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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Issue 6 always marks the beginning of summer and ***Crux*** too takes a break with no issues coming out in July and August. Our workload changes very little though as we try to build up future material in the preview of the fall, which is always a busy time of the year.

We still don't know what exactly the fall will look like for us in academia, but most institutions are running classes in all possible formats: face-to-face, hybrid, fully online synchronous and asynchronous. Remote format allowed seminars and conferences to include a greater variety of speakers and participants as travel and hotel considerations were no longer a concern, replaced instead by more manageable accommodations of time zones. CMS already announced a fully online 2021 Winter meeting and I bet the future will see hybrid conference formats as we embrace and capitalize on the affordances of the online format.

But summer was conference and travel season for many of us and, while I appreciate the ability to attend a conference from my living room, I miss having side-chats with people during coffee breaks, the social time where new ideas and collaborations are born. I also greatly miss the opportunity to tag on a week (or two) of vacation before or after a conference to explore a new place. To avoid cabin fever, for our stay-cation my family is taking advantage of the rivers and the lakes in our area to try out paddle boarding and kayaking. Course prep and an extensive reading and podcast list to get through – thank goodness for a deck and warm summer evenings.

To give our readers something more to do over the next two months, this pre-summer issue contains our usual set of problems and solutions, supplemented by *The Last Problem* (when first passed down from Chris Fisher to Ed Barbeau, the comment was "it was given to me by an enemy") and 25 bonus problems.

I hope this summer everyone is able to take time off and hopefully even see some friends and family in a safe way. Send us pictures of where you will be spending this summer!

Kseniya Garaschuk

MathemAttic

No. 26

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2021**.

MA126. Let A, B, C, X, Y represent distinct, non-zero digits. Consider the following subtraction (and specific example, taking $(A, B, C, X, Y) = (4, 5, 2, 9, 8)$):

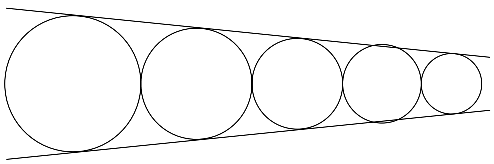
$$\begin{array}{rccc} & A & B & C \\ - & C & B & A \\ \hline & 1 & X & Y \end{array} \qquad \text{Example:} \qquad \begin{array}{rccc} & 4 & 5 & 2 \\ - & 2 & 5 & 4 \\ \hline & 1 & 9 & 8 \end{array}$$

How many ordered quintuplets (A, B, C, X, Y) are there that satisfy the subtraction shown above?

MA127. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.

MA128. I invested \$100. Each day, including the 1st day, my investment first increased in value by $p\%$, then decreased in value. The 1st day's decrease was one-quarter of the 1st day's increase. The 2nd day's decrease was two-quarters of the 2nd day's increase. In general, the nth day's decrease was n-quarters of the nth day's increase. (Note that, from day 5 on, the decrease exceeded the increase.) If my investment first became worthless on the 1000th day, what was the value of p?

MA129. Five marbles of various sizes are placed in a conical funnel of circular cross section. Each marble is in contact with the adjacent marble(s) and with the funnel wall. The smallest marble has a radius of 8 mm. The largest marble has a radius of 18 mm. Determine the radius, measured in mm, of the middle marble.



MA130. Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.

. . . .

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2021**. *La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*

MA126. Soient A, B, C, X, Y des entiers distincts, non nuls. Considérer le schéma de soustraction ci-bas, y inclus le cas particulier $(A, B, C, X, Y) = (4, 5, 2, 9, 8)$:

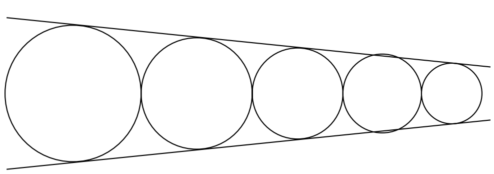
$$\begin{array}{rccc} & A & B & C \\ - & C & B & A \\ \hline & 1 & X & Y \end{array} \qquad \text{Example:} \qquad \begin{array}{rccc} & 4 & 5 & 2 \\ - & 2 & 5 & 4 \\ \hline & 1 & 9 & 8 \end{array}$$

Combien de tels quintuplets ordonnés (A, B, C, X, Y) satisfient le schéma ?

MA127. Si $\log_{10} 2 = a$ et $\log_{10} 3 = b$, déterminer $\log_5 12$.

MA128. J'investis 100\$. Chaque jour, incluant le premier, mon fond augmente premièrement par un pourcentage $p\%$, puis perd de la valeur. Le 1ier jour, cette perte est le quart de l'augmentation du 1ier jour. Le 2ième jour, cette perte est de deux quarts de l'augmentation du 2ième jour. De façon générale, la perte le $n$ième jour est n quarts de l'augmentation du $n$ième jour. (Noter qu' partir du 5ième jour, la perte dépasse l'augmentation.) Si mon fond atteint une valeur nulle le 1000ième jour, déterminer la valeur de p.

MA129. Cinq billes de diverses tailles sont placées dans un entonnoir de forme conique circulaire. Chaque bille est en contact avec toute bille voisinante et avec l'entonnoir. La plus petite bille a un rayon de 8 mm, tandis que la plus grosse a un rayon de 18 mm. Déterminer le rayon en mm de la bille qui se trouve au milieu.



MA130. Démontrer qu'il existe un nombre infini d'entiers positifs k tels que k^k peut être représenté comme somme de cubes de deux entiers positifs.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(1), p. 4–6.

MA101. Standard six-sided dice have their dots arranged so that the opposite faces add up to 7. If 27 standard dice are arranged in a $3 \times 3 \times 3$ cube on a solid table what is the maximum number of dots that can be seen from one position?

Originally problem I21 from the 2014 competition of Australian Math Trust.

We received 5 submissions, of which 4 were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

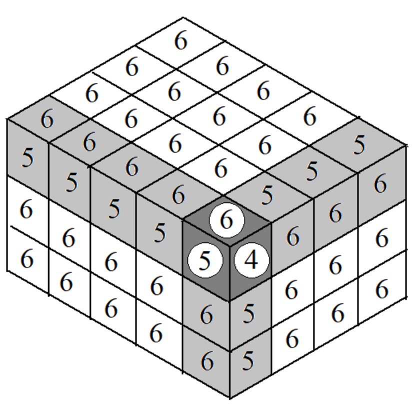
More generally, we will answer the analogous question for abc standard dice arranged in an $a \times b \times c$ cuboid. The maximum number of faces that can be seen are three faces that share a common vertex. There are

$$(a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1)$$

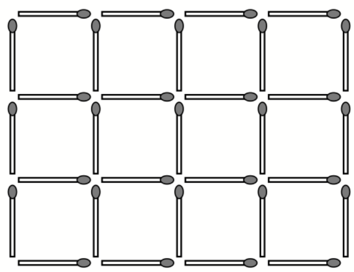
cubes with one face showing, $(a-1)+(b-1)+(c-1)$ cubes with two faces showing, and one cube with three faces showing. This gives

$$\begin{aligned}
&6\left((a-1)(b-1) + (a-1)(c-1) + (b-1)(c-1)\right)\\
&+ (6+5)\left((a-1)+(b-1)+(c-1)\right) + (6+5+4)\\
&= 6(ab+ac+bc) - a - b - c
\end{aligned}$$

as the maximum number of dots. For $a = b = c = 3$, we have 153 dots. The figure below shows the case when $a = 3, b = 4$, and $c = 5$. The white cubes have one exterior face, the light gray cubes have two, and the dark gray cube has three.



MA102. As shown in the diagram, you can create a grid of squares 3 units high and 4 units wide using 31 matches. I would like to make a grid of squares a units high and b units wide, where $a < b$ are positive integers. Determine the sum of the areas of all such rectangles that can be made, each using exactly 337 matches.



Originally problem I29 from the 2014 competition of Australian Math Trust.

We received 8 submissions all of which were correct and complete. We present the solution by Taes Padhihary, modified by the editor.

There are $(a+1)$ rows of horizontal matches, each containing b matches. Similarly, there are $(b+1)$ columns of vertical matches, each containing a matches. So the total number of matches is $(a+1)b+(b+1)a = 2ab+a+b$. Therefore,

$$\begin{aligned} 2ab+a+b = 337 &\implies 4ab+2a+2b = 674 \\ &\implies 4ab+2a+2b+1 = 675 \\ &\implies (2a+1)(2b+1) = 675. \end{aligned}$$

Now, note that $675 = 1 \times 675,\ 3 \times 225,\ 5 \times 135,\ 9 \times 75,\ 15 \times 45,\ 25 \times 27$ and then $27 \times 25, \ldots$ and so on. Given this, we obtain $(a,b) = (0,337)$, $(1,112)$, $(2,67)$, $(4,37)$, $(7,22)$, $(12,13)$ and vice-versa. Except the first one, all are valid. Thus the sum of the areas is

$$112+134+148+154+156 = 704.$$

MA103. What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

Originally problem S26 from the 2014 competition of Australian Math Trust.

We received 7 submissions of which 5 were correct and complete. We present the solution by William Alexander Digout.

Let's start by giving the name ISC to the property where a three-digit number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit. Let abc be a number with the ISC property, where $abc = 100a+10b+c$. Then $100a+10b+c = a+b^2+c^3$. To narrow down our list of possible numbers containing the ISC property, we will look at the value of c.

c	c^3	a
0	0	0 or 1
1	1	0 or 1
2	8	0 or 1
3	27	0 or 1
4	64	0 or 1
5	125	1 or 2
6	216	2 or 3
7	343	3 or 4
8	512	5 or 6
9	729	7 or 8

Since $b^2 \leq 81$, we can eliminate $c = 0, 1, 2$ as $a + b^2 + c^3 < 100$. We can also assume that if $c = 3$ or $c = 4$, then $a = 1$ since $a = 0$ yields a two-digit number.

If $c = 3$, then $c^3 = 27$ and $a = 1$ implies that $b^2 \equiv 5 \pmod{10}$, where $b = 5$. But $1 + 5^2 + 3^3 = 53 < 100$, therefore $c \neq 3$.

If $c = 4$, then $c^3 = 64$ and $a = 1$ implies that $b^2 \equiv 9 \pmod{10}$, where $b = 3$ or $b = 7$. But $1 + 3^2 + 4^3 = 74 < 100$ and $1 + 7^2 + 4^3 = 114 \neq 174$. Therefore $c \neq 4$.

If $c = 5$, then $c^3 = 125$ and $a = 1$ or $a = 2$.

If $a = 2$, then $b^2 \equiv 8 \pmod{10}$, which is absurd since b is an integer. Then $a = 1$ implies that $b^2 \equiv 9 \pmod{10}$, so $b = 3$ or $b = 7$. We have $1 + 3^2 + 5^3 = 135 = 100(1) + 10(3) + 1(5)$ and $1 + 7^2 + 5^3 = 175 = 100(1) + 10(7) + 1(5)$. Thus 135 and 175 both have the ISC property.

If $c = 6$, then $c^3 = 216$ and $a = 2$ or $a = 3$. If $a = 3$, then $b^2 \equiv 7 \pmod{10}$, which is absurd. If $a = 2$, then $b^2 \equiv 8 \pmod{10}$, which is also impossible since b is an integer. Therefore $c \neq 6$.

If $c = 7$, then $c^3 = 343$ and $a = 3$ or $a = 4$. If $a = 3$, then $b^2 \equiv 1 \pmod{10}$, so $b = 1$ or $b = 9$. But $3 + 1^2 + 7^3 = 347 \neq 317$ and $3 + 9^2 + 7^3 = 427 \neq 397$. Therefore $c \neq 7$.

If $c = 8$, then $c^3 = 512$ and $a = 5$ or $a = 6$. If $a = 6$, then $b^2 \equiv 0 \pmod{10}$, so $b = 0$. But $6 + 0^2 + 8^3 = 518 \neq 608$. Thus $a = 5$ implies that $b^2 \equiv 1 \pmod{10}$, so $b = 1$ or $b = 9$. We have $5 + 1^2 + 8^3 = 518 = 100(5) + 10(1) + 1(8)$ and $5 + 9^2 + 8^3 = 598 = 100(5) + 10(9) + 1(8)$. Therefore 518 and 598 both have the ISC property.

If $c = 9$, then $c^3 = 729$ and $a = 7$ or $a = 8$. If $a = 7$, then $b^2 \equiv 3 \pmod{10}$, which is absurd. If $a = 8$, then $b^2 \equiv 2 \pmod{10}$, which is equally absurd. Thus $c \neq 9$.

Of the 900 three-digit numbers, only four have the ISC property. Since we want the largest of the these numbers, we conclude that 598 is the largest three-digit number that is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit.

MA104. The sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$$

is defined by $a_1 = 2$ and $a_{n+1} = 2^{a_n}$ for all $n \geq 1$. What is the first term in the sequence greater than 1000^{1000}?

Originally problem S25 from the 2014 competition of Australian Math Trust.

We received 6 submissions, all of which were correct and complete. We present the solution by Taes Padhihary, modified by the editor.

We want the smallest n for which $a_n > 1000^{1000} = 10^{3000}$. We know

$$a_4 = 2^{16} = 65536 << 10^{3000}.$$

As $2^{10} > 10^3$, we have that

$$a_5 = 2^{65536} = (2^{10})^{6553} \cdot 2^6 > 10^{19659} \cdot 64,$$

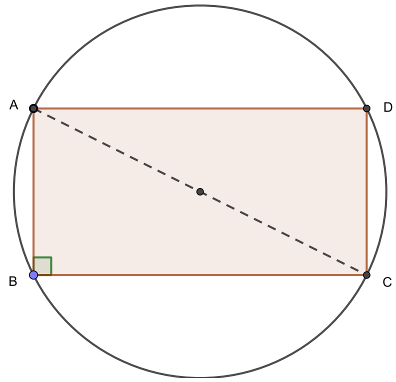
which is safely much greater than 10^{3000}. Hence, the fifth term is the first term of the sequence greater than 1000^{1000}.

