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IN THIS ISSUE / DANS CE NUMÉRO

Upcoming special issue in memory of Richard Guy

Canadian mathematical legend Richard Guy passed away on March 9th, 2020 at the age of 103. To honour his memory, we will have a special issue in fall 2020.

We encourage submissions of problems and articles as well as tributes and reminiscences. If you would like to contribute to the issue, please send the materials to `crux-editors@cms.math.ca` by August 1st.

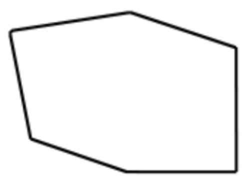
MathemAttic
No. 13

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2020**.

MA61. A hexagon has consecutive angle measures of 90°, 120°, 150°, 90°, 120° and 150°. If all of its sides are 4 units in length, what is the area of the hexagon?



MA62. A positive integer n is called "savage" if the integers $\{1, 2, \ldots, n\}$ can be partitioned into three sets A, B and C such that

i) the sum of the elements in each of A, B and C is the same,

ii) A contains only odd numbers,

iii) B contains only even numbers, and

iv) C contains every multiple of 3 (and possibly other numbers).

Now consider the following:

(a) Show that 8 is a savage integer.

(b) Prove that if n is an even savage integer, then $\dfrac{n+4}{12}$ is an integer.

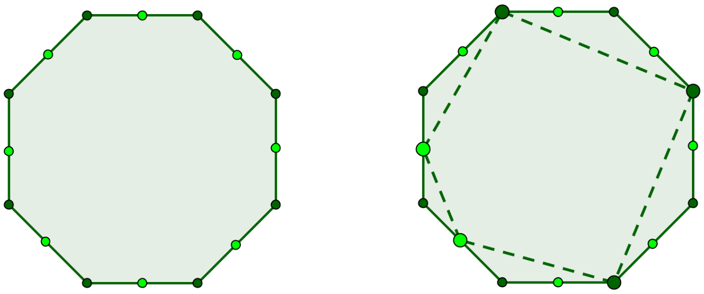
MA63. One way to pack a 100 by 100 square with 10 000 circles, each of diameter 1, is to put them in 100 rows with 100 circles in each row. If the circles are repacked so that the centres of any three tangent circles form an equilateral triangle, what is the maximum number of additional circles that can be packed?

MA64. A regular octagon is shown in the first diagram below, with the vertices and midpoints of the sides marked.

An "inner polygon" is a polygon formed by traversing the octagon in a clockwise manner, selecting some of the marked points as you go, ensuring that each side of the original octagon contains exactly one selected point. Then each selected point is connected to the next with a line segment, and the last is connected to the first to complete the inner polygon.

An example of an inner polygon is shown in the second diagram.

How many inner polygons does the regular octagon have?



MA65. There are four unequal, positive integers a, b, c, and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$?

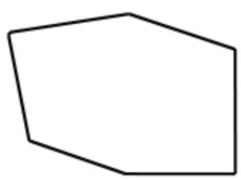
Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA61. Un hexagone a des angles, dans l'ordre, de 90°, 120°, 150°, 90°, 120° et 150°. Si tous les côtés sont de longueur 4, quelle est la surface de l'hexagone?



MA62. Un entier positif n est dit "sauvage" si les entiers $\{1, 2, \ldots, n\}$ peuvent être partitionnés en trois ensembles A, B et C de façon à ce que

i) les sommes des éléments dans A, B et C sont les mêmes,

ii) A contient seulement des entiers impairs,

iii) B contient seulement des entiers pairs et

iv) C contient tous les multiples de 3 (et possiblement d'autres nombres).

Alors:

(a) Démontrer que 8 est un entier sauvage.

(b) Démontrer que si n est un entier sauvage pair, alors $\dfrac{n+4}{12}$ est un entier.

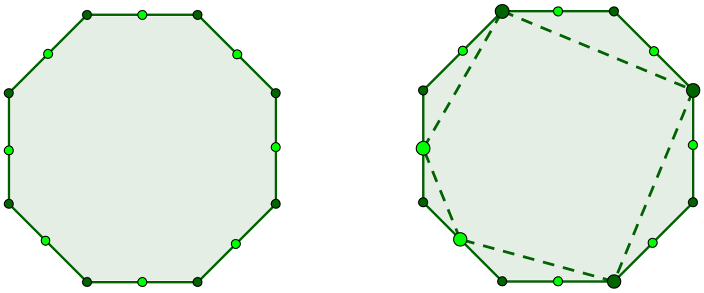
MA63. Une façon de placer 10,000 cercles de diamètre 1 dans un carré de taille 100 par 100 serait de placer 100 cercles dans chacune des 100 rangées. Si par contre on replace les cercles de façon à ce que les centres de trois cercles tangents forment un triangle équilatéral, quel est le nombre maximum de cercles additionnels pouvant être placés?

MA64. Un octagone régulier est indiqué au premier diagramme ci-bas, où sont marqués les sommets et les mi points des côtés.

Un "polygone interne" est un polygone formé en parcourant l'octagone dans le sens des aiguilles d'une montre, choisissant certains des points marqués, tout en s'assurant que chaque côté de l'octagone original contient exactement un point choisi. Et puis, chaque point choisi est relié au prochain avec un segment de ligne, le dernier étant relié au premier.

Un exemple d'un polygone interne est indiqué au deuxième diagramme.

Combien de polygones internes l'octagone a-t-il ?



MA65. Soient a, b, c et N, quatre entiers positifs distincts tels que $N = 5a+3b+5c$. De plus, $N = 4a+5b+4c$ et N se situe entre 131 et 150. Quelle est la valeur de $a+b+c$?

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(8), p. 447–449.

MA36. Let A and B be sets with the property that there are exactly 144 sets which are subsets of at least one of A or B. How many elements does the union of A and B have?

Originally problem 13 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 5 submissions, of which 3 were correct and complete. We present the solution by Digby Smith, slightly modified by the editor.

Let M be the number of elements in A and N the number of elements in B; without loss of generality, assume $N \leq M$. Let K be the number of elements in the intersection of A and B, and note $0 \leq K \leq N$.

The number of subsets (including the empty set) of a set with n distinct elements is 2^n, so the number of sets which are subsets of at least one of A and B is $2^M + 2^N - 2^K$. We have

$$144 = 2^M + 2^N - 2^K < 2^M + 2^M = 2^{M+1}$$

which gives us $8 \leq M + 1$, so $7 \leq M$. On the other hand,

$$144 = 2^M + 2^N - 2^K \geq 2^M,$$

so $7 \geq M$, and we conclude that $M = 7$.

Hence there are $144 - 128 = 16$ nonempty subsets which are subsets of B but not of A. There are $2^N - 2^K$ sets which are subsets of B but not of A, so we must have $N > K$ and also

$$2^K(2^{(N-K)} - 1) = 2^4.$$

If $N - K > 1$ then $2^{(N-K)} - 1$ is an odd number greater than 1, which cannot divide 2^4. Thus we must have $K = 4$ and $N = 5$. Hence the number of elements in the union of A and B is $M + N - K = 8$.

MA37. Both 4 and 52 can be expressed as the sum of two squares as well as exceeding another square by 3:

$$4 = 0^2 + 2^2 \quad \text{and} \quad 4 - 3 = 1^2,$$
$$52 = 4^2 + 6^2 \quad \text{and} \quad 52 - 3 = 7^2.$$

Show that there are an infinite number of such numbers that have these two characteristics.

Originally problem 123 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 7 submissions, of which 6 were correct and complete. We present the solution by Anita Hessami Pilehrood.

We need to show that there is a k such that $k = a^2 + b^2$ and $k = c^2 + 3$ for nonnegative integers a, b, c. Let $a = 2n$, $b = 2n^2 - 2$, and $c = 2n^2 - 1$ for a positive integer n. Then

$$a^2 + b^2 = 4n^2 + 4n^4 + 4 - 8n^2 = 4n^4 - 4n^2 + 4$$

and

$$c^2 + 3 = 4n^4 + 1 - 4n^2 + 3 = 4n^4 - 4n^2 + 4.$$

Therefore $a^2 + b^2 = c^2 + 3$ and hence there is an infinite sequence of numbers $k = 4(n^4 - n^2 + 1)$, $n = 1, 2, 3, \ldots$, that have these two characteristics.

MA38. Consider a 12×12 chessboard consisting of 144 1×1 squares. If three of the four corner squares are removed, can the remaining area be covered by placing 47 1×3 tiles?

Originally problem 33 of "A Mathematical Orchard: Problems and Solutions" by Mark I. Krusemeyer, George T. Gilbert, and Loren C. Larson.

We received 4 solutions, of which 3 were complete and correct. We present the solution by the Missouri State University Problem Solving Group.

Number the squares of the (intact) chessboard as shown in the figure.

3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3
3	1	2	3	1	2	3	1	2	3	1	2
2	3	1	2	3	1	2	3	1	2	3	1
1	2	3	1	2	3	1	2	3	1	2	3

Each number appears 48 times and any 1×3 tile will cover exactly one of each number, regardless of its position or orientation. Therefore, in order to tile the board with three corner squares removed, we must remove one of each square with a given number. In particular, we cannot remove the upper-left and lower-right corner squares, which are both labeled 3. Similarly, rotating the numbering scheme 90°, we conclude that we cannot remove the lower-left and upper-right corner squares. But removing any three corner squares will result in two diagonally opposite corners being removed and this gives us a contradiction.

MA39. Point E is selected on side AB of triangle ABC in such a way that $AE : EB = 1 : 3$ and point D is selected on side BC so that $CD : DB = 1 : 2$. The point of intersection of AD and CE is F. Determine the value of $\frac{EF}{FC} + \frac{AF}{FD}$.

Originally MAA Problem Book II (1961–1965), Question 37, 1965 examination.

We received 8 submissions, all correct. We present the solution provided by Anita Hessami Pilehrood.

Let the area of ΔCFD be y. Then $[DFB] = 2y$ since $\dfrac{DB}{DC} = 2$ and ΔDFB and ΔCFD share the height dropped from vertex F. Similarly, let the area of ΔAFE be x. Then the area of ΔEFB equals $3x$ since $\dfrac{EB}{AE} = 3$ and both triangles have a common height dropped from vertex F.

Now let's consider ΔBCE and ΔECA. Since $\dfrac{BE}{AE} = 3$ and these two triangles have a common height from vertex C, we have $\dfrac{[BCE]}{[ECA]} = 3$ which implies

$$\frac{y + 2y + 3x}{[CFA] + x} = 3,$$

and thus $[CFA] = y$.

Similarly in ΔBAD and ΔCAD, we have $\dfrac{BD}{DC} = 2$ and a common height from vertex A. Thus,

$$\frac{[BAD]}{[CAD]} = 2 \quad \text{implies} \quad \frac{4x + 2y}{2y} = 2, \quad \text{so} \quad \frac{x}{y} = \frac{1}{2}.$$

We have

$$\frac{EF}{FC} = \frac{[EAF]}{[FAC]} = \frac{x}{y} = \frac{1}{2},$$
$$\frac{AF}{FD} = \frac{[ACF]}{[FCD]} = \frac{y}{y} = 1,$$

and therefore

$$\frac{EF}{FC} + \frac{AF}{FD} = \frac{1}{2} + 1 = \frac{3}{2}.$$

MA40. In racing over a given distance d at uniform speeds, A can beat B by 20 yards, B can beat C by 10 yards, and A can beat C by 28 yards. Determine the distance d in yards.

Originally MAA Problem Book II (1961–1965), Question 37, 1961 examination.

We received 7 submissions, all of which were complete and correct. We present the solution of Aaratrick Basu, lightly edited.

Let v_A be the speed of A, v_B be the speed of B, and v_C be the speed of C.

