with Mathematical Mayhem*, **34**:5 (Sep 2008), 312-313.

[2] H. Dörrie, *100 great problems of elementary mathematics: Their history and solution*, Dover Publications, 1965; Problem 39, pages 188-93.

[3] F. G.-M., *Exercices de géométrie—comprenant l'exposé des méthodes géometriques et 2000 questions résolues*, 4ième édition, Maison A. Mame & Fils, Tours 1907.

[4] https://www.cut-the-knot.org/Curriculum/Geometry/NewtonTheorem.shtml.