MA105. Eighteen points are equally spaced on a circle, from which you will choose a certain number at random. How many do you need to choose to guarantee that you will have the four corners of at least one rectangle?

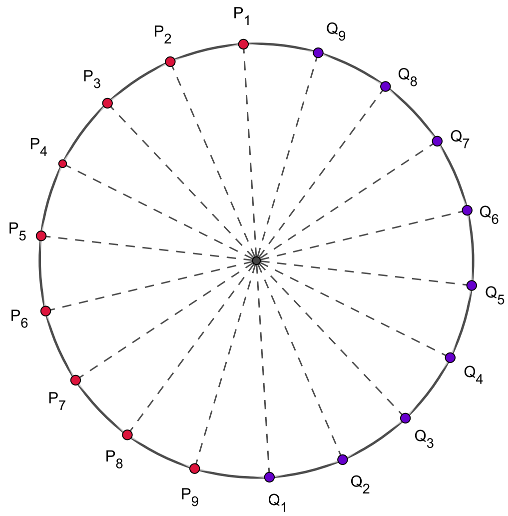
Originally problem J27 from the 2014 competition of Australian Math Trust.

We received 6 solutions. We present the solution of Prithwijit De, modified by the editor.

Suppose a rectangle $ABCD$ is inscribed in a circle (see diagram below). Since $\angle ABC = 90°$, the diagonal AC must be a diameter of the circle, and similarly so does the diagonal BD. Conversely, if on a given circle A and C are points which are diametrically opposite, and so are B and D, then $ABCD$ is a rectangle.



Given 18 points equally spaced around the circle, they are the endpoints of 9 diameters. We label one endpoint of a diameter by P_i and the other by Q_i for $i = 1, \dots, 9$, as in the diagram below.



If the chosen set of points contains two pairs (P_i, Q_i) and (P_j, Q_j) for $i \neq j$ then P_i, P_j, Q_i, Q_j are the vertices of a rectangle, as described earlier. The smallest number of points which we need to choose to guarantee that we have two such pairs is 11. Clearly, if we were to choose only 10 points then it could happen that we chose, say, all the P's and only one Q endpoint, and from those we cannot choose the vertices of a rectangle. However, applying the Pigeonhole Principle, we can show that a set of 11 points contains at least two pairs of points which are diametrically opposite. Therefore, if we choose 11 points, we can guarantee that this set contains the vertices of at least one rectangle.

PROBLEM SOLVING VIGNETTES

No. 17

Shawn Godin

Geometric Constructions II

Welcome back. In the last column [2021 : 47(5), p. 232–237] we looked at some geometric constructions using a compass and straightedge. In this column we will look at some geometric construction problems from the course I took with Professor Honsberger. Note that some of these are quite a bit more complex than the ones we looked at last column. Hopefully, there will be problems that can be enjoyed by both the beginner and experienced constructor.

#31. A segment AB is given and a line m crossing it. Determine the point C on m such that m bisects angle ACB.

#32. Two circles A and B are given and a vector $\vec{k}$. Determine a point P on A and a point Q on B such that PQ is equal and parallel to $\vec{k}$.

#33. Points A and B are given on the same side of the line XY. Determine the point C on XY such that angle ACX is double angle BCY (suppose the line XY runs from left to right and that A and B are above it; for definiteness, label, from left to right, the line XY, and the points A and B).

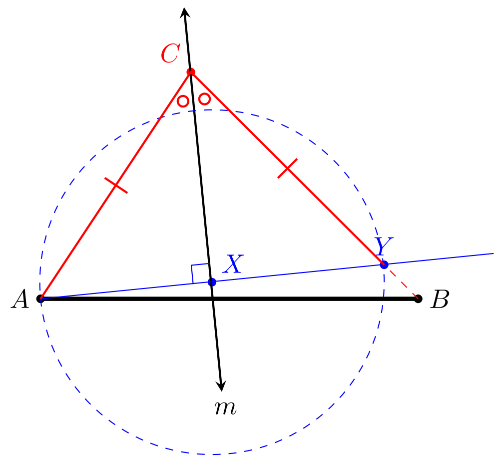
#34. Non-intersecting chords AB and CD of a circle are given. Determine a point X on the circle such that AX and BX determine E and F on CD making EF equal in length to a given segment k.

#35. A and B are points inside a given acute angle PQR. Construct an isosceles triangle XYZ with X on PQ, Y and Z on QR, and A and B, respectively, on the sides XY and XZ.

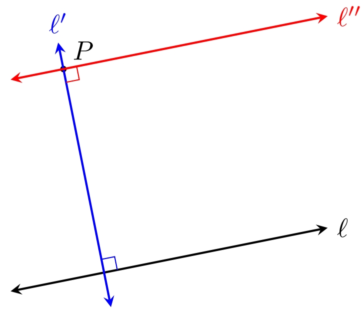
Let's look first at problem #31, which involves angle bisectors, a topic we discussed in the previous column. In several of the constructions in that column we used a fact from an earlier column [2019: 45(1), pp. 13-16] that the angle bisector of the apex angle of an isosceles triangle coincides with the perpendicular bisector of the base. Maybe we can use that result here to find a segment somehow related to the given points A and B for which the given line m is its perpendicular bisector?

If we drop a perpendicular from one of the points, say A, to m meeting at point X, then draw a circle centred at X through A, it will meet the perpendicular to

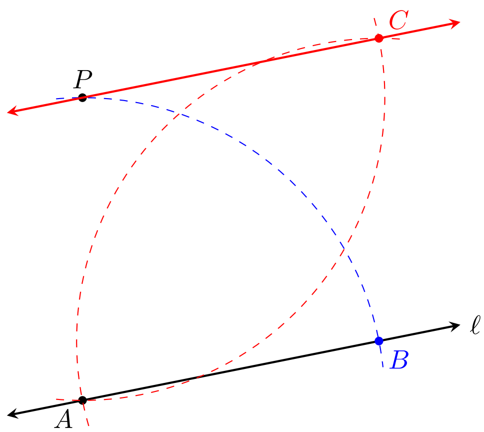
m at another point Y. Therefore, if we pick any point, Z, on m, then ZAY is isosceles. In order to bring B into the mix, if we draw the line through B and Y, it will intersect m at the desired point C, that is, since ΔCAY is isosceles and m is the perpendicular bisector of AY, it is the angle bisector of $\angle ACY = \angle ACB$.



Before we attack another problem, we need an algorithm to construct a line parallel to a given line, ℓ, through a given point, P (construction challenge #4 from the last column). We can do this in any number of ways. Since we know how to construct a line through P perpendicular to ℓ, we can do this yielding line ℓ', and then repeat the process constructing a line perpendicular to ℓ' through P. This new line, ℓ'', will have to be parallel to ℓ by the parallel line theorem (interior angles are supplementary, alternate angles are equal, corresponding angles are equal . . . take your pick).

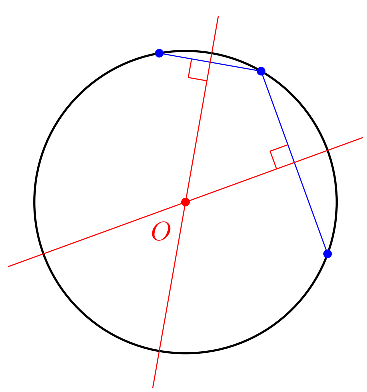


The standard algorithm for constructing a parallel line through a given point constructs a rhombus with one vertex P and one side on ℓ. As a rhombus is also a parallelogram, one of the sides will be parallel to ℓ and we are done. To proceed, we pick any point, A, on ℓ and construct an arc, centred at A that passes through P and intersects ℓ at B. Next we construct arcs centred at P and B that pass through A. These arcs intersect at another point D. By construction $PA = AB = BC = CP$, hence $ABCP$ is a rhombus and therefore the line through P and C is our desired line.



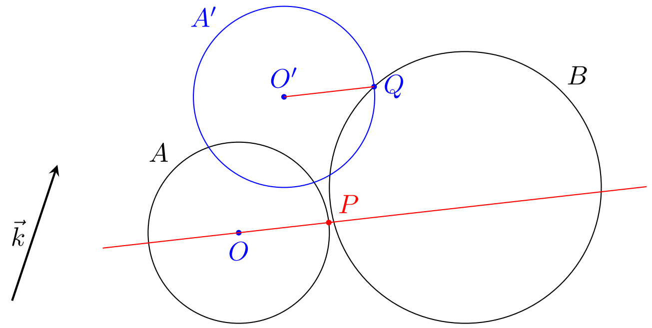
Of course there are other ways to construct the desired parallel line. For example, if we had drawn two points A and B on ℓ, then constructed a circle centred at P with radius equal to AB and another circle centred at A with radius equal to PB, then these two circles will intersect at two points X and Y. The line through one of these points and P is our desired line. I will leave it to the reader to try this construction and to come up with the justification.

Next we will consider problem #32. We will assume that the circles are given, without the centres. It is useful to be able to determine the centre of a given circle. We will use the fact that the perpendicular bisector of a segment is the locus of points that are equidistant from the end points of the segment. Thus, on a circle, if we construct the perpendicular bisector of any chord it will pass though the centre of the circle. Hence, if we pick three points on the circle, and construct two chords using these points as our endpoints, the perpendicular bisectors of the two chords will intersect at the centre of the circle. This is linked to the fact that the perpendicular bisectors of the three sides of a triangle are concurrent at the circumcentre of the triangle.

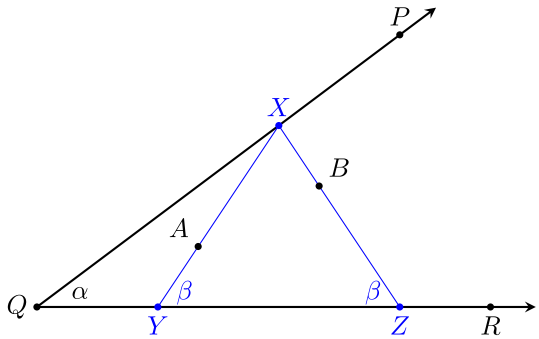


It is often the case in construction problems that we do not know where to start, so it sometimes helps to start with the finished picture and work backwards. For problem #32 we draw circles A and B, stick points P on A, Q on B, and declare $\overrightarrow{PQ}$ to be the vector $\vec{k}$ that Professor Honsberger gave us. With just of touch of inspiration we recall that vectors determine translations, and the translation determined by $\vec{k}$ takes the circle A with its centre O and point P to a new circle with centre O' and point Q for which $OO'QP$ is a parallelogram (because OO' is equal and parallel to PQ). So the construction is clear:

To translate A, we would need to construct its centre, O, then draw a line through O parallel to $\vec{k}$ and finally use our compass to mark the length of $\vec{k}$ and mark the distance from O along the parallel line in the direction of $\vec{k}$ to get the image O'. If we construct the circle, A', with centre O' that is congruent to A, it might intersect circle B in as many as two points. Label either of the points Q. If we construct a line through O parallel to $O'Q$, then it will intersect A at two points, one of which we claim is P (on the same side of O as Q is of O'). Since $O'Q = OP$ and $O'Q \parallel OP$, then $OO'QP$ is a parallelogram and hence $\overrightarrow{PQ} = \overrightarrow{OO'} = \vec{k}$. Thus problem #32 has two solutions if the translated circle meets B in two points, one solution if it is tangent to B, and no solution at all if it misses B.



Finally, we will take a partial look at #35. Imagine the construction is finished, then we have something like the diagram below.



Let $\angle PQR = \alpha$ and $\angle XYZ = \angle XZY = \beta$. At the start of the construction we have not been given β, but we are given α, so that α can be reproduced by construction when needed. From our diagram we can determine that

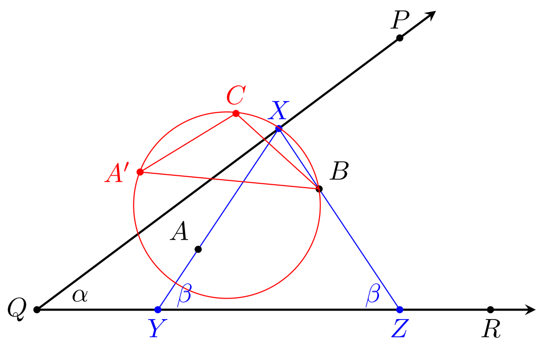
$$\angle YXZ = 180^\circ - 2\beta \quad \text{and} \quad \angle QXY = \beta - \alpha,$$

hence

$$\angle QXZ = 180^\circ - \beta - \alpha.$$

If we could produce a ray XF above QP such that $\angle FXQ = \angle QXZ = \beta - \alpha$, then $\angle FXB = 180^\circ - 2\alpha$, which is the measure of the apex of an isosceles triangle with base angles α. Now *that* triangle we can construct.

Our desired ray is easily constructed. If we reflect A in QR, its image, A', lies on the desired ray. Then, if we form segment $A'B$, we can construct isosceles triangle ABC with $\angle CAB = \angle CBA = \alpha$ making sure C is on the same side of $A'B$ as X. If we construct the circumcircle of ABC it will intersect QP in two points, one of which is above both A and B. I claim the upper point is the desired point X. I will leave the verification of this and the details of the construction as an exercise.



Hopefully you enjoyed this exploration of classical constructions. The constructions from this column were meant to be a bit more challenging, so if you struggled with them don't worry, think about them, come back to them, you will get there eventually. The ideas behind them can be useful in constructing figures in dynamic geometry software that retain desired properties. I suggest you grab a compass and ruler and get constructing!

My thanks again goes to longtime *Crux* editor Chris Fisher for his feedback on this column and the previous one. His comments helped make both articles better.

The Last Problem

In days of yore, the Canadian Mathematical Bulletin had a problem section. It was retired in 1983 with a fascinating problem that came by a circuituous route, from A.E. Brouwer to Chris Fisher and finally to Ed Barbeau, then editor of the problems section. Although there existed a solution, it involved work on partially ordered sets beyond its immediate context, and a more natural solution was sought.