As per the problem, A beats B by 20 yards, i.e,

$$\frac{d}{v_A} = \frac{d-20}{v_B} \tag{1}$$

Similarly, we have

$$\frac{d}{v_B} = \frac{d-10}{v_C}, \tag{2}$$

$$\frac{d}{v_A} = \frac{d-28}{v_C}. \tag{3}$$

Then from (1) and (3) we get

$$\frac{d-20}{v_B} = \frac{d-28}{v_C} \implies \frac{v_B}{v_C} = \frac{d-20}{d-28}.$$

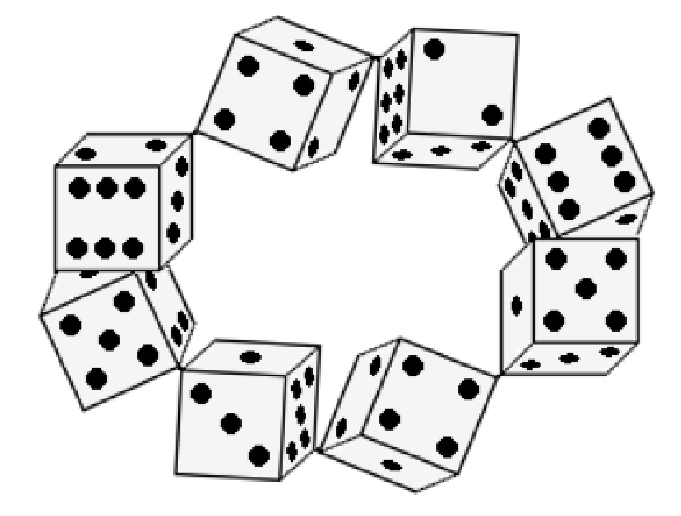
With (2) this becomes

$$\frac{d}{d-10} = \frac{d-20}{d-28} \implies d^2 - 30d + 200 = d^2 - 28d,$$

which reveals $d = 100$. Hence, we have that A, B, C were racing over $d = 100$ yards.

Bracelet made of cubes

Take eight unit cubes, or playing dice, and mark them with dots so that the sums of dots on opposite faces are all equal to 7 (that is, the opposite faces have 1 and 6 dots, 2 and 5 dots, 3 and 4 dots). Then, for each cube, drill an all-the-way-through diagonal hole from the vertex where faces with 1, 2 and 3 dots meet to the vertex where faces with 4, 5 and 6 dots meet. Take a strong thread and string all 8 cubes together through their holes in the direction they were drilled. Tie the thread to get a beautiful bracelet made of cubes:



Now, perform the following tasks:

1. fold this bracelet into a $2 \times 2 \times 2$ cube;
2. fold this bracelet into a $2 \times 2 \times 2$ cube so that the sum of dots on each of its faces is 14;
3. prove that you cannot fold this bracelet into a $2 \times 2 \times 2$ cube so that the sum of dots on each of its faces is 13.

Puzzle by Nikolai Avilov.

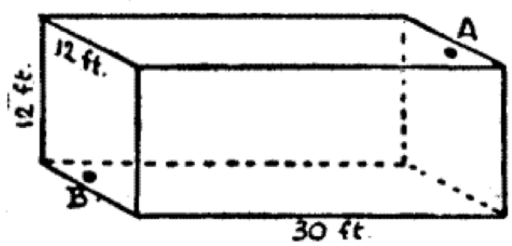
TEACHING PROBLEMS

No.9

Erick Lee

The Spider and the Fly

Inside a rectangular room, measuring 30 feet in length and 12 feet in width and height, a spider is at a point on the middle of one of the end walls, 1 foot from the ceiling, as at A; and a fly is on the opposite wall, 1 foot from the floor in the centre, as shown at B. What is the shortest distance that the spider must crawl in order to reach the fly, which remains stationary? Of course the spider never drops or uses its web, but crawls fairly.

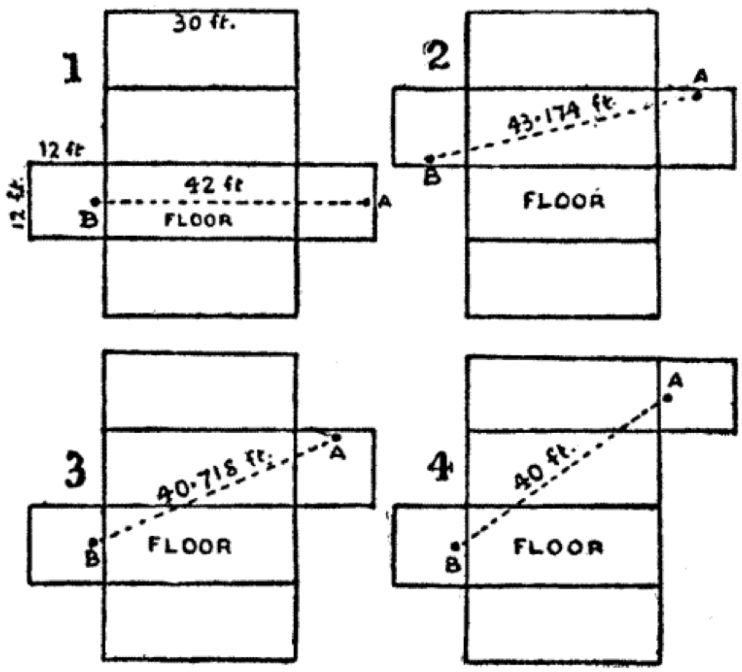


This problem was created by Henry Ernest Dudeney. It is problem 75 from his book *The Canterbury Puzzles* published in 1908. Dudeney was an English mathematician and prolific creator of logic puzzles and recreational mathematics problems. From 1910 until his death in 1930, Dudeney wrote a monthly column in *The Strand Magazine* entitled "Perplexities" which featured mathematical brain-teasers. As *The Canterbury Puzzles* was published over 100 years ago, it is freely available on The Project Gutenberg website at `http://www.gutenberg.org/files/27635/27635-h/27635-h.htm`.

When introducing this problem to students, I draw a spider and a fly each on their own index card and tape them to the appropriate spots on the wall in the classroom which is the shape of a rectangular prism, although not exactly the same dimensions as the given problem. I then describe the problem of the spider and the fly using the classroom to physically model the problem. Students often struggle to visualize problems in three dimensions despite living in a three-dimensional world. Many students will quickly determine that the spider should take the "straight path" directly up to the ceiling (1 ft), directly across the ceiling (30 ft), and down the opposite wall (11 ft). This will give a total distance of 42 ft.

After the majority of the class has come to this conclusion, we have a discussion. I ask them, "How do you know that this is the shortest path?" Students often respond that a straight line is the shortest distance between two points. To challenge their thinking, we discuss how a straight line might look different in three dimensions than in two dimensions. The great circle routes that airplanes fly often seem counterintuitive when students visualize the Mercator projection maps commonly found in classrooms. Where we live in Nova Scotia, we can look into the sky at almost any time of the day and see airplanes high in the sky flying from the Northeastern United States to European destinations. This only makes sense when looking at the great circle route on a spherical globe.

I challenge the class to brainstorm a variety of different routes that the spider might take and to calculate the distances for each of these new routes. To help in their brainstorming, I suggest that they examine possible routes on a net drawing of the room instead of a three dimensional drawing. Some students might find it helpful to model the room with a manipulative which allows them to link polygons together (such as *Polydrons*) on which they could label the walls, floor and ceiling of the room as well as the position of the spider and the fly. This would allow students to see how the path might change depending on how they create the net of the room. You might challenge the students to find a net that results in the spider crossing three sides of the room, four sides of the room or even five sides of the room and to see how these different nets result in different distance paths. Eventually, students will find the solution of the shortest path. Below are four different nets that Dudeney showed in *The Canterbury Puzzles*.

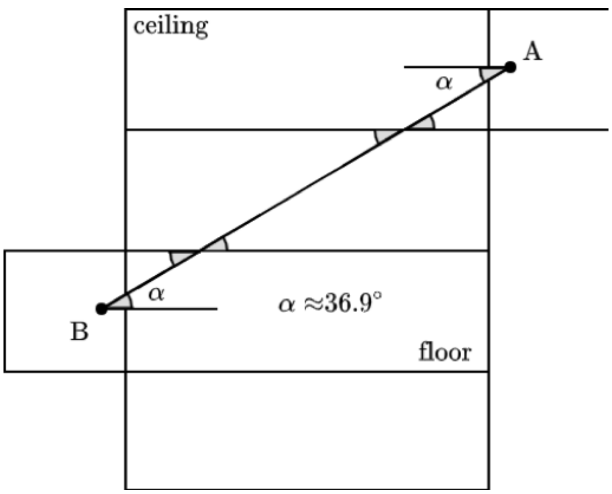


The distances for each of the solutions above can be found by applying the Pythagorean Theorem. Students at times have difficulty imaging the shortest path given in solution number 4. Returning to the pictures of the spider and the fly taped to the walls, a long piece of yarn is used to show the path of the spider along the sides of the classroom.

In his book *The Pythagorean Theorem: A 4,000-Year History*, Eli Maor describes how we could accurately trace the spider's path using trigonometry. In case 4 above, the spider's horizontal distance is 1 + 30 + 1 = 32 feet and the vertical distance travelled is 6 + 12 + 6 = 24 feet.

$$\tan\alpha = \frac{24}{32}$$
$$\arctan\frac{24}{32} = \alpha$$
$$\alpha \approx 36.9^\circ$$

The diagram below shows how the spider travels using this angle across the sides the room.



To imagine why this is the shortest path think about the shape of the room as a cylinder with hemispherical ends instead of a rectangular prism (like a hot dog instead of a block of wood). Imagine the piece of yarn wrapping around this shape from the spider to the fly's position. Now imagine if this "hot dog" shape slowly changed shape, or "deflated", until it was the rectangular prism. The curving path from the cylinder would now be the angled path of the spider around the classroom.

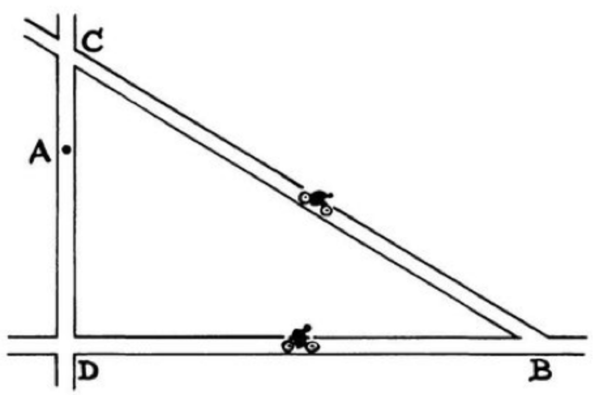
There are several extensions to this problem that could be explored:

- How do the dimensions of the room affect the shortest path that the spider takes? Would the path from the problem we solved be the shortest path for a room with any dimensions?
- How does the spider's height on the wall affect this problem? For which starting heights, h, would there be a different shortest route?
- Investigate the study of geodesics. How do geodesics apply to this problem?

A Follow Up Problem – The Russian Motorcyclists

The following is another problem from Henry Ernest Dudeney which was published in *The Strand Magazine*, Volume 53 (1917).

> Two Army motorcyclists, on the road at Adjbkmlprzll, wish to go to Brczrtwxy, which, for the sake of brevity, are marked in the accompanying map as A and B. Now, Pipipoff said: "I shall go to D, which is six miles, and then take the straight road to B, another fifteen miles." But Sliponsky thought he would try the upper road by way of C. Curiously enough, they found on reference to their cyclometers that the distance either way was exactly the same. This being so, they ought to have been able easily to answer the General's simple question, "How far is it from A to C?" It can be done in the head in a few moments, if you only know how. Can the reader state correctly the distance?



There are several ways to solve this problem with a bit of algebra and the application of the Pythagorean Theorem. Dudeney cryptically states that, "It can be done in the head in a few moments, if you only know how." Can you deduce the clever solution method that Dudeney is referring to?