Here is the problem: A $m \times n$ rectangular array is made up of the positive integers $1, 2, 3, \ldots, mn$ arranged in such a way that each row and each column is monotonically decreasing. In particular, mn must appear in the upper left corner and 1 in the lower right corner. An operation of the array is as follows. The number in the lower right corner is circled. Once any number is circled, the smaller of two of its neighbours, one immediately to the left in the same row and the other immediately above in the same column, is also circled. If there is only one such number, it is circled. In this way, a track of $m + n - 1$ circled numbers from the lower right to the upper left is obtained. Now the number in the lower right is transferred to the upper left position and the rest of the circled numbers are displaced one position along the track. The uncircled numbers are not moved. The same operation is then repeated, with the understanding that, once any number k is transferred from the lower right position to the upper left position, it is treated as though its magnitude were $mn + k$.

An example of such an array with $m = 3$, $n = 4$ is given along with the results of the first three operations:

$$\begin{array}{cccc} 12 & 11 & 7 & 6 \\ 10 & 8 & 5 & 3 \\ 9 & 4 & 2 & 1 \end{array} \longrightarrow \begin{array}{cccc} 1 & 11 & 7 & 6 \\ 12 & 10 & 5 & 3 \\ 9 & 8 & 4 & 2 \end{array} \longrightarrow \begin{array}{cccc} 2 & 1 & 11 & 6 \\ 12 & 10 & 7 & 5 \\ 9 & 8 & 4 & 3 \end{array} \longrightarrow \begin{array}{cccc} 3 & 1 & 11 & 6 \\ 2 & 12 & 10 & 5 \\ 9 & 8 & 7 & 4 \end{array}$$

Prove or disprove:

(a) After mn operations, each number in the array is restored to its initial position;

(b) If i moves down on the jth move, then j moves down on the ith move;

(c) If i moves right on the jth move, then j moves right on the ith move.

Send your comments, investigations, solutions to crux.eic@gmail.com

OLYMPIAD CORNER

No. 394

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **August 15, 2021**.

OC536. The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

OC537. A, B, C are collinear with B betweeen A and C. K_1 is the circle with diameter AB, and K_2 is the circle with diameter BC. Another circle touches AC at B and meets K_1 again at P and K_2 again at Q. The line PQ meets K_1 again at R and K_2 again at S. Show that the lines AR and CS meet on the perpendicular to AC at B.

OC538. Let us consider a polynomial $P(x)$ with integer coefficients satisfying $P(-1) = -4$, $P(-3) = -40$, and $P(-5) = -156$. What is the largest possible number of integers x satisfying $P(P(x)) = x^2$?

OC539. A pair of real numbers (a, b) with $a^2 + b^2 \leq \frac{1}{4}$ is chosen at random. If p is the probability that the curves with equations $y = ax^2 + 2bx - a$ and $y = x^2$ intersect, then identify the integer that is closest to $100p$.

OC540. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where r is a rational number and n a positive integer. Find all triplets $(a, b, c) \in \mathbb{Q}_+ \times \mathbb{Q}_+ \times \mathbb{N}$ for which there exist infinitely many positive integers n satisfying $S_a(n) = (S_b(n))^c$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC536. Soit BC le plus long côté du triangle ABC où, de plus, $AB = CA$. Le point N se trouve sur le côté BC, de façon à ce que $BN = AB$. Enfin, la ligne perpendiculaire à AB, passant par N, rencontre AB en M. Démontrer que la ligne MN divise le triangle ABC en deux parties de même surface et périmètre.

OC537. Les points A, B et C sont alignés, avec B entre A et C. K_1 est le cercle ayant AB comme diamètre et K_2 est le cercle ayant BC comme diamètre. Un autre cercle touche AC en B et rencontre K_1 de nouveau en P et K_2 de nouveau en Q. La ligne PQ rencontre K_1 de nouveau en R et K_2 de nouveau en S. Démontrer que les lignes AR et CS se rencontrent en un point se trouvant sur la perpendiculaire à AC en B.

OC538. Soit un polynôme $P(x)$ à coefficients entiers tel que $P(-1) = -4$, $P(-3) = -40$ et $P(-5) = -156$. Déterminer le plus grand nombre possible d'entiers x vérifiant $P(P(x)) = x^2$.

OC539. Une paire de nombres réels (a, b) vérifiant $a^2 + b^2 \leq \frac{1}{4}$ est choisie de façon aléatoire. Si p est la probabilité que les courbes $y = ax^2 + 2bx - a$ et $y = x^2$ se rencontrent, identifier l'entier le plus près de $100p$.

OC540. Soit $S_r(n) = 1^r + 2^r + \cdots + n^r$, où r est un nombre rationnel et n un entier positif. Déterminer tous les triplets $(a, b, c) \in \mathbb{Q}_+ \times \mathbb{Q}_+ \times \mathbb{N}$ pour lesquels il existe un nombre infini d'entiers positifs n tels que $S_a(n) = (S_b(n))^c$.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(1), p. 25–26.

OC511. All the proper divisors of some composite natural number n, increased by 1, are written out on a blackboard. Find all composite natural numbers n for which the numbers on the blackboard are all the proper divisors of some natural number m. (Note: here 1 is not considered a proper divisor.)

Originally from 2017 Russia Mathematics Olympiad, 5th Problem, Grade 10, Final Round.

We received 6 *solutions. We present the solution by Oliver Geupel.*

For $n = 4$, only the number 3 is written on the blackboard, which is the unique proper divisor of $m = 9$. For $n = 8$, the numbers written on the blackboard are 3 and 5, which are the proper divisors of $m = 15$.

We show that there are no other solutions.

Suppose that n has the desired property. If n has an odd proper divisor d then the even number $d + 1$ is written on the blackboard, so that $d + 1 \mid m$. Hence $2 \mid m$. But 2 is not written because 1 is not considered to be a proper divisor of n. Thus, n is a power of 2. If $n \geq 16$ then the numbers on the blackboard include 3, 5, and 9, so that $45 \mid m$. Then, 15 is a proper divisor of m. This contradicts the fact that $14 \nmid n$, because n is a power of 2. Hence the result.

OC512. A convex quadrilateral $ABCD$ is given. We denote by I_A, I_B, I_C and I_D the centers of the inscribed circles ω_A, ω_B, ω_C and ω_D of the triangles DAB, ABC, BCD and CDA, respectively. It is known that $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$. Prove that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$.

Originally from 2017 Russia Mathematics Olympiad, 8th Problem, Grade 11, Final Round.

We received 3 *correct solutions. We present the solution by UCLan Cyprus Problem Solving Group.*

Let $\alpha_1 = \angle BAC$ and $\alpha_2 = \angle CAD$. Then

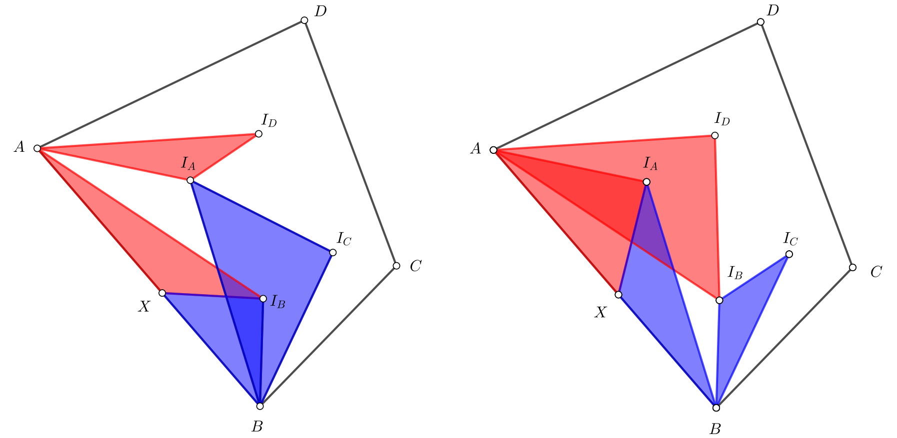
$$\angle I_AAI_D = \angle BAI_D - \angle BAI_A = (\alpha_1 + \alpha_2 - \angle I_DAD) - \frac{\alpha_1 + \alpha_2}{2} = \frac{\alpha_1}{2} = \angle BAI_B\,.$$

Similarly, we have $\angle I_BBI_C = \angle ABI_A$.

Let X be a point on AB such that $\angle AXI_B = \angle AI_AI_D$. Then the triangles AXI_B

and AI_AI_D are similar. Thus

$$\frac{AX}{AI_A} = \frac{AI_B}{AI_D}\,.$$



Since also $\angle XAI_A = \angle I_BAI_D$, it follows that the triangles XAI_A and I_BAI_D are similar. We deduce that $\angle AI_BI_D = \angle AXI_A$.

Since $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$ and $\angle AXI_B = \angle AI_AI_D$ then

$$\angle BI_AI_C = 180^\circ - \angle AI_AI_D = 180^\circ - \angle AXI_B = \angle BXI_B\,.$$

Since also $\angle I_BBI_C = \angle ABI_A$, then $\angle XBI_B = \angle I_ABI_C$. So the triangles BI_BX and BI_CI_A are similar. A similar argument as above shows that the triangles BXI_A and BI_BI_C are also similar. Thus $\angle BXI_A = \angle BI_BI_C$. Therefore

$$\angle AI_BI_D + \angle BI_BI_C = \angle AXI_A + \angle BXI_A = 180^\circ\,.$$

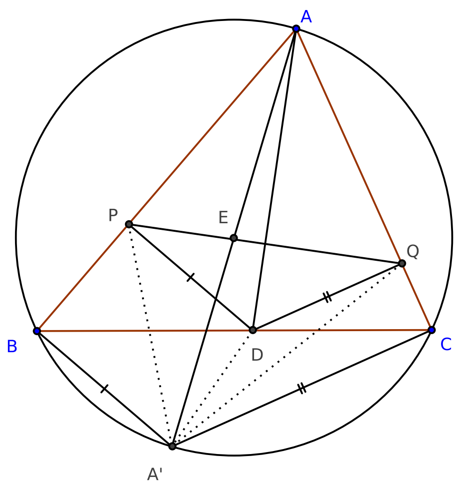
It therefore follows that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$ as required.

OC513. In an acute triangle ABC the angle bisector of $\angle BAC$ intersects BC at point D. Points P and Q are orthogonal projections of D on lines AB and AC. Prove that $Area(APQ) = Area(BCQP)$ if and only if the circumcenter of ABC lies on line PQ.

Originally from 2017 Poland Mathematics Olympiad, 2nd Problem, Second Round.

We received 8 solutions. We present the solution by Oliver Geupel.

Let the point A' be the reflection of the point A in the circumcenter of the triangle ABC, and let E be the point where the line AA' intersects the line PQ.



By Thales's Theorem, we have $\angle A'BA = 90^\circ = \angle DPA$; whence the line $A'B$ is parallel to the line DP. Thus, $[PA'D] = [PBD]$. Similarly, $[QA'D] = [QCD]$. It follows that

$$\begin{aligned}[PA'Q] &= [PDQ] + [PA'D] + [QA'D] \\ &= [PDQ] + [PBD] + [QCD] = [BCQP].\end{aligned}$$

Therefore, $[APQ] = [BCQP]$ is equivalent to $[PAQ] = [PA'Q]$, which is satisfied if and only if $EA = EA'$, that is, if E is the circumcenter of $\triangle ABC$.

OC514. Consider the set $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \mid ab = cd \right\}$.

(a) Give an example of a matrix $A \in M$ such that $A^{2017} \in M$ and $A^{2019} \in M$, but $A^{2018} \notin M$.

(b) Prove that if $A \in M$ and there exists an integer $k \geq 1$ such that $A^k \in M$, $A^{k+1} \in M$ and $A^{k+2} \in M$, then $A^n \in M$ for all integers $n \geq 1$.

Originally from 2018 Romania Mathematics Olympiad, 2nd Problem, Grade 11, District Round.

We received 6 *solutions. We present the solution by UCLan Cyprus Problem Solving Group.*

(a) It is enough to find a matrix A such that $A \in M, A^2 \notin M$ and $A^3 = I$. Indeed we would then have $A^{2017} = A \in M, A^{2018} = A^2 \notin M$ and $A^{2019} = I \in M$. The matrix

$$A = \begin{pmatrix} 1 & -\sqrt{6} \\ \sqrt{3/2} & -2 \end{pmatrix}$$

satisfies the required properties. Indeed it is immediate that $A \in M$. Furthermore, $\mathrm{tr}(A) = -1$ and $\det(A) = 1$, so A has characteristic equation $x^2 + x + 1 = 0$ and therefore satisfies $A^3 = I$. Finally,

$$A^2 = -I - A = \begin{pmatrix} -2 & \sqrt{6} \\ -\sqrt{3/2} & 1 \end{pmatrix} \notin M\,.$$

(b) Lemma 1. If $A \in M$ and $k \in \mathbb{C}$, then $kA \in M$.

Proof of Lemma 1. Immediate. □

Given matrices

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \quad \text{and} \quad A_2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

we call the pair (A_1, A_2) coupled, if $a_1b_2 + a_2b_1 = c_1d_2 + c_2d_1$. We have the following result about coupled matrices:

Lemma 2. If $A_1, A_2 \in M$ and $\lambda_1 A_1 + \lambda_2 A_2 \in M$ for some $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$, then the pair (A_1, A_2) is coupled.

Conversely, if $A_1, A_2 \in M$ and the pair (A_1, A_2) is coupled, then $\lambda_1 A_1 + \lambda_2 A_2 \in M$ for every $\lambda_1, \lambda_2 \in \mathbb{C}$.

Proof of Lemma 2. Since

$$\lambda_1 A_1 + \lambda_2 A_2 = \begin{pmatrix} \lambda_1 a_1 + \lambda_2 a_2 & \lambda_1 b_1 + \lambda_2 b_2 \\ \lambda_1 c_1 + \lambda_2 c_2 & \lambda_1 d_1 + \lambda_2 d_2 \end{pmatrix}$$

then $\lambda_1 A_1 + \lambda_2 A_2 \in M$ if and only if

$$(\lambda_1 a_1 + \lambda_2 a_2)(\lambda_1 b_1 + \lambda_2 b_2) = (\lambda_1 c_1 + \lambda_2 c_2)(\lambda_1 d_1 + \lambda_2 d_2)\,.$$

Equivalently, $\lambda_1 A_1 + \lambda_2 A_2 \in M$ if and only if

$$\lambda_1^2(a_1b_1 - c_1d_1) + \lambda_2^2(a_2b_2 - c_2d_2) + \lambda_1\lambda_2(a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1) = 0\,.$$

So under the condition that $A_1, A_2 \in M$, we have that $\lambda_1 A_1 + \lambda_2 A_2 \in M$ if and only if

$$\lambda_1\lambda_2(a_1b_2 + a_2b_1 - c_1d_2 - c_2d_1) = 0\,.$$

The statement of the lemma follows. □

Assume now that $A \in M$ and $A^k, A^{k+1}, A^{k+2} \in M$ for some integer $k \geqslant 1$. Let $x^2 - ax - b$ be the characteristic equation of A.

Case 1: If $a = 0$, then $A^2 = bI$. By induction $A^{2n} = b^n I$ and $A^{2n+1} = b^n A$. Since $I, A \in M$, by Lemma 1 $A^n \in M$ for every integer $n \geqslant 1$.

Case 2: If $b = 0$, then $A^2 = aA$. By induction $A^{n+1} = a^n A$. Since $A \in M$, by Lemma 1 $A^n \in M$ for every integer $n \geqslant 1$.

Case 3: Assume $ab \neq 0$. Let $A_1 = A^k$ and $A_2 = A^{k+1}$. Then we have $A^{k+2} = aA_2 + bA_1$. Since $A^k, A^{k+1}, A^{k+2} \in M$ and $ab \neq 0$, by Lemma 2 the pair (A^k, A^{k+1}) is coupled.

It will be enough to show that for each natural number n, the matrix A^n is a linear combination of A^k and A^{k+1}. Indeed then by Lemma 2 it will follow that $A^n \in M$.

This follows easily by induction and the facts that $A^{n+2} = aA^{n+1} + bA^n$ and $A^{n-1} = -\frac{a}{b}A^n + \frac{1}{b}A^{n+1}$.

OC515. Let a, b, c, d be natural numbers such that $a+b+c+d = 2018$. Find the minimum value of the expression:

$$E = (a-b)^2 + 2(a-c)^2 + 3(a-d)^2 + 4(b-c)^2 + 5(b-d)^2 + 6(c-d)^2.$$

Originally from 2018 Romania Mathematics Olympiad, 2nd Problem, Grade 8, Final Round.

We received 8 *solutions.*

We present the solution by Roy Barbara.

The minimum value of E is 14, reached when a, b, c, d are (in any order) 504, 504, 505, 505.

More generally, let a, b, c, d be natural numbers such that $a+b+c+d = n$, where n is an even positive integer. Set

$$E = (a-b)^2 + 2(a-c)^2 + 3(a-d)^2 + 4(b-c)^2 + 5(b-d)^2 + 6(c-d)^2.$$

Then,

$$\min E = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4 \\ 14 & \text{if } n \equiv 2 \pmod 4 \end{cases}.$$

Indeed, if $m = 4k$, where $k \in \mathbb{Z}^+$, then $E = 0$ is reached with $a = b = c = d = k$. From now on, we assume $m = 4k + 2$, where $k \in \mathbb{N}$. Among the 6 gaps $|a-b|$, $|a-c|$, $|a-d|$, $|b-c|$, $|b-d|$, $|c-d|$, consider m gaps, $2 \leq m \leq 6$, ordered as $d_1 \geq d_2 \geq \ldots \geq d_m$. It should be clear that

$$E \geq \sum_{r=1}^{m} r d_r^2.$$

Set $x = \min(a, b, c, d)$, $y = \max(a, b, c, d)$ and $\delta = yx \geq 0$. That $\delta = 0$ is impossible, otherwise we would get $a = b = c = d$, so that $4 \mid m$, a contradiction. Hence $\delta \geq 1$.

(i) Suppose $\delta \geq 3$. Since $x, y \in \{a, b, c, d\}$, let z be one of the two remaining variables. Set $d = \max(|z-x|, |z-y|)$. Since $\delta \geq 3$, then $d \leq 1$ is clearly impossible. Hence, $d \geq 2$. Since $\delta \geq d$, we get

$$E \geq 1 \cdot \delta^2 + 2 \cdot \delta^2 \geq 1 \cdot 3^2 + 2 \cdot 2^2 = 17.$$

(ii) Suppose $\delta = 2$. For the remaining variables, we have $z, t \in [x, y] = [x, x+2]$. If one at least of z or t is equal to x or to $x+2$, we get 3 gaps of '2'. Hence,

$$E \geq 1 \cdot 2^2 + 2 \cdot 2^2 + 3 \cdot 2^2 = 24.$$

If $z = t = x+1$, the gaps are all 2, 1, 1, 1, 1, 0. Hence

$$E \geq 1 \cdot 2^2 + 2 \cdot 1^2 + 3 \cdot 1^2 + 4 \cdot 1^2 + 5 \cdot 1^2 = 18.$$

(iii) Suppose $\delta = 1$. For the remaining variables, we have $z, t \in [x, x+1]$. If we had $z = t = x$ or $z = t = x+1$, we would get $m = x + y + z + t = 4x + 1$ or $4x + 3$, contradicting $m \equiv 2 \pmod 4$. Hence, z, t are x, $x+1$, and hence a, b, c, d are (in some order) x, x, $x+1$, $x+1$ (where $x = k$). Any of the 6 cases yields $\min E = 14$.

We conclude that $\min E = 14$.

A Remarkable Point of the Circumcircle

Michel Bataille

We consider a triangle ABC inscribed in a circle Γ with centre O and denote by M the midpoint of BC. The median AM intersects Γ at A_1 ($A_1 \neq A$) and A_1' is the point of Γ diametrically opposite to A_1. We define S as the reflection of A_1 in the perpendicular bisector ℓ of BC (Figure 1). Note that S is a point of Γ (since Γ is its own reflection in ℓ).

The purpose of this note is to show that S has quite a number of interesting properies, making it a remarkable point. Of course, S is associated with the vertex A and two similar points of Γ are associated with the other vertices (see problem 2 at the end).

For simplicity, we always suppose $AB \neq AC$ and $\angle BAC \neq 90^\circ$. The reader will easily adapt the results and proofs if ABC is isosceles or right-angled at A.

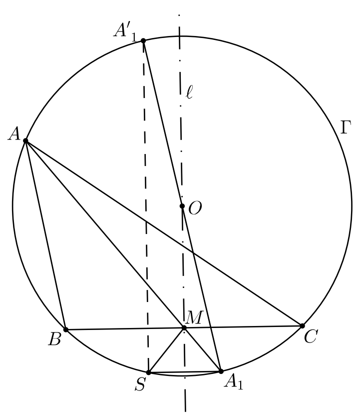


Figure 1

To become familiar with S, here are some very simple properties of S (Figure 1):

(i) The line MS is the reflection of the median AA_1 in ℓ as well as in BC. (This follows from the fact that ℓ and BC are perpendicular at M).

(ii) S is the second point of intersection of Γ with the parallel to BC through A_1.

(iii) The line SA_1' is perpendicular to BC. (Because A_1A_1' is a diameter of Γ, SA_1' is perpendicular to SA_1, hence to BC).

From (iii), SA_1' intersects BC at the projection U of S onto BC. Since in addition S is on the circumcircle of ΔABC, this suggests to consider the Simson line of S, that is, the line through the collinear projections of S onto the sidelines of the triangle ABC ([**2**], p. 43 or [**3**], p. 42). We are led to our first theorem.

Theorem 1 S is the only point of the circumcircle Γ whose Simson line is perpendicular to the median AM.

For the proof we use classical results about the Simson line. First, there is only one point of Γ whose Simson line has a given direction, hence it is sufficient to show that the Simson line of S is perpendicular to AM. But we know that if the perpendicular to BC through S intersects Γ at D ($D \neq S$), then the Simson line of S is parallel to AD (see **[2]** p. 128-9 or **[3]**, p. 50). From (iii), D coincides with A_1' and it just remains to observe that AA_1' is perpendicular to AA_1 since A_1A_1' is a diameter of Γ (Figure 2).

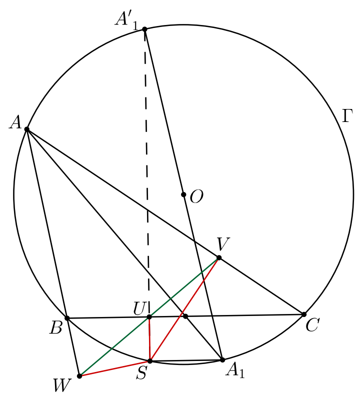


Figure 2

Another property connected to the Simson line of S is the object of Theorem 5 below.

Exercise 1 Show that S is the reflection in BC of the foot of the perpendicular to the median AM from the orthocenter.

A close examination of Figure 2 will provide a simple proof of our next theorem.

Theorem 2 The line AS is the symmedian of ΔABC through the vertex A.

Let V and W be the projections of S onto CA and AB, respectively. The points A, V, S, W lie on the circle with diameter AS, hence

$$\angle(AB, AS) = \angle(AW, AS) = \angle(VW, VS).$$

Now, VW is perpendicular to AM (Theorem 1) and VS is perpendicular to AC, hence $\angle(VW, VS) = \angle(AM, AC)$ and therefore $\angle(AB, AS) = \angle(AM, AC)$. The result follows.

Exercise 2 Let AS intersect BC at Q. Show that $\angle(QA, QC) = \angle(BA, BA_1)$.

Exercise 3 Prove that the midpoint of AS lies on the circumcircle of ΔBOC. (Hint: see **[1]** p. 149)

An important consequence of Theorem 2 is the following characterization of S.

Theorem 3 S is the harmonic conjugate of A with respect to points B and C on the circle Γ.

Recall that this means that for some P on Γ, the lines PA and PS are harmonic conjugates with respect to PB, PC (and then this holds for any point P of Γ).

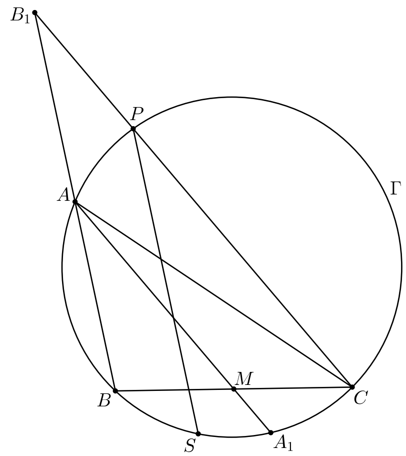


Figure 3

Let the parallel to AB through S intersect Γ at P with $P \neq S$ (Figure 3). Using the concyclicity of A, C, P, S, the parallelism of AB and PS, and Theorem 2 in succession we obtain

$$\angle(CP, CA) = \angle(SP, SA) = \angle(AB, AS) = \angle(AM, AC)$$

and so CP is parallel to the median AM. Let lines CP and AB intersect at B_1. Since M is the midpoint of BC and MA is parallel to CP, A is the midpoint of BB_1. Since PS is parallel to BB_1, it follows that PS is the harmonic conjugate of PA with respect to the lines PB and $PB_1 = PC$, as desired.

Exercise 4 Use Theorem 3 to prove that the line AS passes through the pole of BC with respect to Γ (see [**1**] p. 146)

Our next theorem shows that S lies on another important circle.

Theorem 4 S is the only point of $\Gamma - \{A\}$ such that $\dfrac{SB}{SC} = \dfrac{AB}{AC}$.

When $AB \neq AC$, the locus of all points N such that $\frac{NB}{NC} = \frac{AB}{AC}$ is a circle $\mathcal{C}_A$ (Apollonius' circle associated with the vertex A). Thus, a corollary of Theorem 4 is that $\mathcal{C}_A$ and Γ intersect at A and S. Note that a diameter of $\mathcal{C}_A$ is DD' where D and D' are the feet on BC of the bisectors of $\angle BAC$.

It suffices to prove that $\frac{SB}{SC} = \frac{AB}{AC}$. Since

$$\angle(AB, AS) = \angle(AM, AC) \quad \text{and} \quad \angle(SB, SA) = \angle(CB, CA) = \angle(CM, CA),$$

the triangles ABS and AMC are similar. It follows that $\frac{SB}{MC} = \frac{AB}{AM}$. Similarly, we have $\frac{SC}{MB} = \frac{AC}{AM}$ and the result is obtained by expressing that $MB = MC$.

Exercise 5 What is the symmedian of ΔSBC through S?

Exercise 6 Let $BC = a, CA = b, AB = c$ and $m_a = AM$. Prove that

$$SA = \frac{bc}{m_a}, \quad SB = \frac{ac}{2m_a}, \quad SC = \frac{ab}{2m_a}, \quad SM = \frac{a^2}{4m_a}.$$

In our last theorem, we return to the Simson line of S.

Theorem 5 S is the only point of Γ whose projections U, V, W onto the sidelines BC, CA, AB, respectively, are such that U is the midpoint of VW.

Let S' be a point of Γ, with $S' \neq A$ and let U', V', W' be its respective projections onto BC, CA, AB. Since the triangle $U'W'B$ is inscribed in the circle with diameter BS', the Law of Sines gives $U'W' = BS' \cdot \sin B$. Similarly, $U'V' = CS' \cdot \sin C$, hence

$$U'V' = U'W' \;\Leftrightarrow\; \frac{S'B}{S'C} = \frac{\sin C}{\sin B} \;\Leftrightarrow\; \frac{S'B}{S'C} = \frac{AB}{AC}$$

and therefore $U'V' = U'W'$ is equivalent to $S' = S$.

We conclude with two problems.

Problem 1 [Two more circles through S]

Let A' be the reflection of A in BC and let γ_B (resp. γ_C) be the circle passing through A' and tangent to BC at B (resp. at C). Prove that S is on γ_B and γ_C.