Erick Lee is a Mathematics Support Consultant for the Halifax Regional Centre for Education in Dartmouth, NS. Erick blogs at `https://pbbmath.weebly.com/` *and can be reached via email at* `elee@hrce.ca` *and on Twitter at* `@TheErickLee`

OLYMPIAD CORNER

No. 381

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **May 15, 2020**.

OC471. There are $n > 3$ distinct natural numbers less than $(n-1)!$ written on a blackboard. For each pair of these numbers, Sergei divided the bigger number by the smaller with the remainder and wrote on his notebook the resulting incomplete quotient. For example, so if he divided 100 by 7, he got $100 = 14 \cdot 7 + 2$ and wrote 14 in the notebook. Prove that among the numbers in the notebook there are two that are equal.

OC472. Let $P(x)$ be a polynomial of degree $n \geq 2$ with nonnegative coefficients and let a, b and c be the side lengths of an acute-angled triangle. Prove that the numbers $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$ and $\sqrt[n]{P(c)}$ are also the side lengths of an acute-angled triangle.

OC473. In square $ABCD$, let M be the midpoint of AB, let P be the projection of point B onto line CM and let N be the midpoint of segment CP. The angle bisector of $\angle DAN$ intersects line DP at point Q. Prove that quadrilateral $BMQN$ is a parallelogram.

OC474. Given a right triangle ABC with hypotenuse AB, let D be the foot of the altitude drawn from point C, let M and N be the intersections of the angle bisectors of $\angle ADC$ and $\angle BDC$, respectively, with sides AC and BC. Prove that

$$2 \cdot AM \cdot BN = MN^2.$$

OC475. Let $N > 1$ be an integer. Denote by x the smallest positive integer with the following property: there exists a positive integer y strictly less than $x-1$ such that x divides $N+y$. Prove that x is either p^n or $2p$, where p is a prime number and n is a positive integer.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC471. Sur un tableau noir sont écrits $n > 3$ nombres naturels distincts, tous inférieurs à $(n-1)!$. Serge choisit deux de ces nombres et divise le plus gros par le plus petit, puis il inscrit la partie entière de la division dans son carnet. Par exemple, si les nombres avaient été 100 et 7, il aurait obtenu $100 = 14 \cdot 7 + 2$ et il aurait inscrit 14 dans son carnet. Il fait ceci pour chaque paire de nombres au tableau. Démontrer que parmi les nombres au carnet se retrouvent deux nombres égaux.

OC472. Soit $P(x)$ un polynôme de degré $n \geq 2$ à coefficients non négatifs et soient a, b et c les longueurs des côtés d'un triangle acutangle. Démontrer que les nombres $\sqrt[n]{P(a)}$, $\sqrt[n]{P(b)}$ et $\sqrt[n]{P(c)}$ sont aussi les longueurs des côtés d'un triangle acutangle.

OC473. Pour un carre $ABCD$, soit M le mi point de AB, soit P la projection du point B sur la ligne CM, et soit N le mi point du segment CP. Or, la bissectrice de $\angle DAN$ intersecte la ligne DP au point Q. Démontrer que le quadrilatère $BMQN$ est un parallélogramme.

OC474. Pour un certain triangle rectangle ABC d'hypoténuse AB, soit D le pied de l'altitude émanant du point C, et soient M et N les intersections des bissectrices de $\angle ADC$ et $\angle BDC$ avec les côtés AC et BC, respectivement. Démontrer que

$$2 \cdot AM \cdot BN = MN^2.$$

OC475. Soit $N > 1$ un entier. Dénoter par x le plus petit entier positif avec la propriété suivante : il existe un entier positif y plus petit que $x - 1$ tel que x divise $N + y$. Démontrer que x est soit de la forme p^n soit de la forme $2p$, où p est un nombre premier et n est un entier positif.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(8), p. 463–465.

OC446. Given the numbers $2, 3, \ldots, 2017$ and the natural number $n \leq 2014$, Ivan and Peter play the following game: Ivan selects n numbers from the given ones, then Peter selects 2 numbers from the remaining numbers, then all the selected $n+2$ numbers are ranked in value:

$$a_1 < a_2 < \ldots < a_{n+2}.$$

If there exists i, $1 \leq i \leq n+1$ for which a_i divides a_{i+1}, then Peter wins, otherwise Ivan wins. Find all n for which Ivan has a winning strategy.

Originally Bulgaria Math Olympiad, 3rd Problem, Grade 11, Second Round 2017.

We received 1 submission. We present the solution by Oliver Geupel.

We show that Ivan has a winning strategy if and only if $n \geq 10$.

First, suppose $n \geq 10$. Consider the intervals $I_k = \left[2^{k-1}+1,\ 2^k\right]$, where $1 \leq k \leq 10$, and $I_{11} = \left[2^{10}+1,\ 2017\right]$. Note that $\{2, 3, \ldots, 2017\} = \bigcup_{k=1}^{11} I_k$. Every interval has the property that its upper endpoint is less than twice its lower endpoint. Hence, for any two distinct integers in the same interval, it cannot happen that one of them divides the other one. From this observation, it follows that Ivan wins if his selection includes the lower endpoints of all intervals (except for the number 2). As consequence, it is enough for Ivan to include the ten numbers 3, 5, 9, 17, 33, 65, 129, 257, 513, and 1025 in his selection.

It remains to show that Ivan has no winning strategy when $n < 10$. Consider the disjoint intervals $J_k = \left[2^{k-1}+2,\ 2^k+1\right]$ where $1 \leq k \leq 10$. By the pigeonhole principle, there exists an index k such that J_k is disjoint to Ivan's selection. So Peter can avoid choosing any numbers from the interval J_k which Ivan has already avoided, and can choose the numbers $2^{k-1}+1$ and 2^k+2 (if not selected by Ivan). Thus these two numbers become a_i and a_{i+1} for some i, so Peter wins the game.

OC447. Let $m > 1$ be an integer and let $N = m^{2017}+1$. Positive numbers $N, N-m, N-2m, \ldots, m+1, 1$ are written in a row. At each step, the leftmost number and all of its divisors (if any) are erased. This process continues until all the numbers are erased. What are the numbers deleted at the last step?

Originally Bulgaria Math Olympiad, 2nd Problem, Grades 9-12, Final Round 2017.

We received 1 submission. We present the solution by Oliver Geupel.

We prove that the number deleted at the last step is

$$M = \frac{N}{m+1} + m = m^{2016} + 1 - m^2(m-1)\sum_{k=0}^{1006} m^{2k}.$$

First, $M \equiv 1 \pmod{m}$ and $M \leq N$, which implies that M is a member of the row. Second, M is not a proper divisor of any number in the row, because numbers $2M$, $3M$, $4M$, $\ldots$, mM are not congruent to $1 \pmod{m}$, and $(m+1)M > N$. Thus, the number M is the leftmost remaining number at some step.

Next, we show that all numbers in the row that are strictly smaller than M are erased before M. Let $r < M$ be a number in the row. It is enough to prove that there exists a positive integer k such that the number $(km+1)r$ is in the row and $(km+1)r > M$; consequently, r is erased before M. Equivalently, we have to show that at least one member of the arithmetic progression $\{r + kmr\}_{k=1,2,\ldots}$ is in the interval $(M, N]$. For example, for the largest $r < M$ in the row, $r = N/(m+1)$, it is enough to select $k = 1$: $(km+1)r = N$. For an arbitrary $r < N/(m+1)$ in the row

$$mr \leq \frac{m}{m+1} \cdot N - m = N - \left(\frac{N}{m+1} + m\right) = N - M.$$

Therefore, the increment of arithmetic progression, $\{r + kmr\}_{k=1,2,\ldots}$, is strictly smaller than the length of interval $(M, N]$, and at least one member of the arithmetic progression must belong to interval $(M, N]$.

Hence M is the number erased at the last step.

OC448. Let $x_1 \leq x_2 \leq \ldots \leq x_{2n-1}$ be real numbers whose arithmetic mean is equal to A. Prove that

$$2\sum_{i=1}^{2n-1} (x_i - A)^2 \geq \sum_{i=1}^{2n-1} (x_i - x_n)^2 .$$

Originally Poland Math Olympiad, 3rd Problem, Second Round 2017.

No solutions were received.

OC449. A sequence $(a_1, a_2, \ldots, a_k)$ consisting of pairwise distinct squares of an $n \times n$ chessboard is called a *cycle* if $k \geq 4$ and the squares a_i and a_{i+1} have a common side for all $i = 1, 2, \ldots, k$, where $a_{k+1} = a_1$. Subset X of this chessboard's squares is *mischievous* if each cycle on it contains at least one square in X. Determine all real numbers C with the following property: for each integer $n \geq 2$, on an $n \times n$ chessboard there exists a mischievous subset consisting of at most Cn^2 squares.

Originally Poland Math Olympiad, 2nd Problem, Final Round 2017.

No solutions were received.

OC450. Find all pairs (x, y) of real numbers satisfying the system of equations

$$x \cdot \sqrt{1-y^2} = \frac{1}{4}\left(\sqrt{3}+1\right),$$
$$y \cdot \sqrt{1-x^2} = \frac{1}{4}\left(\sqrt{3}-1\right).$$

Originally Germany Math Olympiad, 3rd Problem, Grades 11-12, Second Day, 3rd Round 2017.

We received 19 submissions. We present two solutions.

Solution 1, by the Missouri State University Problem Solving Group.

Suppose x, y are solutions. From the given equations, $0 < x, y < 1$. So we may set $x = \sin\alpha$ and $y = \sin\beta$ for some $0 < \alpha, \beta < \pi/2$. Then

$$4\sin\alpha\cos\beta = \sqrt{3}+1,$$
$$4\cos\alpha\sin\beta = \sqrt{3}-1.$$

Add and subtract the two equations and divide by 4 to get

$$\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sqrt{3}/2,$$
$$\sin\alpha\cos\beta - \cos\alpha\sin\beta = 1/2.$$

Hence solving the original system reduces to solving

$$\sin(\alpha+\beta) = \sqrt{3}/2,$$
$$\sin(\alpha-\beta) = 1/2,$$

with $0 < \alpha, \beta < \pi/2$. The angles α and β are given by

$$\alpha+\beta = \pi/3 + 2\pi n, \qquad \alpha-\beta = \pi/6 + 2\pi m$$
$$\alpha+\beta = 2\pi/3 + 2\pi n, \qquad \alpha-\beta = \pi/6 + 2\pi m$$
$$\alpha+\beta = \pi/3 + 2\pi n, \qquad \alpha-\beta = 5\pi/6 + 2\pi m$$
$$\alpha+\beta = 2\pi/3 + 2\pi n, \qquad \alpha-\beta = 5\pi/6 + 2\pi m$$

or by

$$\alpha = \pi/4 + (n+m)\pi, \qquad \beta = \pi/12 + (n-m)\pi,$$
$$\alpha = 5\pi/12 + (n+m)\pi, \qquad \beta = \pi/4 + (n-m)\pi,$$
$$\alpha = 7\pi/12 + (n+m)\pi, \qquad \beta = -\pi/4 + (n-m)\pi,$$
$$\alpha = 3\pi/4 + (n+m)\pi, \qquad \beta = -\pi/12 + (n-m)\pi.$$

for some integers n, m. Since $0 < \alpha, \beta < \pi/2$, then either $\alpha = \pi/4$, $\beta = \pi/12$, or $\alpha = 5\pi/12$, $\beta = \pi/4$. The two solutions of the initial system are

$$(x, y) = \left(\sin\frac{\pi}{4},\ \sin\frac{\pi}{12}\right) = \left(\frac{\sqrt{2}}{2},\ \frac{\sqrt{6}-\sqrt{2}}{4}\right)$$

and

$$(x, y) = \left(\sin\frac{5\pi}{12},\ \sin\frac{\pi}{4}\right) = \left(\frac{\sqrt{6}+\sqrt{2}}{4},\ \frac{\sqrt{2}}{2}\right).$$

Solution 2, by David Manes.