Problem 2 [About S and its analogues]

Define $S_a = S$ and let S_b and S_c be constructed from B and C, respectively, as $S = S_a$ is from vertex A. Let m_a, m_b, and m_c denote the lengths of the medians from A, B, and C, respectively. Let K be the symmedian point, R the circumradius, and r the inradius of ΔABC. Prove that

$$\text{(a)} \qquad m_a^2\,\overrightarrow{KS_a} + m_b^2\,\overrightarrow{KS_b} + m_c^2\,\overrightarrow{KS_c} = \overrightarrow{0}$$

$$\text{(b)} \qquad [ABC] \cdot [S_aS_bS_c] \leq \frac{27(rR)^2}{4},$$

where $[\cdot]$ denotes area.

References

[1] Bataille M., Characterizing a Symmedian, *Crux Mathematicorum*, Vol. 43 (4).

[2] Honsberger R., *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, MAA, 1995

[3] Sortais Y. et R., *La géometrie du triangle*, Hermann, 1987

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **August 15, 2021**.

4651. *Proposed by Michel Bataille.*

The complex numbers z_1 and z_2 represent points on or inside the unit circle of the Euclidean plane such that both $\mathrm{Re}(z_1 + z_2) \geq 1$ and $\mathrm{Im}(z_1 + z_2) \geq 1$. Find the extremal values of $\mathrm{Re}(z_1 z_2)$ and the pairs (z_1, z_2) at which they are attained.

4652. *Proposed by Nguyen Viet Hung.*

Let ABC be an equilateral triangle with centroid O and let M be any point inside of the triangle. D, E, F are feet of altitudes from M onto the sides BC, CA, AB respectively. Prove that

$$(MD - ME)^4 + (ME - MF)^4 + (MF - MD)^4 = \frac{81}{8} MO^4.$$

4653. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r and circumradius R. It is known (e.g. Item 2.48 on page 31 of *"Geometric Inequalities"* by Bottema et al.) that

$$\sec^2 \frac{A}{2} + \sec^2 \frac{B}{2} + \sec^2 \frac{C}{2} \geq 4.$$

Prove that

$$\sec^2 \frac{A}{2} + \sec^2 \frac{B}{2} + \sec^2 \frac{C}{2} \leq \frac{2R}{r}.$$

4654. *Proposed by Andrei Eckstein and Leonard Giugiuc.*

Consider positive real numbers $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

where $n \geq 3$. Prove that

$$\sum_{i<j} a_i a_j \geq \frac{n(n-1)}{2}.$$

4655. *Proposed by Daniel Brin.*

Let $A = (a_{ij})$ be a matrix of order n where $n > 1$ is odd. Let $C = (-1)^{i+j}M_{ij}$ denote the cofactor matrix of A where M_{ij} are the minors of A. If X is an $n \times n$ matrix such that $XMX = C$, find the sum of all the entries of X.

4656. *Proposed by Abdollah Zohrabi.*

If a, b, c and d are positive real numbers such that $abcd = 1$, prove that

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq 2(ab + cd)(bd + ac)(cb + da).$$

4657. *Proposed by George Stoica.*

Let us consider the equation $f(x) + f(2x) = 0$, $x \in \mathbb{R}$.

(i) Prove that, if f is continuous at 0, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Construct a function f, discontinuous at every $x \in \mathbb{R}$, that solves the given equation.

4658. *Proposed by Mihaela Berindeanu.*

In the right triangle ABC, let D be the foot of the altitude on the hypotenuse BC, and let I_1 and I_2 be the incenters of triangles ABD and ADC, respectively. Prove that the line I_1I_2 meets AB at a point on the circle BDI_1.

4659. *Proposed by Tien Nguyen.*

For each positive integer n, find $\gcd(a_n, b_n)$ such that

$$(4 + \sqrt{5})^n = a_n + b_n\sqrt{5},$$

where a_n and b_n are positive integers.

4660. *Proposed by Thanh Tung Vu, modified by the Editorial Board.*

a) Given a triangle ABC with its orthocenter H, define the three circles

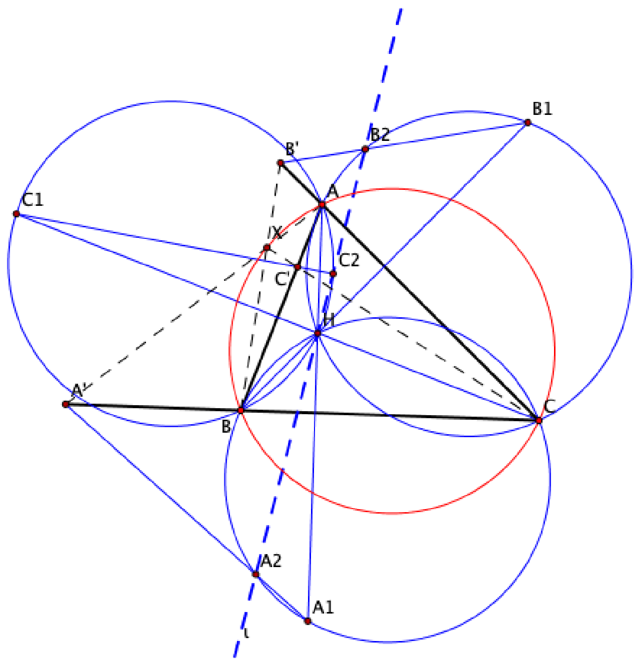
$$\alpha = (HBC), \quad \beta = (HCA), \quad \text{and} \quad \gamma = (HAB).$$

For a fixed line ℓ through H let
A_1 and A_2 be the points where α again meets ℓ and AH,
B_1 and B_2 be the points where β again meets ℓ and BH,
C_1 and C_2 be the points where γ again meets ℓ and CH.

Finally, define

$$A' = BC \cap A_1A_2, \quad B' = CA \cap B_1B_2, \quad C' = AB \cap C_1C_2.$$

Prove that the cevians AA', BB', CC' are concurrent at some point X of the circumcircle of ΔABC.



b)* Establish the corresponding result with the orthocenter H replaced by an arbitrary point P not on a side of ΔABC; prove that the locus of resulting point X as ℓ turns about P is an ellipse that circumscribes ΔABC.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 août 2021**. *La rédaction souhaite remercier Frédéric Morneau-Guérin, professeur à l'Université TÉLUQ, d'avoir traduit les problèmes.*

4651. *Soumis par Michel Bataille.*

Soient z_1 et z_2 des nombres complexes situés sur ou à l'intérieur du cercle unité du plan complexe et tels que $\mathrm{Re}(z_1 + z_2) \geq 1$ et $\mathrm{Im}(z_1 + z_2) \geq 1$. Trouvez les valeurs extrêmes de $\mathrm{Re}(z_1 z_2)$ ainsi que les paires (z_1, z_2) pour lesquelles celles-ci sont atteintes.

4652. *Soumis par Nguyen Viet Hung.*

Soit ABC un triangle équilatéral de centre de gravité O. Soit encore M un point quelconque situé à l'intérieur du triangle. On désigne par D, E, F les pieds respectifs des droites perpendiculaires aux côtés BC, CA, AB et passant par le point M. Montrez que

$$(MD - ME)^4 + (ME - MF)^4 + (MF - MD)^4 = \frac{81}{8} MO^4.$$

4653. *Soumis par George Apostolopoulos.*

Soit ABC un triangle dont le rayon du circle inscrit est r et celui du cercle circonscrit est R. Il est établi (voir par exemple Item 2.48 à la page 31 de *"Geometric Inequalities"* de Bottema) que

$$\sec^2 \frac{A}{2} + \sec^2 \frac{B}{2} + \sec^2 \frac{C}{2} \geq 4.$$

Montrez que

$$\sec^2 \frac{A}{2} + \sec^2 \frac{B}{2} + \sec^2 \frac{C}{2} \leq \frac{2R}{r}.$$

4654. *Soumis par Andrei Eckstein et Leonard Giugiuc.*

Considérons des nombres réels positifs $a_1, a_2, \ldots, a_n$ tels que

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

où $n \geq 3$. Montrez que

$$\sum_{i<j} a_i a_j \geq \frac{n(n-1)}{2}.$$

4655. *Soumis par Daniel Brin.*

Soit $A = (a_{ij})$ une matrice d'ordre n où $n > 1$ est impair. Soit $C = (-1)^{i+j} M_{ij}$ la comatrice (ou matrice des cofacteurs) de A, où les M_{ij} sont les mineurs de A. Étant donné X une matrice $n \times n$ vérifiant $XMX = C$, trouvez la somme de toutes les composantes de X.

4656. *Soumis par Abdollah Zohrabi.*

Si a, b, c et d désignent des nombres réels positifs tels que $abcd = 1$, montrez que

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq 2(ab + cd)(bd + ac)(cb + da).$$

4657. *Soumis par George Stoica.*

Considérons l'équation $f(x) + f(2x) = 0$, où $x \in \mathbb{R}$.

(i) Montrez que si f est continue en 0 alors $f(x) = 0$ pour tout $x \in \mathbb{R}$.

(ii) Construisez une fonction f qui est discontinue en tout point $x \in \mathbb{R}$ et qui est une solution de l'équation ci-dessus.

4658. *Soumis par Mihaela Berindeanu.*

Considérons un triangle rectangle ABC. Soit D le pied de la hauteur projetée sur l'hypothénuse BC. Notons respectivement par I_1 et I_2 les centres des cercles inscrits aux triangles ABD et ADC. Montrez que la droite I_1I_2 rencontre AB en un point du cercle BDI_1.

4659. *Soumis par Tien Nguyen.*

Pour tout entier positif n, trouvez PGCD(a_n, b_n) tel que

$$(4 + \sqrt{5})^n = a_n + b_n\sqrt{5},$$

oú a_n et b_n sont entiers positifs.

4660. *Soumis par Thanh Tung Vu puis modifié par le comité de rédaction.*

a) Étant donné un triangle ABC d'orthocentre H, définissons les trois cercles suivant :

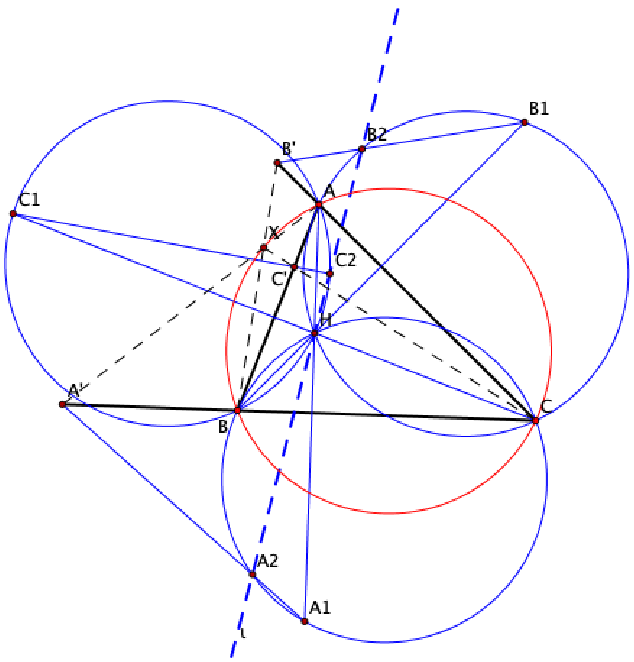
$$\alpha = (HBC), \quad \beta = (HCA), \quad \text{et} \quad \gamma = (HAB).$$

Fixons une droite ℓ passant par le point H et considérons
A_1 et A_2 les points où α rencontre à nouveau ℓ et AH,
B_1 et B_2 les points où β rencontre à nouveau ℓ et BH,
C_1 et C_2 les points où γ rencontre à nouveau ℓ et CH.

Enfin, définissons

$$A' = BC \cap A_1A_2, \quad B' = CA \cap B_1B_2, \quad C' = AB \cap C_1C_2.$$

Montrez que les céviennes AA', BB', CC' sont concourantes en un point X du cercle circonscrit à ΔABC.



b)* Tâchez d'établir le résultat correspondant où, cette fois, l'orthocentre H est remplacé par un point arbitraire P n'étant pas situé sur l'un des côtés de ΔABC. Montrez que le lieu des points X obtenus lorsque ℓ pivote autour de P est une ellipse circonscrivant ΔABC.

BONUS PROBLEMS

*These problems appear as a bonus. Their solutions will **not** be considered for publication.*

76. *Proposed by Michel Bataille.*

Let ABC be a triangle, M the midpoint of BC and Γ_b, Γ_c the circumcircles of $\Delta AMB, \Delta AMC$, respectively. Let t be the tangent to Γ_c at the point N diametrically opposite to M. If the lines MA, MC intersect t at A', C', respectively, prove that the tangent to Γ_b at M bisects $A'C'$.

77. *Proposed by Nguyen Viet Hung.*

Given a positive integer k. Evaluate

$$\lim_{n\to\infty}\left(\frac{1^k}{n^{k+1}+1}+\frac{2^k}{n^{k+1}+2}+\cdots+\frac{n^k}{n^{k+1}+n}\right).$$

78. *Proposed by George Stoica.*

Let $S_n = \sum_{i=[n/2]+1}^{n} a_i$, where $[\ ]$ denote the integer part. If $\lim\limits_{n\to\infty} S_n$ exists, must $\lim\limits_{n\to\infty} a_i$ equal to zero?

79. *Proposed by Leonard Giugiuc.*

Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $n \geq 5$. Prove that

$$(n-2)(a_1^2+a_2^2+\cdots+a_n^2+1)+2a_1a_2\cdots a_n \geq 2\sum_{i<j} a_ia_j.$$

80. *Proposed by George Apostolopoulos.*

Let a, b and c be positive real numbers with $a^2+b^2+c^2=12$. Prove that

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right) \geq 12.$$

81. *Proposed by Minh Ha Nguyen.*

Let ABC be a triangle with $BC = a, CA = b$ and $AB = c$, where m_a, m_b and m_c are the lengths of medians from A, B and C, respectively. Prove that

$$m_a + m_b + m_c = \frac{\sqrt{3}}{2}(a+b+c),$$

if and only if one of $m_a = \frac{\sqrt{3}}{2}a$, $m_b = \frac{\sqrt{3}}{2}b$ or $m_b = \frac{\sqrt{3}}{2}b$ holds.