The two pairs (x, y) of real numbers that satisfy the system of equations are

$$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2-\sqrt{3}}}{2}\right) \quad \text{and} \quad \left(\frac{\sqrt{2+\sqrt{3}}}{2}, \frac{\sqrt{2}}{2}\right).$$

We can check that these values of x and y satisfy the two equations.

Since $1 - y^2 \geq 0$ and $1 - x^2 \geq 0$ it follows that $-1 \leq x, y \leq 1$. Moreover, $(\sqrt{3}+1)/4 > 0$, $(\sqrt{3}-1)/4 > 0$, $\sqrt{1-x^2} > 0$, and $\sqrt{1-y^2} > 0$. Therefore, $x, y > 0$. Hence, if x and y solve the system then $0 < x, y < 1$. Squaring each of the two equations, we obtain

$$x^2\left(1-y^2\right) = \frac{1}{8}\left(2+\sqrt{3}\right),$$
$$y^2\left(1-x^2\right) = \frac{1}{8}\left(2-\sqrt{3}\right).$$

Adding and then subtracting the two equations yields

$$2x^2 + 2y^2 - 4x^2y^2 = 1,$$
$$x^2 - y^2 = \frac{\sqrt{3}}{4}.$$

From the second equation, we obtain $y^2 = x^2 - \sqrt{3}/4$. Rearranging the terms in the first equation, we obtain $2x^2\left(1-2y^2\right) = 1 - 2y^2$. Hence, either $x^2 = 1/2$ or $1 - 2y^2 = 0$.

First, if $x = \pm\sqrt{2}/2$ then $y^2 = x^2 - \sqrt{3}/4 = (2-\sqrt{3})/4$, and $y = \pm\sqrt{2-\sqrt{3}}/2$. Note that $\sqrt{2}/2$ and $\sqrt{2-\sqrt{3}}/2$ belong to the interval $(0, 1)$. We obtain the first solution $x = \sqrt{2}/2$ and $y = \sqrt{2-\sqrt{3}}/2$.

Second, if $1 - 2y^2 = 0$, then $y = \pm\sqrt{2}/2$ and $x^2 = y^2 + \sqrt{2}/4 = (2+\sqrt{3})/4$. Therefore, $x = \pm\sqrt{2+\sqrt{3}}/2$. Since $0 < x, y < 1$, we find the second solution $x = \sqrt{2+\sqrt{3}}/2$ and $\sqrt{2}/2$. This solves the system.

Editor's comments. All submissions followed one of the two techniques presented above: trigonometric or algebraic approach.

FOCUS ON...

No. 40

Michel Bataille

Inequalities *via* auxiliary functions (I)

Introduction

In attempts at proving an inequality, a resort to the study of an auxiliary function often arises naturally. Most of the time, choosing an appropriate function and using calculus to obtain its variations lead to a solution. The goal of this number is to illustrate the method through various examples.

From their very definition, convex functions are connected to inequalities and consequently regularly appear in the treatment of inequalities. In this first part, we will leave them aside, devoting our next number to their use.

A series of simple examples

We start with some cases when the auxiliary function is readily deduced from the proposed inequality itself.

Our first example is problem **2970** [2004 : 368, 371 ; 2005 : 414]:

> If m and n are positive integers such that $m \geq n$, and if $a, b, c > 0$, prove that
>
> $$\frac{a^m}{b^m + c^m} + \frac{b^m}{c^m + a^m} + \frac{c^m}{a^m + b^m} \geq \frac{a^n}{b^n + c^n} + \frac{b^n}{c^n + a^n} + \frac{c^n}{a^n + b^n}.$$

It is quite natural to introduce the function f defined on $[0, \infty)$ by

$$f(x) = \frac{a^x}{b^x + c^x} + \frac{b^x}{c^x + a^x} + \frac{c^x}{a^x + b^x}.$$

The key is that we may suppose $a \geq b \geq c$ (since $f(x)$ remains unchanged when a, b, c are permuted). The derivative of f is easily calculated:

$$f'(x) = \sum_{cyclic} \frac{a^x}{(b^x + c^x)^2} \left(b^x \ln\left(\frac{a}{b}\right) + c^x \ln\left(\frac{a}{c}\right) \right)$$

$$= a^x b^x \ln\left(\frac{a}{b}\right) \left(\frac{1}{(b^x + c^x)^2} - \frac{1}{(c^x + a^x)^2} \right) + b^x c^x \ln\left(\frac{b}{c}\right) \left(\frac{1}{(c^x + a^x)^2} - \frac{1}{(a^x + b^x)^2} \right)$$

$$+ c^x a^x \ln\left(\frac{a}{c}\right) \left(\frac{1}{(b^x + c^x)^2} - \frac{1}{(a^x + b^x)^2} \right).$$

Since $a \geq b \geq c$, the numbers $\ln\left(\frac{a}{b}\right), \ln\left(\frac{b}{c}\right), \ln\left(\frac{a}{c}\right)$ are nonnegative and for $x \geq 0$ we have $(b^x + c^x)^2 \leq (c^x + a^x)^2 \leq (a^x + b^x)^2$, whence $f'(x) \geq 0$ (the three summands above are nonnegative). Thus, f is nondecreasing on $[0, \infty)$ and $f(u) \geq f(v)$ whenever $u \geq v \geq 0$, which is more general than the required result.

The auxiliary function is also chosen at once in the next example, problem **3889** [2013 : 413 ; 2014 : 404]:

> Prove that
> $$e^{\pi} > \left(\frac{e^2+\pi^2}{2e}\right)^e.$$

First we take logarithms, transforming the given inequality into

$$\frac{\pi}{e} > \ln\left(\frac{e}{2}\right) + \ln\left(1+\frac{\pi^2}{e^2}\right), \tag{1}$$

and define the auxiliary function f by $f(x) = x - \ln(1+x^2)$. Its derivative $f'(x) = \frac{(x-1)^2}{x^2+1}$ is positive for $x \in (1, \infty)$, hence f is increasing on $[1, \infty)$. As a result, $f\left(\frac{\pi}{e}\right) > f(1) = 1 - \ln(2)$ and (1) follows.

In our last example, problem **2933** [2004 : 173 ; 2005 : 186], the auxiliary function is less obvious:

> Prove, without the use of a calculator, that $\sin(40°) < \sqrt{\frac{3}{7}}$.

Here the key observation is that $\sin(3 \times 40°)$ is well-known. Recalling the formula $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, we are led to introduce the function f defined by $f(x) = 3x - 4x^3$. From the derivative $f'(x) = 3(1-2x)(1+2x)$, we see that f is decreasing on the interval on $(1/2, 1]$.

Both $\sin(40°)$ and $\sqrt{\frac{3}{7}}$ lie in this interval and, in addition to $\sin(3 \times 40°) = \sin(120°) = \frac{\sqrt{3}}{2}$, a short calculation gives $f(\sqrt{\frac{3}{7}}) = \frac{9}{7}\sqrt{\frac{3}{7}}$. From $\frac{7}{4} > \frac{81}{49}$, we deduce that $\frac{3}{4} > \frac{3}{7} \cdot \frac{81}{49}$, hence $\frac{\sqrt{3}}{2} > \frac{9}{7}\sqrt{\frac{3}{7}}$. Thus, $f(\sin(40°)) > f\left(\sqrt{\frac{3}{7}}\right)$ and $\sin(40°) < \sqrt{\frac{3}{7}}$ follows.

Auxiliary functions in succession

The resort to two or more auxiliary functions frequently occurs, for instance when another study of function is needed to obtain the sign of a derivative. A good example is extracted from problem **3228** [2007 : 169, 172 ; 2008 : 178]:

> For $x \in (0, \frac{\pi}{2})$, prove that
> $$\frac{x}{\sin x} \leq \frac{\pi}{2+\cos x}.$$

The inequality is equivalent to $\psi(x) \geq 0$ where $\psi(x) = \pi\sin x - 2x - x\cos x$. We calculate

$$\psi'(x) = (\pi - 1)\cos x + x\sin x - 2,$$
$$\psi''(x) = x\cos x - (\pi - 2)\sin x,$$
$$\psi'''(x) = -x\sin x - (\pi - 3)\cos x.$$

Since $\psi'''(x) < 0$ for $x \in [0, \frac{\pi}{2}]$, the function ψ'' is decreasing. Remarking that $\psi''(0) = 0$, we deduce that $\psi''(x) < 0$ for $x \in (0, \frac{\pi}{2}]$ and ψ' is decreasing as well. Since $\psi'(0) > 0$, $\psi'(\frac{\pi}{2}) < 0$ we have $\psi'(\alpha) = 0$ for some unique $\alpha \in (0, \frac{\pi}{2})$.

It follows that ψ is increasing on $(0, \alpha)$ and decreasing on $(\alpha, \frac{\pi}{2})$. Observing that $\psi(0) = \psi(\frac{\pi}{2}) = 0$, we may conclude that $\psi(x) > 0$ for $x \in (0, \frac{\pi}{2})$.

Our next example is problem **4061** [2015 : 302, 303 ; 2016 : 318]. We offer a variant of solution making use of two independent auxiliary functions.

> Let ABC be a non-obtuse triangle none of whose angles are less than $\frac{\pi}{4}$. Find the minimum value of $\sin A \sin B \sin C$.

We begin by obtaining inequalities about two auxiliary functions:

(a) For $x \in [\frac{\pi}{4}, \frac{\pi}{3}]$, let $f(x) = \sin^2 x \sin 2x$. Then $f(x) \geq \frac{1}{2}$.

Proof. The derivative f' satisfies

$$f'(x) = 2\sin^2 x(1 + 2\cos 2x) \geq 0$$

(since $\frac{\pi}{2} \leq 2x \leq \frac{2\pi}{3}$), hence $f(x) \geq f(\frac{\pi}{4}) = \frac{1}{2}$. □

(b) Let θ be a fixed real number in $[\frac{\pi}{4}, \frac{\pi}{3}]$. For $x \in [\theta, \frac{\pi-\theta}{2}]$, let

$$g_\theta(x) = \sin x \sin(x + \theta).$$

Then $g_\theta(x) \geq \sin\theta \sin 2\theta$.

Proof. Here $g'_\theta(x) = \sin(2x + \theta) \geq 0$ (since $0 < 3\theta \leq 2x + \theta \leq \pi$), hence

$$g_\theta(x) \geq g_\theta(\theta) = \sin\theta \sin 2\theta.$$

□

Turning to the problem, we may suppose that $C \leq B \leq A$. Then, $\frac{\pi}{4} \leq C \leq \frac{\pi}{3}$ (note that $3C \leq A + B + C = \pi$) and $B \leq \pi - B - C$ so that $C \leq B \leq \frac{\pi - C}{2}$.

Now, applying successively (b) and (a), we obtain

$$\begin{aligned}\sin A \sin B \sin C &= \sin C \sin B \sin(B + C)\\ &= \sin C \cdot g_C(B)\\ &\geq \sin C \cdot \sin C \sin 2C\\ &= f(C) \geq \frac{1}{2}.\end{aligned}$$

In addition, $\sin A \sin B \sin C = \frac{1}{2}$ if $A = \frac{\pi}{2}$, $B = C = \frac{\pi}{4}$. Thus, the desired minimum value is $\frac{1}{2}$.

To conclude, we consider problem **4267** [2017 : 303, 305 ; 2018 : 311]. We propose a solution, which, if longer than the featured one, may show to the beginner how to deal with a difficult inequality in a natural way.

> Let a, b, c and d be real numbers such that $0 < a, b, c \leq 1$ and $abcd = 1$. Prove that
>
> $$5(a+b+c+d) + \frac{4}{abc+abd+acd+bcd} \geq 21.$$

Since $abcd = 1$, the inequality is equivalent to $L \geq 21$ where

$$L = 5\left(a+b+c+\frac{1}{abc}\right) + \frac{4}{abc+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

To prove the inequality $L \geq 21$, we use a chain of auxiliary functions.

Let a, b be fixed in $(0, 1]$ and $f(x) = 5\left(a+b+x+\frac{1}{abx}\right) + \frac{4}{abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}}$ so that $L = f(c)$. We calculate the derivative of f in $(0, 1]$:

$$f'(x) = \frac{1-abx^2}{x^2}\left(\frac{4}{\left(abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}\right)^2} - \frac{5}{ab}\right).$$

Since $0 < a, b, x \leq 1$ we have $1 - abx^2 \geq 0$ and on the other hand $\frac{1}{a}+\frac{1}{b}+\frac{1}{x} \geq 3$, hence

$$\frac{4}{\left(abx+\frac{1}{a}+\frac{1}{b}+\frac{1}{x}\right)^2} \leq \frac{4}{9}$$

while $\frac{5}{ab} \geq 5$. Therefore $f'(x) < 0$ for $x \in (0, 1]$. It follows that f is decreasing on $(0, 1]$ and so

$$f(c) \geq f(1) = 5\left(a+b+1+\frac{1}{ab}\right) + \frac{4}{ab+\frac{1}{a}+\frac{1}{b}+1} = g(b)$$

where $g(x) = 5\left(a+x+1+\frac{1}{ax}\right) + \frac{4}{ax+\frac{1}{a}+\frac{1}{x}+1}$.
Similarly,

$$g'(x) = \frac{1-ax^2}{x^2}\left(\frac{4}{\left(ax+\frac{1}{a}+1+\frac{1}{x}\right)^2} - \frac{5}{a}\right)$$

is negative on $(0, 1]$ and so

$$g(b) \geq g(1) = 5\left(2+a+\frac{1}{a}\right) + \frac{4}{2+a+\frac{1}{a}} = h\left(2+a+\frac{1}{a}\right)$$

where $h(x) = 5x + \frac{4}{x}$. Since $2 + a + \frac{1}{a} \geq 2 + 2 = 4$, we study h on the interval $[4, \infty)$. On this interval, $h'(x) = 5 - \frac{4}{x^2} > 0$ so that h is increasing. Consequently

$$h\left(2+a+\frac{1}{a}\right) \geq h(4) = 21.$$

In conclusion, we have

$$L = f(c) \geq g(b) \geq h\left(2 + a + \frac{1}{a}\right) \geq 21$$

and the required inequality follows.

About the choice of an auxiliary function

To avoid a complicated study, it is sometimes better to delay the introduction of an auxiliary function. We give two examples of such situations. First, here is a variant of solution to problem **3908** [2014 : 29, 31 ; 2015 : 39]:

> Prove that $\dfrac{(n-1)^{2n-2}}{(n-2)^{n-2}} < n^n$ for each integer $n \geq 3$.

The function $x \mapsto x^x - \frac{(x-1)^{2x-2}}{(x-2)^{x-2}}$ does not seem a very good choice! We rewrite the inequality in a more convenient form:

$$\left(1 + \frac{1}{n(n-2)}\right)^n < \left(1 + \frac{1}{n-2}\right)^2.$$

But once again, a function like $x \mapsto \left(1 + \frac{1}{x}\right)^2 - \left(1 + \frac{1}{x(x+2)}\right)^x$ would not lead to a nice study! However, recalling that for any positive real number x and any positive integer n,

$$\left(1 + \frac{x}{n}\right)^n < e^x$$

we see that it is sufficient to prove that

$$\left(1 + \frac{1}{n-2}\right)^2 > e^{1/(n-2)}$$

for $n \geq 3$. At that stage we can efficiently consider $f(x) = (1+x)^2 - e^x$ for $x \in [0,1]$. A quick study of the derivative $f'(x) = 2(1+x) - e^x$ shows that f' is increasing on $[0, \ln 2]$ and decreasing on $[\ln 2, 1]$. Since $f'(0) = 1, f'(1) = 4 - e > 0$, it follows that $f'(x) > 0$ for all $x \in [0,1]$. Therefore f is increasing on $[0,1]$ and $f(x) > f(0) = 0$ whenever $x \in (0,1]$ and the desired inequality follows.

A similar difficulty is to be found in problem **3929** [2014 : 122,124 ; 2015 : 135]:

> Show that for all $0 < x < \pi/2$, the following inequality holds:
>
> $$\left(1 + \frac{1}{\sin x}\right)\left(1 + \frac{1}{\cos x}\right) \geq 5\left[1 + x^4\left(\frac{\pi}{2} - x\right)^4\right].$$

The inequality is $f(x) \geq 4 + 5x^4\left(\frac{\pi}{2} - x\right)^4$ where

$$f(x) = \frac{1}{\sin x} + \frac{1}{\cos x} + \frac{2}{\sin 2x}.$$

Again, it is better to remark that for $0 < x < \pi/2$ we have

$$0 < x\left(\frac{\pi}{2} - x\right) \le \left(\frac{x + \frac{\pi}{2} - x}{2}\right)^2 = \frac{\pi^2}{16},$$

hence

$$4 + 5x^4\left(\frac{\pi}{2} - x\right)^4 \le 4 + 5\left(\frac{\pi^2}{16}\right)^4$$

and therefore it is enough to prove that $f(x) \ge 4 + 5\left(\frac{\pi^2}{16}\right)^4$. Now, we readily obtain that $f'(x)$ has the same sign as $g(x) = \sin^3 x - \cos^3 x - \cos 2x$. A quick study of g then will show that $f(x) \ge f(\pi/4) = 2 + 2\sqrt{2}$ [details are left to the reader] and the conclusion follows from $2 + 2\sqrt{2} > 4 + 5\left(\frac{\pi^2}{16}\right)^4$.

As usual, we end this number with a couple of exercises.

Exercises

1. Let $n \in \mathbb{N}$ and let

$$\Delta(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i - \prod_{i=1}^{n} x_i.$$

If $a_1, a_2, \ldots, a_n \in (0, 1]$ prove that

$$\Delta(a_1, a_2, \ldots, a_n) \ge \Delta\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

2. For $y \in (0, 1]$, let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = y^x + x^y - 1$ and $g : (0, 1] \to \mathbb{R}$ be defined by $g(x) = f(x) - \frac{x}{y} \cdot f'(x)$. From the study of g deduce that $f(x) > 0$ for $x \in (0, 1]$.

3. (inspired by problem **1061** of *the College Mathematics Journal*) Let m be an integer with $m \ge 2$ and r a real number in $[1, \infty)$. Prove that

$$\left(\frac{1 + r^m}{1 + r^{m-1}}\right)^{m+1} \ge \frac{1 + r^{m+1}}{2}.$$

[Hint: determine the sign of $u(x) = (m - 1)(1 + x^{m+1}) - x(1 + x^{m-1})$ for $x \ge 1$.]

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **May 15, 2020**.

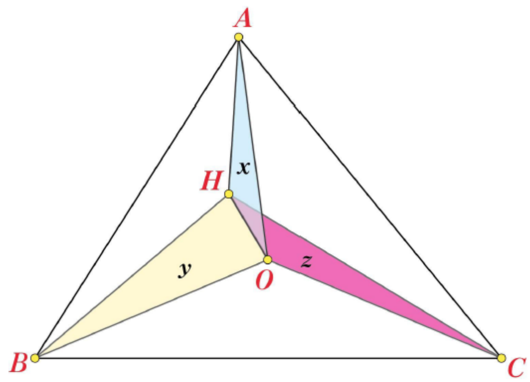
4521. *Proposed by Robert Frontczak.*

Let $m \in \mathbb{N}$, define the sequence $a_n (n \geq 0)$ by $a_0 = m$, $a_1 = a_2 = \cdots = a_m = 1$ and $a_n = \sqrt{a_{n-m-1} \cdot a_{n-m}}$ for $n \geq m+1$. Determine $\lim_{n\to\infty} a_n$.

4522. *Proposed by Miguel Ochoa Sanchez, Leonard Giugiuc and Kadir Altintas.*

Let ABC be an acute triangle with orthocenter H and circumcenter O. Denote Area(AHO)=x, Area(BHO)=y and Area(CHO)=z. Prove that

$$2(x^2y^2 + y^2z^2 + z^2x^2) = x^4 + y^4 + z^4.$$



4523★. *Proposed by Leonard Giugiuc.*

Let n be a natural number such that $n \geq 2$. Further, let $\{a_1, a_2, \ldots, a_n\} \subset [0, 1]$ and $\{b_1, b_2, \ldots, b_n\} \subset [1, \infty)$ such that

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = n + 1.$$

Prove that

$$\frac{1}{n}(n^2+1) \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 \leq n + 3.$$

4524. *Proposed by Lorian Saceanu.*

Let x, y, z be non-negative real numbers at most one of which is zero. Prove that if

$$x^2 + y^2 + z^2 = 2(xy + yz + xz),$$

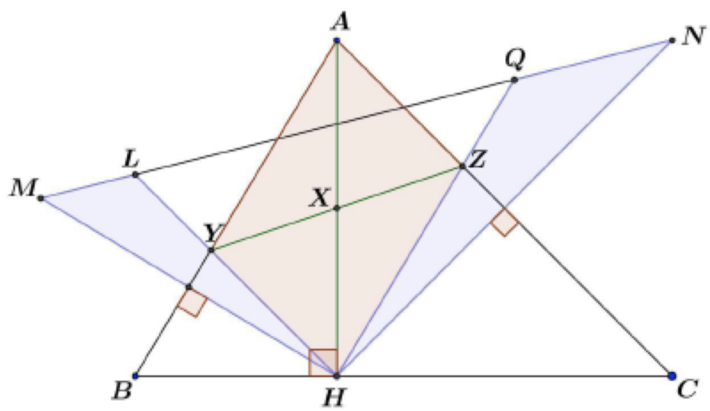
then

$$5 \leq (x+y+z)\left(\sum_{cyclic} \frac{1}{y+z}\right) \leq \frac{27}{5}$$

and determine when equality holds for either bound.

4525. *Proposed by Julio Orihuela and Leonard Giugiuc.*

Let H be the foot of the altitude from vertex A to side BC of the acute triangle ABC; let the circle with center B and radius BH meet the perpendicular from H to AB again at M, and the circle with center C and radius CH meet the perpendicular from H to AC again at N. Moreover, let the line MN meet the first circle again at L and the second circle again at Q, and finally, let Y be the point where HL intersects AB and Z the point where HQ intersects AC. Prove that $AYHZ$ is a parallelogram and $\angle MHL = \angle QHN$.



4526. *Proposed by Michel Bataille.*

Let ABC be a scalene, not right-angled triangle with orthocenter H and let D, E, F be the midpoints of BC, CA, AB, respectively. Points U, V, W, respectively on the lines BC, CA, AB, are such that AU, BV, CW are perpendicular to HD, HE, HF (respectively). Prove that U, V, W are collinear.

4527. *Proposed by George Stoica.*

Let $n \geq 4$ be a positive integer. Prove that the roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$, whose coefficients satisfy $|a_{n-2}|, |a_{n-1}| \leq |a_n| \leq |a_0|$, cannot be all real.

4528. *Proposed by Leonard Giugiuc.*

Let $ABCD$ be a rectangle situated in a plane $\mathcal{P}$. Find

$$\min_{M\in\mathcal{P}}\left(\frac{MA+MC}{MB+MD}\right).$$

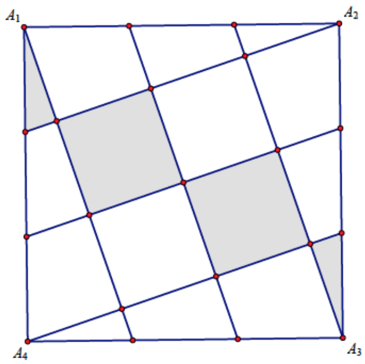
4529. *Proposed by George Apostolopoulos.*

Let a, b, c be the side-lengths of a triangle. Prove that

$$\frac{2a+b}{a+c}+\frac{2b+c}{b+a}+\frac{2c+a}{c+b}\geq\frac{9}{2}.$$

4530. *Proposed by Arsalan Wares.*

Let A be a square with vertices A_k, $k = 1, 2, 3, 4$. On each side of A, mark 2 points which divide the side into 3 equal parts. These 8 points and the vertices of A are connected to one another, dividing A into 16 disjoint regions, as shown in the figure. Determine the ratio of the area of the shaded regions to the area of A.