82. *Proposed by Leonard Giugiuc.*

If a, b, c and d are nonnegative real numbers such that $ab+bc+cd+da>0$, then prove that

$$\frac{a^5+b^5+c^5+d^5}{\sqrt{ab+bc+cd+da}}+6abcd \geq \frac{(ab+bc+cd+da)^2}{2}.$$

83. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r and circumradius R. Equilateral triangles with sides AB, BC and CA are drawn externally to triangle ABC. Let K, L and M be the centroids of the equilateral triangles. Prove that $2r \leq R' \leq R$, where R' denotes the circumradius of the triangle KLM.

84. *Proposed by George Apostolopoulos.*

Let h_a, h_b and h_c be the altitudes, r_a, r_b and r_c the exradii, r the inradius and R the circumradius of a triangle ABC. Prove that

$$\frac{r_a+r_b}{\sqrt{h_a^2+h_b^2}}+\frac{r_b+r_c}{\sqrt{h_b^2+h_c^2}}+\frac{r_c+r_a}{\sqrt{h_c^2+h_a^2}} \leq 3\sqrt{2}\left(\frac{R}{2r}\right)^2.$$

85. *Proposed by Nguyen Viet Hung.*

Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that

$$\left(\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}}\right)^2+\frac{4abc}{(a+b)(b+c)(c+a)} \leq 2.$$

86. *Proposed by Daniel Sitaru.*

Let $x, y, z \in (0,1)$ with $xy+yz+zx=1$. Prove that

$$x^2y^2+y^2z^2+z^2x^2+4x^2y^2z^2 \geq \frac{13}{27}.$$

87. *Proposed by Robert Frontczak.*

Let F_n denote the nth Fibonacci number, defined by $F_n = F_{n-1}+F_{n-2}$ with $F_0=0, F_1=1$. For $n \geq 0$, let A_n be defined by $A_n = \sum_{k=0}^{n} \frac{C_k}{2^k}$, where C_n is the nth Catalan number, that is $C_n = \frac{1}{n+1}\binom{2n}{n}$. Prove that

$$\sum_{n=0}^{\infty} A_n \frac{F_n}{4^n} = \frac{8}{5}\sqrt{10}\left(\frac{a^3}{(a^3+2)(\sqrt{2}a+1)}-\frac{1}{(a+3)(\sqrt{2}+a)}\right),$$

where $a=(1+\sqrt{5})/2$ is the golden ratio.

88. *Proposed by Conar Goran.*

Let $x_1, \dots, x_n > 0$ be real numbers and $s = \sum_{i=1}^n x_i$. Prove that

$$\frac{1}{n}\sum_{i=1}^{n}\frac{x_i^{x_i}}{(1+x_i)^{x_i}} \geq \left(\frac{s}{n+s}\right)^{\frac{s}{n}}.$$

When does equality occur?

89. *Proposed by Lorian Saceanu and Marian Cucoanes.*

Let x, y, z be positive real numbers. Prove that:

$$\frac{x^3+y^3+z^3}{3xyz} + \left[\frac{8xyz}{(x+y)(y+z)(z+x)}\right]^3 \geq 2.$$

90. *Proposed by Michel Bataille.*

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x) \cdot (f(x))^3 = 1$ for all real x and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = 2$. Prove that the equation $f(x) = \frac{1}{2}$ has a unique solution. Assuming that $f'(0) \geq 0$, express this solution as a function of $f(0)$.

91. *Proposed by George Stoica.*

Let P be a polynomial whose coefficients are equal to ± 1. Prove that $\frac{1}{2} < |z_0| < 2$ for any root z_0 of P.

92. *Proposed by Michel Bataille.*

Let m, n be integers such that $2 \leq m < n$. Express

$$\sum_{j=1}^{n-1}\left\lfloor\frac{(2j+1)m+n}{2mn}\right\rfloor - \sum_{j=1}^{m-1}\left\lfloor\frac{(2j+1)n+m}{2mn}\right\rfloor$$

as a function of $\left\lfloor\frac{n}{m}\right\rfloor$, where $\lfloor\rfloor$ denotes the greatest integer function.

93. *Proposed by George Stoica.*

Let $a \geq 2$. Prove that if $f \neq 0$ is a continuous and periodic function, then there is x such that $f(x) + af(x+1) \neq 0$.

94. *Proposed by Daniel Sitaru.*

Find:

$$\Omega = \lim_{n\to\infty}\left(n \cdot \int_0^{\frac{\pi}{2}} \cos^n x dx \cdot \int_0^{\frac{\pi}{2}} \cos^{n+1} x dx\right)$$

95. *Proposed by Michel Bataille.*

In the plane, let ABC be a triangle with $AB \neq AC$ and let $\mathcal{S}$ be the set of all circles passing through B and C. If O is the centre of $\Gamma \in \mathcal{S}$, the point M of Γ is called the *trace* of Γ if A, O, M are collinear in this order. Given $\Gamma_1 \in \mathcal{S}$ with trace M_1, construct $\Gamma_2 \in \mathcal{S}$ with trace M_2 such that $M_2 \neq M_1$ and $M_1M_2 \perp BC$. Discuss the number of solutions.

96. *Proposed by George Stoica.*

Let $y_n \in (0,1)$ for all $n \geq 1$ be such that $\displaystyle\sum_{n=1}^{\infty} y_n = \infty$. Prove that there is a unique sequence $(a_n)_{n\geq 1}$ with $a_n > 0$ for all $n \geq 1$, $\displaystyle\sum_{n=1}^{\infty} a_n = 1$, and such that $a_n = y_n \cdot \displaystyle\sum_{k=n}^{\infty} a_k$ for all $n \geq 1$.

97. *Proposed by Chen Jiahao.*

Consider a triangle ABC with incenter I. Let circle with center J be tangent to the sides AC and AB at D and E, respectively. Prove the following statements.

a) If the circumcircles of EIB and DIC intersect at I and X, then X lies on (J).

b) The circumcircle of BXC is tangent to (J).

98. *Proposed by Mihaela Berindeanu.*

Let $ABCD$ be a square and M be a point on the side BC so that $MC = \dfrac{BC}{4}$. $\angle AMC$ bisector cuts DC in N, $P \in AM$, $NP \perp AM$. The middle points of NP and MN are X, respectively Y. Show that $\measuredangle(NAX) = \measuredangle(YAM)$.

99. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let $\gamma_n = -\ln n + \sum_{k=1}^{n} \dfrac{1}{k}$ with $\lim_{n\to\infty} \gamma_n = \gamma$, the Euler-Mascheroni constant.

a) Find $\lim_{n\to\infty}(\gamma_n - \gamma)n$.

b) Find $\lim_{n\to\infty}(\gamma_n\gamma_{n+1} - \gamma^2)n$.

100. *Proposed by Nguyen Viet Hung.*

Find $\lfloor\sqrt{n^2+1} + \sqrt{n^2+2} + \cdots + \sqrt{n^2+2n}\rfloor$, where $\lfloor x\rfloor$ denotes the greatest integer not exceeding x.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(1), p. 43–47.

4601. *Proposed by Bill Sands.*

One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Clothes do not overlap and each clothespin holds up one piece of clothing. You want to remove all the clothing from the line, obeying the following rules:

(i) you must remove the clothing in the order that they are hanging on the line;

(ii) all the pins holding up a piece of clothing must be removed at the same time;

(iii) the number of clothespins you remove each time must belong to the set $\{n+1, n+2, \dots, n+c\}$, where n and c are given positive integers.

Find the smallest positive integer c so that, for any positive integer n, all sufficiently long lines of clothing can be removed.

There were 3 solutions submitted, all correct. We present all the approaches.

Solution 1, by UCLan Cyprus Problem Solving Group.

The smallest value of c is 6. We first show that the conditions cannot be satisfied for $c = 5$ and thus for any smaller c. The choice set from which the number of pins must be selected is $\{n+1, n+2, \dots, n+5\}$.

Let $n = 6r+3$, so that the choice set is $\{6r+4, 6r+5, 6r+6, 6r+7, 6r+8\}$. Consider an arbitrarily long line with an odd number of items, each held by three pins. To clear the line, we have no choice but to remove a multiple of 3 pins each time, and the only option available to us is the even multiple $6(r+1)$. We must be left with items held by an odd multiple of three pins not exceeding n that cannot be cleared.

Before showing that $c = 6$ is suitable for the procedure, we make the important observation that for any three consecutive integers not exceeding the number of remaining pins on the line, at least one of them represents a possible number of pins that can be removed in accordance with condition (ii).

Let n be a positive integer and the choice set be $\{n+1, \dots, n+6\}$. Suppose that there are $v(n+3)+w$ pins on the line where $v \geq n+3$ and $0 \leq w \leq n+2$. We begin by removing either $n+4$, $n+5$ or $n+6$ pins, and continue clearing the line using one of these three choices. After doing this k times we have removed $k(n+3)+w_k$ pins where $1 \leq w_k - w_{k-1} \leq 3$ for $k \geq 1$. Since $w_0 = 0$ and $w_{n+2} \geq n+2$, there must be a number $a \leq n+2$ for which $w-1 \leq w_a \leq w+1$. Thus, the first step is to remove $a(n+3)+w_a$ pins where $a \leq n+2$ and $w-1 \leq w_a \leq w+1$.

We now increase k from $a+1$ to $v-1$, with at each stage a total of $k(n+3)+w_k$ pins having been removed while ensuring that $w-1 \leq w_k \leq w+1$. Suppose that $w_a = w-1$. Then if possible remove $n+3$, $n+4$ or $n+5$ pins so that a total of $(a+1)(n+3)+w_{a+1}$ pins have been removed altogether, where $w-1 \leq w_{a+1} \leq w+1$. If $w_a = w$, we remove either $n+2$, $n+3$ or $n+4$ pins, while if $w_a = w+1$, we can remove either $n+1$, $n+2$ or $n+3$ pins to the same effect. Repeat this process as long as possible until we end up with removing $(v-1)(n+3)+u$ pins where $w-1 \leq u \leq w+1$, leaving $n+3+w-u$ pins. This is equal to one of $n+2$, $n+3$ and $n+4$, so that one further removal takes away all the pins.

Thus we can remove all items from a line with at least $(n+3)^2$ pins.

Solution 2, by the proposer and Sergey Sadov, done independently.

To show that c cannot be less than 6, let $n \geq 3$ and $c = 5$. Let the finite sequence

$$S = \{3, 1, 1, \ldots, 1, 3; 3, 1, 1, \ldots, 1, 3; \ldots; 3, 1, \ldots, 1, 3; 3\}$$

denote the number of pins in the items in order on the line, where there are m blocks $\{3, 1, 1, \cdots, 1, 3\}$ of $n-1$ integers consisting of two threes separated by $n-3$ ones, these blocks followed by a single 3.

Since the sum of the first $n-2$ terms is n, the corresponding items of clothing cannot be removed. Since the first n or more terms add up to at least $n+6$, we have no choice but to clear the first $n-1$ items by removing $n+3$ pins. Then we must start afresh with the next block and clear the line block by block until there is single item secured by three pins. Thus, there are arbitrarily long lines of items that cannot be cleared.

We now show when $c = 6$, sufficiently long lines of clothing can be removed. Fix n. Call a positive integer m removable if any line with m pins can be cleared. The strategy is to construct a sequence $\{S_k\}$ of blocks, each with $k+5$ integers, starting with $S_1 = \{n+1, n+2, \ldots, n+6\}$ such that each integer in S_k can be reduced to an integer in S_{k-1} by subtracting one of the numbers in S_1. Eventually, the blocks will overlap and together include all the integers from some point on.

Let $S_2 = \{2n+4, 2n+5, \ldots, 2n+10\}$, $S_3 = \{3n+7, 3n+8, \ldots, 3n+14\}$, and, generally for $k \geq 2$,

$$S_k = \{k(n+3)-2, k(n+3)-1, \ldots, k(n+4)+1, k(n+4)+2\}.$$

As in solution 1, we can clear a succession of items from the line by taking away at least one of $\{n+1, n+2, n+3\}$ pins and also by taking away at least one of $\{n+4, n+5, n+6\}$ pins. We follow a two-pronged process to ensure that for large k, there is no gap between S_k and S_{k+1}.

Suppose that we have a line with m pins where $m \in S_k$, $k \geq 2$. If

$$k(n+3)-2 \leq m \leq k(n+3)+(k-1),$$

we can remove one of $n+1$, $n+2$ or $n+3$ to shorten the line, with a number of pins lying between

$$k(n+3)-2-(n+3)=(k-1)(n+3)-2$$

and

$$k(n+3)+(k-1)-(n+1)=(k-1)(n+4)+2$$

inclusive, *i.e.* a number in S_{k-1}.

If $k(n+3)+k \le m \le k(n+4)+2$, we can shorten the line by removing one of $n+4$, $n+5$, $n+6$ pins to obtain a number between

$$k(n+3)+k-(n+6)=(k-1)(n+3)+k-3>(k-1)(n+3)-2$$

and

$$k(n+4)+2-(n+4)=(k-1)(n+4)+2$$

inclusive, *i.e.* within S_{k-1}. We can continue on in this way until we get to a line with a number of pins in S_1 which can then be cleared. Thus, by induction, we see that every integer in each S_k is removable.

The blocks S_k and S_{k+1} will abut or overlap iff $k(n+4)+2 \ge [(k+1)(n+3)-2]-1$, which reduces to $k \ge n-2$. Since the smallest integer in S_{n-2} is

$$(n-2)(n+3)-2=n^2+n-8,$$

then

$$\bigcup_{k=n-2}^{\infty} S_k=[n^2+n-8,\infty).$$

Therefore, each line with at least n^2+n-8 pins can be cleared by taking away a number of pins in S_1 each time.

Two notes from the proposer.