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Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mai 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

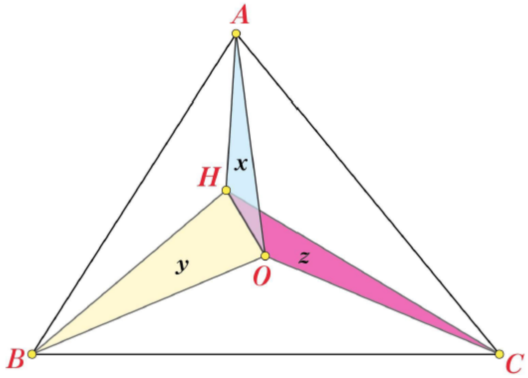
4521. *Proposeé par Robert Frontczak.*

Soit $n \geq 0$ et soit la suite a_n définie par $a_0 = m, m \in \mathbb{N}$, $a_1 = a_2 = \cdots = a_m = 1$ et $a_n = \sqrt{a_{n-m-1} \cdot a_{n-m}}$ pour $n \geq m+1$. Déterminer $\lim_{n\to\infty} a_n$.

4522. *Proposeé par Miguel Ochoa Sanchez, Leonard Giugiuc et Kadir Altintas.*

Soit ABC un triangle acutangle d'orthocentre H et soit O le centre de son cercle circonscrit. Dénoter Surface(AHO)=x, Surface(BHO)=y et Surface(CHO)=z. Démontrer que

$$2(x^2y^2 + y^2z^2 + z^2x^2) = x^4 + y^4 + z^4.$$



4523*. *Proposeé par Leonard Giugiuc.*

Soit n un nombre naturel tel que $n \geq 2$. De plus, soient $\{a_1, a_2, \ldots, a_n\} \subset [0, 1]$ et $\{b_1, b_2, \ldots, b_n\} \subset [1, \infty)$ tels que

$$\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = n + 1.$$

Démontrer que

$$\frac{1}{n}(n^2 + 1) \leq \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 \leq n + 3.$$

4524. *Proposeé par Lorian Saceanu.*

Soient x, y, z des nombres réels non négatifs dont au plus un est zéro. Démontrer que si

$$x^2 + y^2 + z^2 = 2(xy + yz + xz),$$

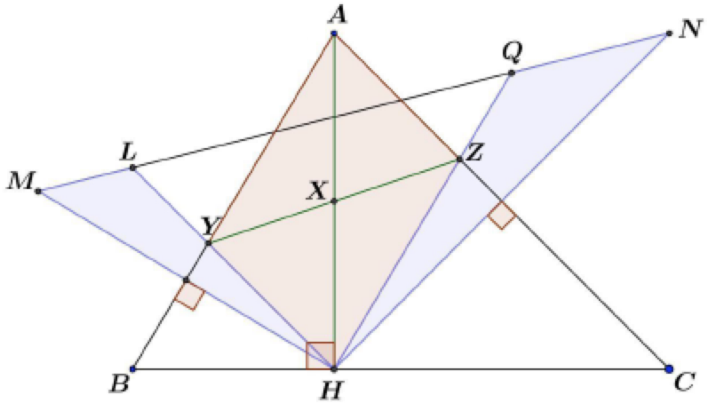
alors

$$5 \leq (x+y+z)\left(\sum_{cyclic} \frac{1}{b+c}\right) \leq \frac{27}{5}$$

et déterminer les situations où égalité tient pour l'une inégalité ou l'autre.

4525. *Proposeé par Julio Orihuela et Leonard Giugiuc.*

Soit H le pied de l'altitude émanant du sommet A vers le côté BC d'un triangle acutangle ABC. Supposons que le cercle de centre B et rayon BH rencontre la perpendiculaire de H vers AB une seconde fois à M, puis que le cercle de centre C et rayon CH rencontre la perpendiculaire de H vers AC une seconde fois à N. De plus, supposer que la ligne MN rencontre le premier cercle une seconde fois à L et le deuxième cercle une seconde fois à Q. Enfin, soient Y le point d'intersection de HL et AB, puis Z le point d'intersection de HQ et AC. Démontrer que $AYHZ$ est un parallélogramme et que $\angle MHL = \angle QHN$.



4526. *Proposeé par Michel Bataille.*

Soit ABC un triangle scalène non rectangle d'orthocentre H et soient D, E, F les mi points de BC, CA, AB respectivement. Les points U, V, W se trouvent sur les lignes BC, CA, AC, respectivement, de façon à ce que AU, BV, CW sont perpendiculaires à HD, HE, HF, respectivement. Démontrer que U, V, W sont colinéaires.

4527. *Proposeé par George Stoica.*

Soit n un nombre naturel tel que $n \geq 4$. Démontrer que les racines du polynôme $a_0 + a_1x + \cdots + a_nx^n$, dont les coefficients vérifient $|a_{n-2}|, |a_{n-1}| \leq |a_n| \leq |a_0|$, ne peuvent pas toutes être réelles.

4528. *Proposeé par Leonard Giugiuc.*

Soit $ABCD$ un rectangle dans le plan $\mathcal{P}$. Déterminer

$$\min_{M\in\mathcal{P}} \left(\frac{MA + MC}{MB + MD}\right).$$

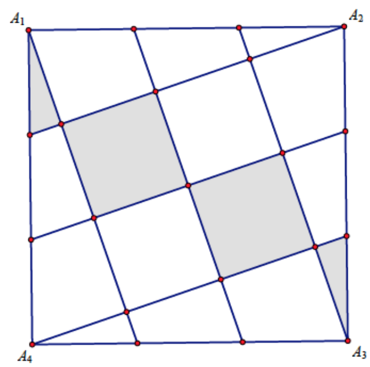
4529. *Proposeé par George Apostolopoulos.*

Soient a, b, c les longueurs des côtés d'un triangle. Démontrer que

$$\frac{2a+b}{a+c} + \frac{2b+c}{b+a} + \frac{2c+a}{c+b} \geq \frac{9}{2}.$$

4530. *Proposeé par Arsalan Wares.*

Soit A un carré de sommets A_k, $k = 1, 2, 3, 4$. Sur chaque côté de A, on note 2 points qui divisent le côté en 3 parties égales. Ces 8 points et les sommets de A sont reliés de façon à diviser A en 16 régions, tel qu'indiqué. Déterminer le ratio de la surface ombragée par rapport à la surface de A.



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(8), p. 476–479.

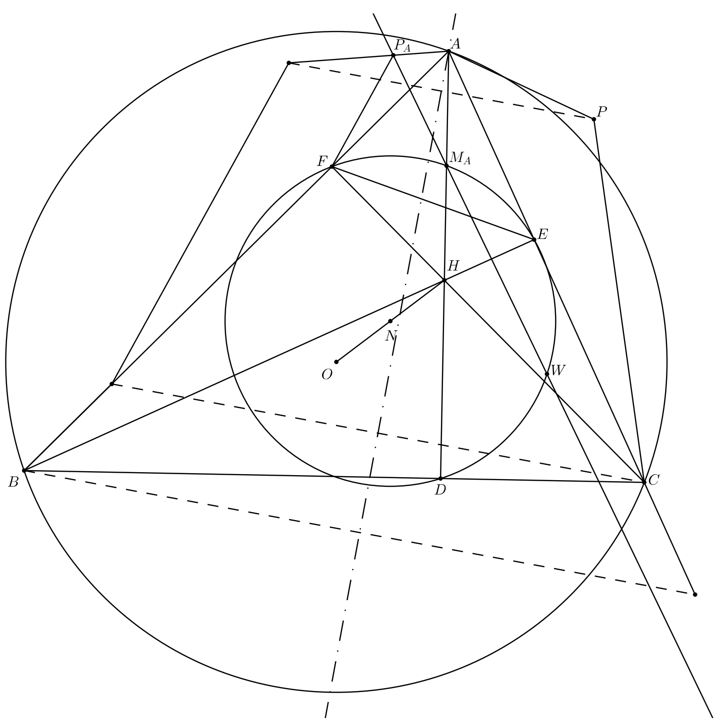
4471. *Proposed by Michael Diao.*

In $\triangle ABC$, let H be the orthocenter. Let M_A be the midpoint of AH and D be the foot from H onto BC, and define M_B, M_C, E and F similarly. Suppose P is a point in the plane distinct from the circumcenter of ΔABC, and suppose that P_A, P_B and P_C are points such that quadrilaterals $PABC$, P_AAEF, P_BDBF and P_CDEC are similar with vertices in that order. Show that M_AP_A, M_BP_B and M_CP_C concur on the circumcircle of $\triangle DEF$.

We received 5 submissions, all of which were correct. Only the proposer avoided the use of coordinates, so we provide two solutions: the proposer's together with one example of a solution via coordinates.

Solution 1, by Michel Bataille.

First, a few remarks (see figure):



(1) the circumcircle of ΔDEF is the nine-point circle whose centre is the midpoint N of OH (O being the circumcentre of ΔABC). Its radius is half the circumradius of ΔABC and it passes through M_A. We add the hypothesis that ΔABC is not right-angled (to keep the triangle DEF non-degenerate).

(2) The statement of the problem assumes as known that triangles ABC and AEF are oppositely similar, a fact for which a proof is as easily found as a reference: Points E, F, B, C lie on the circle with diameter BC (since $\angle BEC = \angle CFB = 90^\circ$), hence, using directed angles modulo π, we have

$$\begin{aligned}\angle(EA, EF) = \angle(EA, EB) + \angle(EB, EF) &= \frac{\pi}{2} + \angle(CB, CF)\\ &= \frac{\pi}{2} + \angle(BC, BF) + \frac{\pi}{2}\\ &= \angle(BC, BA).\end{aligned}$$

Similarly, $\angle(FE, FA) = \angle(CA, CB)$, whence corresponding angles of triangles ABC and AEF are equal. The similarity σ such that $\sigma(A) = A, \sigma(B) = E, \sigma(C) = F$ is indirect (since $\angle(\sigma(CB), \sigma(CA)) = \angle(CA, CB)$); moreover, by definition $\sigma(P) = P_A$.

This said, we embed the problem in the complex plane and suppose without loss of generality that ΔABC is inscribed in the unit circle. We denote by a, b, c, m_a, p, p_a, n the affixes of A, B, C, M_A, P, P_A, N and we set $m = \frac{b+c}{2}$, the affix of the midpoint of BC. Note that $m \neq 0$ since ΔABC is not right-angled. We have

$$h = a + b + c, \quad n = \frac{h}{2} = \frac{a}{2} + m, \quad \text{and} \quad m_a = \frac{a+h}{2} = a + m.$$

The equation of the line AC is $z + ac\overline{z} = a + c$ and the equation of the perpendicular to AC through B is

$$\frac{z-b}{c-a} + \frac{\overline{z} - \overline{b}}{\overline{c} - \overline{a}} = 0,$$

that is, $bz - abc\overline{z} = b^2 - ac$ (using $\overline{a} = \frac{1}{a}$, etc.). From these equations, we deduce the affix e of E:

$$e = \frac{hb - ac}{2b}.$$

The similarity σ transforms the point with affix z into the point with affix $z' = \alpha\overline{z} + \beta$, where α, β satisfy

$$a = \alpha\overline{a} + \beta \quad \text{and} \quad \frac{hb - ac}{2b} = \alpha\overline{b} + \beta.$$

This yields $\alpha = -\frac{a(b+c)}{2} = -am$ and $\beta = a + m$ and therefore,

$$p_a = -am\overline{p} + a + m.$$

Now, let W (affix w) be the point of intersection distinct from M_A of the line M_AP_A and the nine-point circle: for some real number λ, we have

$$w = m_a + \lambda(m_a - p_a) = a + m + \lambda am\overline{p}.$$

Expressing that w must satisfy $|w-n|^2 = \frac{1}{4}$, we obtain

$$\lambda^2 m\overline{m}p\overline{p} + \frac{\lambda}{2}(m\overline{p} + \overline{m}p) = 0.$$

Since $w \neq m_a$, we have $\lambda \neq 0$, hence $\lambda = -\frac{1}{2}\left(\frac{1}{p\overline{m}} + \frac{1}{m\overline{p}}\right)$ and we readily deduce that

$$w = \frac{a+b+c}{2} - \frac{abc\overline{p}}{2p}.$$

Since the affix w is invariant under permutations of a, b, c, the lines M_BP_B and M_CP_C also pass through W and the required result follows.