(1) When $c=6$, we ask whether the number n^2+n-8 is a hard lower bound for the number of pins on a line that can always be cleared. For $3 \le n \le 8$, there are examples of allocations of pins to items of clothing that cannot be cleared following the rules where the total number of pins is n^2+n-9. For example, when $n=5$, the line with nine items held by 21 pins with pin number sequence $\{3,2,3,1,3,1,3,2,3\}$ cannot be cleared if we can remove only 6 to 11 pins each time. It is worth noting that, for each n from 3 to 8, there are maximal length nonclearable lines which are palindromes, as in the above example for $n=5$.

(2) In 2017, a problem was posed on the Alberta High School Mathematics Competition, Part II, that treated the special case $(n,c)=(1,3)$. The candidates were asked to find all finite sequences $\{a_k\}$ where the kth item has $a_k \in \{1,2,3\}$ pins for which the line can be cleared. The competition can be found on the website.

4602. *Proposed by Nguyen Viet Hung.*

Let ABC be an acute triangle. Prove that

$$\frac{h_b h_c}{a^2}+\frac{h_c h_a}{b^2}+\frac{h_a h_b}{c^2}=\frac{r}{2R}+\frac{2h_a h_b h_c}{w_a w_b w_c}.$$

We received 27 submissions, all of which are correct. We present the solution by Marie-Nicole Gras.

The following identities are all well-known:

$$\begin{aligned} &h_a=b\sin C=c\sin B, && w_a=\frac{2bc}{b+c}\cos\frac{A}{2},\\ &r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}, && \sin A=\frac{a}{2R},\\ &abc=4rsR=2(a+b+c)rR. \end{aligned}$$

Using these formulas together with similar ones obtained by permutations, we have

$$\frac{h_b h_c}{a^2}+\frac{h_c h_a}{b^2}+\frac{h_a h_b}{c^2}=\sin B\sin C+\sin C\sin A+\sin A\sin B=\frac{bc+ca+ab}{4R^2}, \quad (1)$$

and

$$\begin{aligned}\frac{2h_a h_b h_c}{w_a w_b w_c}&=\frac{2abc(b+c)(c+a)(a+b)\sin A\sin B\sin C}{8a^2b^2c^2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\\ &=\frac{16(b+c)(c+a)(a+b)\sin\frac{A}{2}\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{B}{2}\sin\frac{C}{2}\cos\frac{C}{2}}{8abc\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}\\ &=\frac{(b+c)(c+a)(a+b)}{abc}\,\frac{r}{2R}. \qquad (2)\end{aligned}$$

By the identity

$$(a+b)(b+c)(c+a)=(a+b+c)(bc+ca+ab)-abc$$

and the formula that $abc=4rsR$, we then obtain

$$\begin{aligned}\frac{2h_a h_b h_c}{w_a w_b w_c}&=\frac{(a+b+c)(bc+ca+ab)-abc}{abc}\,\frac{r}{2R}\\ &=\frac{2(a+b+c)(bc+ca+ab)rR}{4abcR^2}-\frac{r}{2R}\\ &=\frac{bc+ca+ab}{4R^2}-\frac{r}{2R}. \qquad (3)\end{aligned}$$

From (1) and (3) the result follows.

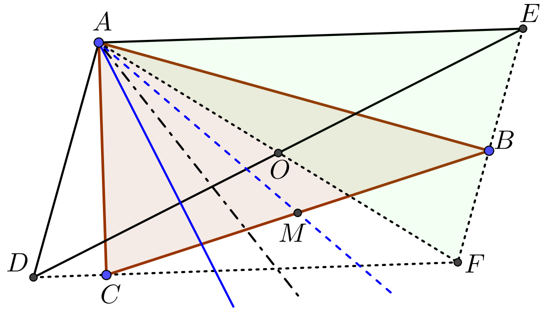
Remark. The condition of $\triangle ABC$ being acute is redundant.

4603. *Proposed by Michel Bataille.*

Let ABC be a triangle. The perpendiculars to AB through A and to AC through C intersect at D. The perpendiculars to AC through A and to AB through B intersect at E. Prove that the altitude from A in ΔDAE is a symmedian of ΔABC.

We received 15 solutions, all correct. The solvers used a variety of analytic methods involving cartesian and barycentric coordinates. Some of the solutions were pure geometric including the following by Sergey Sadov.

Let $F = EB \cap DC$ be the fourth vertex of the parallelogram $EADF$. Denote by O the center of $EADF$.



We have $\angle ADC = \angle AEB$, hence $\triangle ADC \sim \triangle AEB$, hence

$$\frac{AC}{AB} = \frac{AD}{AE} = \frac{EF}{AE}. \tag{1}$$

Now, $\angle BAC = \angle EAC - \angle EAB = 90^\circ - \angle EAB = \angle AEB$. Taking (1) into account, we find that $\triangle CAB \sim \triangle FEA$.

Note that EO is the median of $\triangle FEA$ corresponding to the median AM in the similar triangle CAB. Hence $\angle BAM = \angle AEO$.

The assertion that the A-altitude in $\triangle ADE$ is a symmedian of $\triangle ABC$ is equivalent to the equality of the angles

$$90^\circ - \angle ADE = \angle EAM. \tag{2}$$

To prove it, write, using the above,

$$\angle EAM = \angle EAB + \angle BAM = 90^\circ - \angle AEB + \angle AEO.$$

Finally, since

$$\angle AEB - \angle AEO = \angle BEO = \angle ADE,$$

we obtain (2), as required.

Editor's note. Author of the problem Michel Bataille observed that the result can be used to construct the symmedians of ΔABC only with a set square and ruler.

4604. *Proposed by Nguyen Viet Hung.*

Prove that the triangle ABC is equilateral if and only if

$$a\sin(A-\frac{\pi}{3})+b\sin(B-\frac{\pi}{3})+c\sin(C-\frac{\pi}{3})=0.$$

We received 27 submissions, of which 2 were incomplete. We present the solution by UCLan Cyprus Problem Solving Group.

If the triangle is equilateral then the identity obviously holds. So assume now that the identity holds. Since

$$\begin{aligned}\sin\left(x-\frac{\pi}{3}\right)&=\sin(x)\cos\left(\frac{\pi}{3}\right)+\cos(x)\sin\left(\frac{\pi}{3}\right)\\&=\frac{\sin(x)+\sqrt{3}\cos(x)}{2},\end{aligned}$$

then we have that

$$a\sin(A)+b\sin(B)+c\sin(C)=\sqrt{3}\left(a\cos(A)+b\cos(B)+c\cos(C)\right).$$

Using $a=2R\sin(A),b=2R\sin(B),c=2R\sin(C)$, we get

$$\sin^2(A)+\sin^2(B)+\sin^2(C)=\frac{\sqrt{3}}{2}\left(\sin(2A)+\sin(2B)+\sin(2C)\right).$$

We now see that

$$\begin{aligned}\sin(2A)+\sin(2B)+\sin(2C)&=2\sin(A+B)\cos(A-B)+\sin(2C)\\&=2\sin(C)\left(\cos(A-B)+\cos(C)\right)\\&=2\sin(C)\left(\cos(A-B)-\cos(A+B)\right)\\&=4\sin(A)\sin(B)\sin(C)\\&=\frac{abc}{2R^3}\\&=\frac{2\Delta}{R^2}\end{aligned}$$

where R is the circumradius and Δ is the area of the triangle ABC.

We also have

$$\sin^2(A)+\sin^2(B)+\sin^2(C)=\frac{a^2+b^2+c^2}{4R^2}$$

from which we deduce that

$$a^2+b^2+c^2=4\sqrt{3}\Delta\,.$$

However by Weitzenböck's inequality we have $a^2+b^2+c^2\geqslant 4\sqrt{3}\Delta$ with equality if and only if the triangle is equilateral.

4605. *Proposed by George Stoica.*

Let $\{x_i\}_{i=1}^m$ be any set of non-zero vectors in $\mathbb{R}^n$. Prove the following:

(1) If $\langle x_i, x_j\rangle < 0$ for all $i \neq j$, then $m \leq n+1$.

(2) If $\langle x_i, x_j\rangle \leq 0$ for all $i \neq j$, then $m \leq 2n$.

We received 5 submissions and 4 of them were complete and correct. We present the following 2 solutions.

Solution 1, by Michel Bataille, Sergey Sadov, and the proposer (independently), slightly modified by the editor.

(1) We proceed by inducting on the dimension n of the space.

For $n = 2$, if there are 4 vectors in $\mathbb{R}^2$, then by the pigeonhole principle, the angle formed by 2 of them will be at most $\frac{\pi}{2}$, and thus their inner product will be non-negative. On the other hand, it is easy to construct three vectors in $\mathbb{R}^2$ such that the pairwise inner product is negative, for example $(1, 0)$, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Assuming the result is true for n, now consider the case of $n+1$. Suppose $\{x_i\}_{i=1}^m$ is a set of non-zero vectors in $\mathbb{R}^{n+1}$, such that $\langle x_i, x_j\rangle < 0$ for all $i \neq j$. Without loss of generality, we can assume that x_1 has unit norm. Let P be the orthogonal projection onto $\operatorname{span}\{x_1\}$. Then $Px = \langle x, x_1\rangle x_1$ for all $x \in \mathbb{R}^{n+1}$. Note that the set of vectors $\{(I-P)x_i\}_{i=2}^m$ lies on the hyperplane $x_1^\perp$ and, for any $i \neq j$, we have

$$\langle (I-P)x_i, (I-P)x_j\rangle = \langle x_i, x_j\rangle - \langle Px_i, Px_j\rangle.$$

Since

$$\langle Px_i, Px_j\rangle = \langle \langle x_i, x_1\rangle x_1, \langle x_j, x_1\rangle x_1\rangle = \langle x_i, x_1\rangle\langle x_j, x_1\rangle > 0,$$

it follows that

$$\langle (I-P)x_i, (I-P)x_j\rangle < 0 \text{ for all } i \neq j.$$

By the induction hypothesis we must have $m - 1 \leq n + 1$. So $m \leq n + 2$.

(2) The proof is similar. First observe that for $n = 2$, the largest set of non-zero vectors with non-positive inner products is 4, and one such example is $(1, 1)$, $(1, -1)$, $(-1, -1)$ and $(-1, 1)$. Repeating the proof as in (1) and noting that the set $\{(I-P)x_i\}_{i=2}^m$ contains at most one zero vector (otherwise there are x_i and x_j with $1 < i < j$ such that x_1, x_i, x_j are all in $\operatorname{span}\{x_1\}$, and the inner product between two of them would be positive), we get the desired claim.

From the proof we can deduce the following stronger statement: if $\{x_i\}_{i=1}^{2n}$ is a set of non-zero vectors in $\mathbb{R}^n$, such that $\langle x_i, x_j\rangle \leq 0$ for all $i \neq j$, then

$$\{x_i\}_{i=1}^{2n} = \{e_1, e_2, \ldots, e_n\} \cup \{c_1e_1, c_2e_2, \ldots, c_ne_n\},$$

where $\{e_1, e_2, \ldots, e_n\}$ is an orthogonal basis of $\mathbb{R}^n$, and $c_1, c_2, \ldots, c_n$ are negative scalars.

Solution 2, by Aart Blokhuis, and UCLan Cyprus Problem Solving Group (independently), slightly modified by the editor.

We proceed by complete induction on the dimension n of $\langle x_1, \ldots, x_m \rangle$ and show that $m \leqslant n+1$ in (1) and $m \leqslant 2n$ in (2). In fact we will use induction only for (2) but we will essentially prove both (1) and (2) at the same time.

If $n = 1$ the result is easy as no two x_i's can have the same sign. Assume now that $n > 1$. We may also assume that $m > n$ as otherwise the claim is immediate. Then $x_1, \ldots, x_m$ is linearly dependent, and we can pick a minimal non-empty subset I of $\{1, 2, \ldots, m\}$ such that the set $\{x_i : i \in I\}$ is linearly dependent. Note that $|I| \geqslant 2$ and that $\dim span\{x_i : i \in I\} = |I| - 1$.

Since $\{x_i : i \in I\}$ is a minimal linearly dependent set, we can find nonzero reals $\lambda_i, i \in I$, such that

$$\sum_{i \in I} \lambda_i x_i = 0\,. \tag{1}$$

We claim that all of the λ_i's have the same sign. If this is not the case then we can find disjoint non-empty sets I_1, I_2 with $I_1 \cup I_2 = I$, $\lambda_i > 0$ for each $i \in I_1$, and $\lambda_j < 0$ for each $j \in I_2$.

Let $v = \sum_{i \in I_1} \lambda_i x_i$. By the given assumptions and equation (1), we have

$$\begin{aligned} 0 \leqslant \langle v, v \rangle &= \left\langle \sum_{i \in I_1} \lambda_i x_i, \sum_{j \in I_2} (-\lambda_j) x_j \right\rangle \\ &= \sum_{i \in I_i} \sum_{j \in I_2} \lambda_i (-\lambda_j) \langle x_i, x_j \rangle \leqslant 0\,. \end{aligned}$$

It follows that $v = 0$, which contradicts the minimality of the set I.

So we may assume that all the λ_i's are positive. For each $j \notin I$, we have

$$0 = \langle x_j, 0 \rangle = \left\langle x_j, \sum_{i \in I} \lambda_i x_i \right\rangle = \sum_{i \in I} \lambda_i \langle x_j, x_i \rangle \leqslant 0\,.$$

In case (1), this is impossible as actually the last inequality becomes strict. Therefore $I = \{1, 2, \ldots, m\}$ and $n = |I| - 1 = m - 1$ as required.

In case (2), we have that $\langle x_j, x_i \rangle = 0$ for every $j \notin I$. So each such x_j belongs to the orthogonal complement of $span\{x_i : i \in I\}$, which has dimension $n-(|I|-1)$. Since $|I| \geqslant 2$, then $n-(|I|-1) \leqslant n-1$, so by inductive hypothesis, $m-|I| \leqslant 2[n-(|I|-1)]$. Thus $m \leqslant 2n + 2 - |I| \leqslant 2n$.

Editor's Comment. UCLan Cyprus Problem Solving Group pointed out that a generalization of both (1) and (2) appears as Lemma 1 of Chapter 10 in the book B. Bollobás, *Combinatorics*, Cambridge University Press, 1986.

4606. *Proposed by Garcia Antonio.*

For $a, b, c, n > 0$, show that

$$(a+b)\sqrt{\frac{na+b}{a+nb}}+(b+c)\sqrt{\frac{nb+c}{b+nc}}+(c+a)\sqrt{\frac{nc+a}{c+na}} \geq 2(a+b+c).$$

We received 13 submissions, of which 10 were correct and complete. We present the solution by Theo Koupelis.

Note that $(n+1)(a+b) = (na+b)+(a+nb) \geq 2\sqrt{(na+b)(a+nb)}$, with equality when $(n-1)(a-b) = 0$. Therefore

$$(a+b)\sqrt{\frac{na+b}{a+nb}} = \frac{(a+b)(na+b)}{\sqrt{(na+b)(a+nb)}} \geq \frac{2(na+b)}{n+1}.$$

Similarly

$$(b+c)\sqrt{\frac{nb+c}{b+nc}} \geq \frac{2(nb+c)}{n+1}, \qquad \text{and} \qquad (c+a)\sqrt{\frac{nc+a}{c+na}} \geq \frac{2(nc+a)}{n+1}.$$

Adding these three expressions yields the desired result. Equality holds when $n = 1$ or $a = b = c$.

4607. *Proposed by Ted Barbeau.*

a) Determine all polynomials $q(x)$ that satisfy the functional equation

$$q(x)q(x+1) = q(x^2+x).$$

b) Determine all polynomials $p(x)$ that satisfy the functional equation

$$p(x)p(x+1) = p(x+p(x)).$$

c) $\star$ Prove or disprove the conjecture: Let $p(x)$ be a polynomial solution of the functional equation in (b). Then, if $q(x)$ satisfies the functional equation

$$q(x)q(x+1) = q(x+p(x)),$$

then $q(x) = p(x)^n$ for some nonnegative integer n.

We received 13 submissions and 12 of them were complete and correct. There are a few different approaches to solve the problem.

Solution 1 of (a), by Michel Bataille, Marie-Nicole Gras, and Sergey Sadov (done independently).

If $q(x)$ is a constant, then clearly it has to be either 0 or 1.

Next assume that $q(x)$ has degree $n \geq 1$. Write $q(x) = a_n x^n + r(x)$ where $a_n \neq 0$ and $r(x)$ is either 0 or a polynomial with degree $m < n$. The functional equation becomes

$$(a_n x^n + r(x))(a_n(x+1)^n + r(x+1)) = a_n x^n (x+1)^n + r(x(x+1)). \qquad (1)$$

On the left, the term with highest degree is $a_n^2 x^{2n}$, while on the right it is $a_n x^{2n}$. It follows that $a_n = 1$ and thus equation (1) can be simplified as

$$x^n r(x+1) + (x+1)^n r(x) = r(x(x+1)) - r(x)r(x+1).$$

If $r(x)$ is the zero polynomial, then it is easy to verify that $q(x) = x^n$ is a solution. If $r(x)$ is not the zero polynomial, then on the left the degree is $n + m$ while it is at most $2m$ on the right, contradicting our assumption that $m < n$.

Therefore, the solutions to the functional equation are $0, 1$, and x^n, where n is a positive integer.

Solution 2 of (a), by Roy Barbara, Cal Poly Pomona Problem Solving Group, Antonio Garcia, and UCLan Cyprus Problem Solving Group (done independently).

It is easy to check that constant solutions are 0 and 1.

Next assume that $q(x)$ has degree n. Let α be a root of $q(x)$ of maximum modulus. Then $q(\alpha^2 + \alpha) = q(\alpha)q(\alpha + 1) = 0$, so $\alpha^2 + \alpha$ is a root of $q(x)$. Similarly, $(\alpha - 1)^2 + (\alpha - 1) = \alpha^2 - \alpha$ is also a root of $q(x)$. By the triangle inequality we have

$$|\alpha^2 + \alpha| + |\alpha^2 - \alpha| \geqslant |\alpha + \alpha^2 - \alpha^2 + \alpha| = 2|\alpha|\,.$$

By the definition of α we also have $|\alpha| \geqslant |\alpha^2 + \alpha|$ and $|\alpha| \geqslant |\alpha^2 - \alpha|$. So we must have

$$|\alpha| = |\alpha^2 + \alpha| = |\alpha^2 - \alpha|\,.$$

If $\alpha \neq 0$ then we get $|\alpha + 1| = |\alpha - 1| = 1$. This says that the distance of α from 1 and -1 is equal to 1. This can only happen if $\alpha = 0$.

Therefore all roots of $q(x)$ are equal to 0 and thus $q(x) = Cx^n$ for some constant $C \neq 0$. Substituting in the original functional equation we get $C = 1$, and thus $q(x) = x^n$.

Solution of (b), by the majority of solvers.

If $p(x)$ is a constant, then clearly it has to be either 0 or 1.

Next assume that $p(x)$ has degree $n \geq 1$. Similar to the Solution 1 of (a), we can show that $n = 2$ and the leading coefficient of $p(x)$ is 1. Moreover, it is easy to verify that any monic quadratic $p(x)$ is a solution to the required functional equation. Therefore, the solutions to the functional equation are $0, 1$, and monic quadratic polynomials.

Several solvers pointed out (c) could be easily disproved, for example we can take $p(x) = x^2$ and $q(x) = x$. They also pointed out that (c) is true under extra *assumptions. We feature the solution by Navid Safaei, slightly modified by the editor.*

We first prove the following more general lemma.

Lemma. Let P, Q, R be non-constant polynomials such that the leading coefficients of P and Q have the same sign. Then, for each positive integer k, there is at most one monic polynomial f with degree n, such that $f(P(x)) \cdot f(Q(x)) = f(R(x))$.

Proof of the lemma. Suppose there are distinct monic polynomials f, g with degree n such that $f(P)f(Q) = f(R)$ and $g(P)g(Q) = g(R)$. Suppose $\deg P = a$ and $\deg Q = b$. It follows that $\deg R = a + b$.

Note that $f - g$ is a polynomial with degree $m < n$, and we have

$$f(R) - g(R) = f(P)f(Q) - g(P)g(Q) = f(P)(f(Q) - g(Q)) + g(Q)(f(P) - g(P)).$$

The degrees of $f(R) - g(R)$, $f(P)(f(Q) - g(Q))$, $g(Q)(f(P) - g(P))$ are $m(a+b)$, $na + mb$, $nb + ma$, respectively. Since the leading coefficients of P and Q have the same sign, it follows that the leading coefficients of $f(P)(f(Q) - g(Q))$ and $g(Q)(f(P) - g(P))$ also have the same sign. Thus the right-hand-side of the above equation has degree $\max\{na + mb, nb + ma\} > m(a + b)$, which is the degree of $f(R) - g(R)$, a contradiction. □

Recall the solutions to (b) are $0, 1$, and all monic quadratic polynomials. We consider the case that $p(x) = x^2 + (a - 1)x + b$, where a, b are constants. Then it suffices to solve the following functional equation:

$$q(x)q(x + 1) = q(x^2 + ax + b). \tag{2}$$

Suppose q is a non-constant, then it is easy to verify that $q(x)$ is monic. We can apply the above lemma to show that for each n, there is at most one monic polynomial $q(x)$ with degree n satisfying the functional equation (2). Recall that $p(x)p(x+1) = p(x+p(x))$, then for any positive integer n, if we let $q(x) = (p(x))^n$, we have

$$q(x)q(x + 1) = (p(x)p(x + 1))^n = (p(x + p(x)))^n = q(x + p(x)) = q(x^2 + ax + b).$$

This means if q is a non-constant polynomial such that $\deg q$ is even, then the proposed conjecture is true.

Finally, if q satisfies equation (2) such that $\deg q = k$ is odd, by a similar reasoning as above, q^2 also satisfies equation (2), and we must have $q(x)^2 = (p(x))^k$. This implies that $p(x)$ is a perfect square, i.e., $(a - 1)^2 = 4b$.

To conclude, if $(a-1)^2 = 4b$, then the non-constant solutions to (2) are $(x+\frac{a-1}{2})^n$, where n is any positive integer; if $(a - 1)^2 \neq 4b$, then the non-constant solutions to (2) are $(x^2 + (a - 1)x + b)^n$, where n is any positive integer.

Editor's Comment. Walther Janous pointed out that (a) has appeared a few times in the literature; see for example Section 4.5 of Christopher G. Small, *Functional equations and how to solve them*, Springer, New York, 2007.

4608. *Proposed by Florin Stanescu.*

Calculate
$$\lim_{n\to\infty} \frac{H_{n+1} + H_{n+2} + \cdots + H_{2n}}{nH_n},$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 1$.

We received 29 submissions, of which 24 were complete and correct. We present the solution by Samuel Gómez García.

For each pair of natural numbers n, m we have

$$H_{n+m} = H_n + \sum_{k=n+1}^{n+m} \frac{1}{k}.$$

Thus,

$$\lim_{n\to\infty} \frac{H_{n+1} + H_{n+2} + \cdots + H_{2n}}{nH_n} = \lim_{n\to\infty} \frac{nH_n + \frac{n}{n+1} + \frac{n-1}{n+2} + \cdots + \frac{1}{2n}}{nH_n}. \qquad (1)$$

Since $0 < \dfrac{n-k+1}{n+k} < 1$ for all $1 \leq k \leq n$, we have

$$1 < \frac{nH_n + \frac{n}{n+1} + \frac{n-1}{n+2} + \cdots + \frac{1}{2n}}{nH_n} < 1 + \frac{n}{nH_n} = 1 + \frac{1}{H_n}.$$

Using the fact that $H_n \to \infty$, the Squeeze Lemma gives us that the limit in (1) equals 1, concluding the calculation of the desired limit.

4609. *Proposed by George Apostolopoulos.*

Triangle ABC has internal angle bisectors AD, BE and CF, where points D, E and F lie on the sides BC, AC and AB, respectively. Prove that

$$\frac{AB^4 + BC^4 + CA^4}{DE^4 + EF^4 + FD^4} \geq 16.$$

We received 12 solutions, all of which were correct. We present the solution by Subhankar Gayen.

Let $a = BC$, $b = CA$, $c = AB$ be the side lengths of the triangle ABC. From the angle bisector theorem in triangle ABC, it follows that

$$AF = \frac{bc}{a+b} \text{ and } AE = \frac{bc}{a+c}.$$

By the law of cosines in triangle AEF it follows that

$$\begin{aligned}
EF^2 &= AF^2 + AE^2 - 2AF \cdot AE \cdot \cos A \\
&= \left(\frac{bc}{a+b}\right)^2 + \left(\frac{bc}{a+c}\right)^2 - \frac{2b^2c^2}{(a+b)(a+c)} \cdot \frac{b^2+c^2-a^2}{2bc} \\
&= \left(\frac{bc}{(a+b)(a+c)}\right)^2 \cdot \frac{bc\left(b^2+c^2\right) + 2abc(a+b+c) - \left(b^2+c^2-a^2\right)[bc + a(a+b+c)]}{bc} \\
&= \left(\frac{bc}{(a+b)(a+c)}\right)^2 \cdot \frac{a^2\left(bc+a^2+ab+ac\right) + 2abc(a+b+c) - a(a+b+c)\left(b^2+c^2\right)}{bc} \\
&= \left(\frac{bc}{(a+b)(a+c)}\right)^2 \cdot \frac{a^2(a+b)(a+c) - a(a+b+c)(b-c)^2}{bc} \\
&\le \frac{a^2bc}{(a+b)(a+c)} \le \frac{a^2bc}{2\sqrt{ab}\cdot 2\sqrt{ac}} = \frac{a\sqrt{bc}}{4},
\end{aligned}$$

where we have used the AM-GM inequality in the last step. Thus

$$EF^4 \le \frac{a^2bc}{16}.$$

Using similar bounds for DE^4 and FD^4, we obtain

$$\frac{AB^4 + BC^4 + CA^4}{DE^4 + EF^4 + FD^4} \ge \frac{16\left(a^4+b^4+c^4\right)}{abc(a+b+c)}.$$

We now use the AM-GM inequality repeatedly to get

$$\begin{aligned}
a^4+b^4+c^4 &= \frac{a^4+b^4}{2} + \frac{b^4+c^4}{2} + \frac{c^4+a^4}{2} \\
&\ge a^2b^2 + b^2c^2 + c^2a^2 \\
&= a^2\left(\frac{b^2+c^2}{2}\right) + b^2\left(\frac{c^2+a^2}{2}\right) + c^2\left(\frac{a^2+b^2}{2}\right) \\
&\ge a^2bc + b^2ca + c^2ab \\
&= abc(a+b+c).
\end{aligned}$$

Thus the desired result follows, and equality holds if and only if triangle ABC is equilateral.

4610. *Proposed by Albert Natian.*

Find the smallest positive number x so that the following three quantities a, b and c are all integers:

$$a = a(x) = \sqrt[4]{72 + \sqrt{3x} + \sqrt{16 + 275x} + \sqrt{19 + 288x}}\ ,$$
$$b = b(x) = 5\sqrt[3]{\frac{9x}{20}} + \sqrt{16 + 275x}\ ,$$
$$c = c(x) = 7\sqrt[3]{\frac{2x}{15}} + 2\sqrt{3x}\ .$$

We received 11 solutions, one of which was incorrect. We present the solution by Sergey Sadov.

The dependence of c on x is monotone. Hence, for any $c > 0$ there is a unique positive root $x = x_c$ of the third equation and $x_1 < x_2 < \dots$. If x_1 happens to yield integer values of a and b, then x_1 is the required value.

This is indeed the case, since $x_1 = 3/400$ satisfies the third equation with $c = 1$ and yields

$$\sqrt{3x} = \frac{3}{20}, \quad \left(\frac{2x}{15}\right)^{1/3} = \frac{1}{10}, \quad \left(\frac{9x}{20}\right)^{1/3} = \frac{3}{20}, \quad \sqrt{16 + 275x} = \frac{17}{4},$$
$$\sqrt{19 + 288x} = \frac{23}{5},$$
$$a = 3, \qquad b = 5.$$