Solution 2 by the proposer, revised by the editor.

We work in the Euclidean plane extended by the line at infinity. We shall be using properties of isogonal conjugation with respect to triangle $M_AM_BM_C$: Recall that the isogonal conjugate of a point with respect to $\Delta M_AM_BM_C$ is constructed by reflecting each line joining it to a vertex in the internal angle bisector at the corresponding vertex; the three reflected lines then concur at the isogonal conjugate. This type of conjugation is an involution of the points of the extended plane that are not on a sideline of the triangle; in particular, every point not on an extended side of the triangle is interchanged with its conjugate; moreover, each point other than a vertex on the circumcircle is interchanged with a point at infinity. Details can be found in standard textbooks such as Roger A. Johnson's *Advanced Euclidean Geometry* and Nathan Altshiller Court's *College Geometry*, as well as in standard internet sources.

Note that EF is antiparallel to BC with respect to $\angle BAC$ (where two lines are said to be *antiparallel with respect to an angle* if the image of either line under reflection in the angle bisector is parallel to the other line). For a proof, see the second preliminary remark in Solution 1 above. Because $PABC$ is similar to P_AAEF, the similarity that takes ΔABC to ΔAEF takes the circumcenter O of the former to the circumcenter M_A of the latter, whence the line OP and its image line M_AP_A must also be antiparallel with respect to $\angle BAC$. Since the triangles $M_AM_BM_C$ and ABC are homothetic, it follows that OP and M_AP_A are antiparallel with respect to $\angle M_BM_AM_C$. Letting P_∞ denote the point at infinity of the line OP, we have M_AP_∞ is parallel to OP, so we may conclude, finally, that M_AP_∞ and M_AP_A are isogonal in $\angle M_BM_AM_C$ (in the sense that $\angle M_BM_AM_C$ and $\angle P_AM_AP_\infty$ have the same angle bisectors).

Analogously, M_BP_∞ and M_BP_B are isogonal in $\angle M_CM_BM_A$, and M_CP_∞ and M_CP_C are isogonal in $\angle M_AM_CM_B$. Because these three lines (namely M_AP_∞, M_BP_∞ and M_CP_∞) meet on the line at infinity, their isogonal conjugates with respect to $\triangle M_AM_BM_C$, namely the lines M_AP_A, M_BP_B and M_CP_C, must concur at the isogonal conjugate of P_∞, which must lie on the circumcircle of $\triangle M_AM_BM_C$. But that circle is the nine-point circle of ΔABC, which coincides with the circumcircle of ΔDEF; in other words, the three lines concur at a point of the circumcircle of ΔDEF as claimed.

Editor's comments. The proposer observed that the concurrence point in question is the Poncelet point of the isogonal conjugate of P with respect to $\triangle ABC$. He also stated that by starting with P at the orthocenter, the result becomes,

> The Euler lines of $\triangle AEF$, $\triangle BFD$ and $\triangle CDE$ concur on the nine-point circle.

4472. *Proposed by Liam Keliher.*

Let n be a positive integer. Prove that n divides

$$\prod_{i=0}^{n-1} \left(2^n - 2^i\right).$$

We received 22 submissions, all correct. Several solvers proved a stronger result that the given product is actually divisible by $n!$. *We present the solution of Prithwijit De.*

Observe that $|\mathrm{GL}_n(\mathbb{Z}_2)| = \prod_{i=0}^{n-1}(2^n - 2^i)$ and the group of permutation matrices of order n (itself a group of order $n!$) is a subgroup of $\mathrm{GL}_n(\mathbb{Z}_2)$. Therefore

$$n! | \prod_{i=0}^{n-1}(2^n - 2^i).$$

Editor's Comment. Bataille pointed out that this result is well-known as it was problem 5 of the 4th National Mathematical Olympiad of Turkey, appearing in ***Crux*** before: see [2000 : 390] and [2002 : 503].

4473. *Proposed by Nguyen Viet Hung.*

Let $\lfloor a \rfloor$ denote the greatest integer not exceeding a. For every positive integer n,

(a) find the last digit of $\lfloor (2+\sqrt{3})^n \rfloor$,

(b) find $\gcd(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1)$, where $\gcd(a, b)$ denotes the greatest common divisor of a and b.

We received 15 correct solutions. Most followed the tack of the general solution presented below.

(a) Let $u = 2+\sqrt{3}$ and $v_n = u^n + u^{-n}$ for $n \geq 1$. Then $0 < u^{-1} = 2-\sqrt{3} < 1$ and

$$v_n - 1 < u^n < v_n$$

for each positive integer n. Since u and u^{-1} are the roots of the quadratic equation $x^2 = 4x - 1$, the sequence $\{v_n\}$ satisfies the recursion

$$v_{n+2} = 4v_{n+1} - v_n$$

with initial conditions $v_1 = 4$ and $v_2 = 14$. Thus $\{v_n\}$ is a sequence of even integers and $\lfloor u^n \rfloor = v_n - 1$.

Modulo 10, $\{v_n\}$ has the period 3 cycle $\{4, 4, 2\}$, so that $\lfloor u^n \rfloor$ ends in the digit 3 when $n \equiv \pm 1 \pmod 3$ and in the digit 1 when $n \equiv 0 \pmod 3$.

(b) Since gcd $(v_1, v_2) = 2$ and $v_{n+2} = 4v_{n+1} - v_n$ for $n \geq 1$, an induction argument establishes that

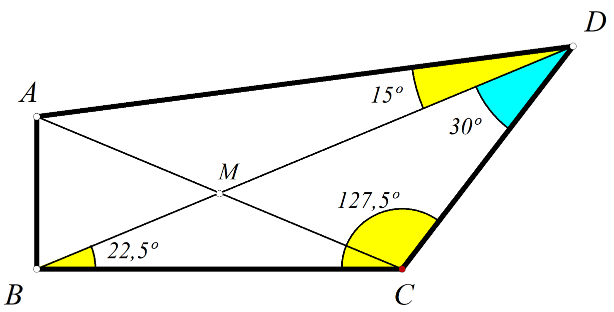
$$\gcd(\lfloor (2+\sqrt{3})^{n+1} \rfloor + 1, \lfloor (2+\sqrt{3})^n \rfloor + 1) = \gcd(v_{n+1}, v_n) = 2.$$

4474. *Proposed by Kadir Altintas and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral such that $\angle ABC = \frac{\pi}{2}$, $\angle ADB = \frac{\pi}{12}$, $\angle BDC = \frac{\pi}{6}$ and $\angle DBC = \frac{\pi}{8}$. Prove that BD passes through the midpoint of AC.

We received 23 submissions, all correct, and from the rich variety of solutions we have chosen two that were relatively light on the use of trigonometric identities.

Solution 1, by Cristobal Sánchez-Rubio.



Let M be the point of intersection of the lines AC and BD. Since ΔABC is a right triangle, it is enough to prove that the triangle MCB is isosceles; specifically, since we have $\angle DBC = \angle MBC = 22.5°$, we must show that $\angle MCB = 22.5°$. By the sine law applied to triangle ABD,

$$\frac{AB}{\sin 15°} = \frac{BD}{\sin 97.5°} = \frac{BD}{\cos 7.5°},$$

and therefore,

$$AB = \frac{BD \sin 15°}{\cos 7.5°}.$$

By the sine law for triangle BCD,

$$\frac{BC}{\sin 30°} = \frac{BD}{\sin 127.5°},$$

and therefore

$$BC = \frac{BD \sin 30°}{\cos 37.5°}.$$

Then
$$\frac{AB}{BC} = \frac{\sin 15^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ \cos 7.5^\circ} = \frac{2\sin 7.5^\circ \cdot \cos 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ \cos 7.5^\circ} = \frac{2\sin 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ}.$$
But
$$\begin{aligned} 2\sin 7.5^\circ \cdot \cos 37.5^\circ &= \sin(7.5^\circ + 37.5^\circ) + \sin(7.5^\circ - 37.5^\circ) \\ &= \sin 45^\circ - \sin 30^\circ \\ &= \frac{\sqrt{2}}{2} - \frac{1}{2}, \end{aligned}$$
whence, finally,
$$\frac{AB}{BC} = \frac{2\sin 7.5^\circ \cdot \cos 37.5^\circ}{\sin 30^\circ} = \sqrt{2} - 1 = \tan 22.5^\circ.$$
In other words, $\angle MCB = 22.5^\circ$, so that $MB = MC$, as desired.

Solution 2, by Ivko Dimitrić.

Let M be the point of intersection of AC and BD, and let P and Q be the feet of the perpendiculars from A and C to BD, respectively. Then, from $\triangle ABP$ and $\triangle APD$ we have
$$\frac{BD}{AP} = \frac{BP}{AP} + \frac{PD}{AP} = \cot\frac{3\pi}{8} + \cot\frac{\pi}{12} = \tan\frac{\pi}{8} + \cot\frac{\pi}{12},$$
and from $\triangle BCQ$ and $\triangle CDQ$,
$$\frac{BD}{CQ} = \frac{BQ}{CQ} + \frac{QD}{CQ} = \cot\frac{\pi}{8} + \cot\frac{\pi}{6}.$$
Using the relevant half-angle formulas we have
$$\tan\frac{\pi}{8} = \sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}} = \sqrt{\frac{(2-\sqrt{2})^2}{(2+\sqrt{2})(2-\sqrt{2})}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2} - 1$$
and $\cot\frac{\pi}{8} = \sqrt{2} + 1$, whereas
$$\cot\frac{\pi}{12} = \sqrt{\frac{1+\cos\frac{\pi}{6}}{1-\cos\frac{\pi}{6}}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2 + \sqrt{3}.$$
Then,
$$\tan\frac{\pi}{8} + \cot\frac{\pi}{12} = (\sqrt{2}-1) + (2+\sqrt{3}) = (\sqrt{2}+1) + \sqrt{3} = \cot\frac{\pi}{8} + \cot\frac{\pi}{6}.$$
Hence, $\dfrac{BD}{AP} = \dfrac{BD}{CQ}$, implying $AP = CQ$. As a consequence, the right triangles APM and CQM are congruent, implying $AM = CM$; that is, BD passes through the midpoint of AC.

4475. *Proposed by Michel Bataille.*

Let a, b be real numbers with $a, b, a+b, a-b \neq 0$. Prove the inequality

$$\frac{\sinh(2(a+b))}{a+b} + \frac{\sinh(2(a-b))}{a-b} \geq 4\left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right).$$

There were 4 correct and one incomplete solution. The correct solutions all used an integration argument along the lines of the following.

We first establish that

$$(\cosh x)(\cosh y) \geq \sinh x + \sinh y$$

for real x and y. This can be done either by using the identity $\cosh^2 t = 1 + \sinh^2 t$ and squaring, or by making the substitutions

$$(\sinh x, \cosh x) = (\tan u, \sec u) \quad \text{and} \quad (\sinh y, \cosh y) = (\tan v, \sec v)$$

with $\pi/2 < u, v < \pi/2$, and noting that the inequality is equivalent to

$$1 \geq \sin(u+v).$$

Let $(x, y) = (2ta, 2tb)$. For $0 \leq t \leq 1$,

$$\begin{aligned}\frac{1}{2}[\cosh(2t(a+b)) + \cosh(2t(a-b))] &= \cosh(2ta)\cosh(2tb)\\ &\geq \sinh(2ta) + \sinh(2tb).\end{aligned}$$

Integrate this inequality from 0 to 1 with respect to t to obtain

$$\begin{aligned}\frac{\sinh(2(a+b))}{4(a+b)} + \frac{\sinh(2(a-b))}{4(a-b)} &\geq \frac{\cosh(2a)-1}{2a} + \frac{\cosh(2b)-1}{2b}\\ &= \left(\frac{\sinh^2(a)}{a} + \frac{\sinh^2(b)}{b}\right),\end{aligned}$$

as desired.

4476. *Proposed by Leonard Giugiuc.*

Prove that for any real numbers a, b and c, we have

$$3\sqrt{6}(ab(a-b) + bc(b-c) + ca(c-a)) \leq ((a-b)^2 + (a-c)^2 + (b-c)^2)^{3/2}.$$

We received 10 submissions, all correct. We present a composite of nearly the same solutions by Michel Bataille and Marie-Nicole Gras.

Let

$$\begin{aligned}p &= \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right] = a^2 + b^2 + c^2 - ab - bc - ca,\\ q &= ab(a-b) + bc(b-c) + ca(c-a) = a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2.\end{aligned}$$

We need to prove that
$$3\sqrt{6}\,q \le (2p)^{\frac{3}{2}}. \qquad (1)$$
If $q < 0$, then (1) clearly holds. If $q \ge 0$, then
$$(1) \iff 54q^2 \le (2p)^3 \iff 27q^2 \le 4p^3. \qquad (2)$$
Consider the polynomial
$$\begin{aligned}P(x) &= (x-a+b)(x-b+c)(x-c+a)\\ &= x^3 + \big[(a-b)(b-c)+(b-c)(c-a)+(c-a)(a-b)\big]x - (a-b)(b-c)(c-a)\\ &= x^3 + (-a^2-b^2-c^2+ab+bc+ca)x - (-a^2b-b^2c-c^2a+ab^2+bc^2+ca^2)\\ &= x^3 - px + q,\end{aligned}$$
of discriminant
$$\Delta = -4p^3 + 27q^2.$$
Since $P(x)$ has 3 real roots by its definition, we must have $\Delta \le 0$, so $4p^3 - 27q^2 \ge 0$, from which (2) follows, completing the proof.

4477. *Proposed by Warut Suksompong.*

Given a positive integer n, let $a_1 \ge \cdots \ge a_n \ge 0$ and $b_1 \ge \cdots \ge b_n \ge 0$ be integers such that

1. $a_1 + \cdots + a_i \ge b_1 + \cdots + b_i$ for all $i = 1, \ldots, n-1$;
2. $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.

Assume that there are n boxes, with box i containing a_i balls. In each move, Alice is allowed to take two boxes with an unequal number of balls, and move one ball from the box with more balls to the other box. Prove that Alice can perform a finite number of moves after which each box i contains b_i balls.

There were 4 correct solutions, all along the lines of the following.

The proof is by induction, the result being trivial for $n = 1$.

Assume it holds for at most $n-1$ boxes, with $n \ge 2$. If $a_j = b_j$ for some j with $1 \le j \le n$, we can remove box j from consideration. The conditions hold for the remaining $n-1$ boxes and we can invoke the induction hypothesis to rearrange the balls among them.

Henceforth, let $a_i \ne b_i$ for each i.

Since $a_1 > b_1$ and
$$a_n = b_n - [(a_1 + \cdots + a_{n-1}) - (b_1 + \cdots + b_{n-1})] \le b_n,$$
then $a_n < b_n$ and there is a positive integer $k \le n-1$ for which $a_k > b_k$ and $a_{k+1} < b_{k+1}$. Start removing balls one at a time from box k and placing them in box $k+1$.

If $a_k - b_k \le b_{k+1} - a_{k+1}$, then transfer a total of $a_k - b_k$ balls from box $k+1$, leaving b_k balls in box k and

$$a_{k+1} + a_k - b_k \le a_{k+1} + b_{k+1} - a_{k+1} = b_{k+1}$$

balls in box $k+1$.

Since

$$(a_1 + \cdots + a_{k-1}) + b_k + (a_{k+1} + a_k - b_k) = a_1 + \cdots + a_{k+1} \ge b_1 + \cdots + b_{k+1},$$

we see that the consequent arrangement of balls in the boxes satisfies the two conditions.

If $a_k - b_k > b_{k+1} - a_{k+1}$, transfer $b_{k+1} - a_{k+1}$ balls from box k to box $k+1$, leaving

$$a_k - (b_{k+1} - a_{k+1}) > a_k - (a_k - b_k) = b_k$$

balls in box k and b_{k+1} balls in box $k+1$.

Since

$$(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] > (b_1 + \cdots + b_{k-1}) + b_k$$

and

$$\begin{aligned}(a_1 + \cdots + a_{k-1}) + [a_k - (b_{k+1} - a_{k+1})] + b_{k+1} &= a_1 + \cdots + a_{k-1} + a_k + a_{k+1} \\ &\ge b_1 + \cdots + b_{k+1},\end{aligned}$$

the consequent arrangement of balls in the boxes satisfies the conditions.

In either case, we have a rearrangement of balls for which the number of balls in one of the kth and $(k+1)$th boxes is equal to the corresponding value of b, so we can apply the induction step.

4478. *Proposed by Florin Stanescu.*

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2) - f(b^2) \le (f(a) + b)(a - f(b))$$

for all $a, b \in \mathbb{R}$.

There were 11 correct and 3 incorrect submitted solutions.

The only solutions are $f(x) = x$ and $f(x) = -x$.

Make the substitutions $(a, b) = (x, y)$ and $(a, b) = (y, x)$ to obtain

$$f(x^2) - f(y^2) \le xy - f(x)f(y) + xf(x) - yf(y)$$

and

$$f(y^2) - f(x^2) \le xy - f(x)f(y) - xf(x) + yf(y).$$

Adding these inequalities leads to $f(x)f(y) \leq xy$ for all real x and y. In particular, $f(0)^2 \leq 0$, so that $f(0) = 0$.

Substituting $(a, b) = (x, 0)$ and $(a, b) = (0, x)$ yields

$$f(x^2) \leq xf(x) \quad \text{and} \quad -f(x)^2 \leq -xf(x),$$

whence $f(x^2) = xf(x)$ for all real x. In particular, $f(1) = -f(-1)$.

Since

$$f(1)f(x) \leq x \quad \text{and} \quad -f(1)f(x) = f(-1)f(x) \leq -x,$$

then $f(1)f(x) = x$ and $f(1)^2 = 1$. When $f(1) = 1$, then $f(x) \equiv x$, and when $f(1) = -1$, then $f(x) \equiv -x$.

4479. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with $\angle A = 90^\circ$ and let H be the foot of the altitude from A. Prove that

$$\frac{6}{(AB + AC)^2} - \frac{1}{2 \cdot AH^2} \leq \frac{1}{BC^2}.$$

We received 32 solutions, including two from the featured solver. We present the solution by Miguel Amengual Covas.

Denote the length of the hypotenuse of the given triangle by a and the legs by b and c. Then the area of ΔABC may be expressed as $bc/2$, and also as $a \cdot AH/2$. Equating these and solving for AH, we get

$$AH = \frac{bc}{a}.$$

When this is substituted into the proposed inequality, the proposed inequality becomes

$$\frac{6}{(b + c)^2} - \frac{a^2}{2b^2c^2} \leq \frac{1}{a^2}. \tag{1}$$

We subsitute $b^2 + c^2$ for a^2 in (1), obtaining

$$\frac{6}{(b + c)^2} - \frac{b^2 + c^2}{2b^2c^2} \leq \frac{1}{a^2},$$

or, equivalently,

$$b^6 + 2b^5c - 7b^4c^2 + 8b^3c^3 - 7b^2c^4 + 2bc^5 + c^6 \geq 0.$$

This in turn is equivalent to

$$(b - c)^2 \left(b^4 + 4b^3c + 4bc^3 + c^4\right) \geq 0,$$

whose validity is obvious. Equality occurs if and only if $b = c$.

4480. *Proposed by Leonard Giugiuc.*

Find all the solutions to the system

$$\begin{cases} a+b+c+d=4, \\ a^2+b^2+c^2+d^2=6, \\ a^3+b^3+c^3+d^3=\frac{94}{9}, \end{cases}$$

in $[0,2]^4$.

We received 11 submissions, all correct. We present the solution by Digby Smith.

Note first that

$$\begin{aligned} ab+ac+ad+bc+bd+cd &= \frac{1}{2}\left((a+b+c+d)^2-(a^2+b^2+c^2+d^2)\right) \\ &= \frac{1}{2}(4^2-6)=5, \end{aligned} \tag{1}$$

and

$$\begin{aligned} (a+b+c+d)^3 &= a^3+b^3+c^3+d^3+3a^2(b+c+d)+3b^2(c+d+a) \\ &\quad +3c^2(d+a+b)+3d^2(a+b+c)+6(abc+abd+acd+bcd) \\ &= 3(a+b+c+d)(a^2+b^2+c^2+d^2)-2(a^3+b^3+c^3+d^3) \\ &\quad +6(abc+abd+acd+bcd) \\ \Longrightarrow 64 &= 3(4)(6)-2\frac{94}{9}+6(abc+abd+acd+bcd) \\ \Longrightarrow abc+abd+acd+bcd &= \frac{1}{6}(64+\frac{188}{9}-72)=\frac{58}{27}. \end{aligned} \tag{2}$$

Next, let $k=abcd$ and $p(x)$ be the polynomial function defined by

$$p(x)=(x-a)(x-b)(x-c)(x-d).$$

Then by (1) and (2) we have

$$\begin{aligned} p(x) &= x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2 \\ &\quad -(abc+bcd+cda+dab)x+abcd \\ &= x^4-4x^3+5x^2-\frac{58}{27}x+k. \end{aligned}$$

Since

$$\begin{aligned} p'(x) &= 4x^3-12x^2+10x-\frac{58}{27} \\ &= \frac{1}{27}(108x^3-324x^2+270x-58) \\ &= \frac{2}{27}(3x-1)(18x^2-48x+29), \end{aligned}$$

solving $p'(x) = 0$ yields
$$x = \frac{1}{3}, \frac{4}{3} \pm \frac{\sqrt{6}}{2}.$$
Since $p(x)$ is a 4th degree polynomial with positive leading coefficient and $p'(x)$ has 3 distinct real roots in (0,2), it follows that in order for a, b, c, d to be solutions of the given equations, where $0 \le a, b, c, d \le 2$, we must have
$$p(0) \ge 0,\ p(\frac{1}{3}) \le 0,\ p(\frac{4}{3} - \frac{\sqrt{6}}{2}) \ge 0,\ p(\frac{4}{3} + \frac{\sqrt{6}}{2}) \le 0,\ p(2) \ge 0.$$
Evaluating, we find $p(\frac{1}{3}) = p(2) = k - \frac{8}{27}$. Hence, $k = \frac{8}{27}$, from which we obtain
$$\begin{aligned} p(x) &= x^4 - 4x^3 + 5x^2 - \frac{58}{27}x + \frac{8}{27} \\ &= \frac{1}{27}(27x^4 - 108x^3 + 135x^2 - 58x + 8) \\ &= \frac{1}{27}(3x-1)^2(3x-4)(x-2). \end{aligned}$$
Therefore, the solutions in $[0,2]^4$ are the 12 permutations of $(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}, 2)$.