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Crux Mathematicorum with Mathematical Mayhem

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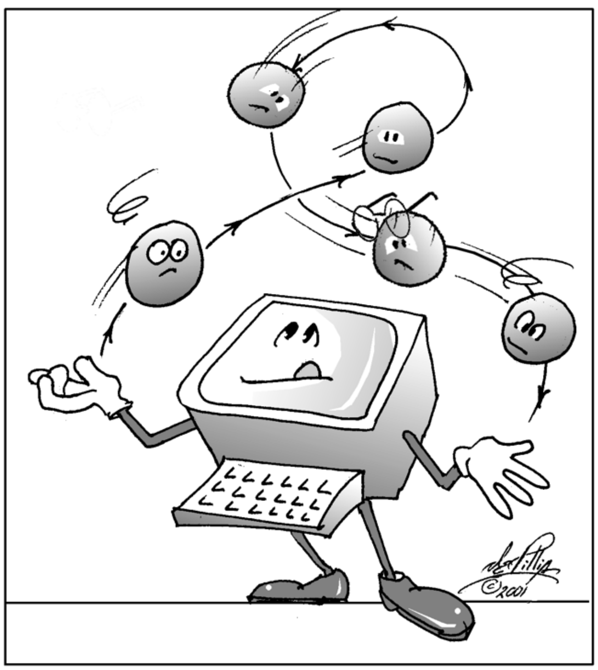
EDITORIAL

Ron Graham passed away on July 6th as the year 2020 claimed the life of another mathematical legend.

Richard Guy, John Conway, Ron Graham. I'm lucky to have met each of these mathematicians in person, even share a glass of wine with John Conway. To me, this is a short list of candidates for "If you could have dinner with anyone, who would it be?" But enough is enough 2020. I'd like to have at least a hypothetical opportunity to still meet some of my favourite mathematicians in person.

Kseniya Garaschuk



The problem with juggling is that the balls go where you throw them. Just as the problem with programming is that the computer does exactly what you tell it.
Ron Graham with cartoon by John de Pillis.

MathemAttic

No. 16

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 15, 2020**.

MA76. The sum of two real numbers is n and the sum of their squares is $n+19$, for some positive integer n. What is the maximum possible value of n?

MA77. In a regular decagon, all diagonals are drawn. If a diagonal is chosen at random, what is the probability that it is neither one of the shortest nor one of the longest?

MA78. Let $T(n)$ be the digit sum of a positive integer n; for example, $T(5081) = 5+0+8+1 = 14$. Find the number of three-digit numbers that satisfy $T(n) + 3n = 2020$.

MA79. Suppose BD bisects $\angle ABC$, $BD = 3\sqrt{5}$, $AB = 8$ and $DC = \frac{3}{2}$. Find $AD + BC$.

MA80. Suppose $ABCD$ is a parallelogram. Let E and F be two points on BC and CD, respectively. If $CE = 3BE$, $CF = DF$, DE intersects AF at K and $KF = 6$, find AK.

. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 septembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA76. La somme de deux nombres réels est n tandis que la somme de leurs carrés est $n + 19$, où n est un entier positif. Déterminer la plus grande valeur possible pour n.

MA77. On considère toutes les diagonales d'un décagone régulier. Si une d'entre elles est choisie aléatoirement, quelle est la probabilité qu'elle est ni la plus courte ni la plus longue?

MA78. Soit $T(n)$ la somme des chiffres d'un entier positif n; par exemple, $T(5081) = 5 + 0 + 8 + 1 = 14$. Déterminer le nombre d'entiers à trois chiffres tels que $T(n) + 3n = 2020$.

MA79. Supposer que BD bissecte $\angle ABC$, puis que $BD = 3\sqrt{5}$, $AB = 8$ et $DC = \frac{3}{2}$. Déterminer $AD + BC$.

MA80. Soit $ABCD$ un parallélogramme. Soient aussi E et F deux points, sur BC et CD respectivement. Supposer que $CE = 3BE$ et $CF = DF$, puis que DE intersecte AF en K et que $KF = 6$. Déterminer AK.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(1), p. 4–7.

MA51. Find all non-negative integers x, y, z satisfying the equation

$$2^x + 3^y = 4^z.$$

Proposed by Nguyen Viet Hung.

We received 10 complete and correct and 2 incomplete submissions. We present the solution by the Sigma Problem Solving Group.

We use $4^z = 2^{2z}$ to rewrite the equation from the problem as

$$3^y = 2^{2z} - 2^x.$$

Since $3^y > 0$, we obtain $2z > x$ and thus $z \neq 0$. If $x \neq 0$ as well then the right hand side of the equation above is even, while the left hand side is odd. Thus $x = 0$ and we have

$$3^y = 2^{2z} - 1 \qquad \Longleftrightarrow \qquad 3^y = (2^z + 1)(2^z - 1).$$

Since $2^z + 1$ and $2^z - 1$ cannot both be multiples of 3 we must have $2^z - 1 = 1$. Thus the only solution to the problem is $x = 0, y = 1, z = 1$.

MA52. The diagram shows part of a tessellation of the plane by a quadrilateral. Khelen wants to colour each quadrilateral in the pattern.

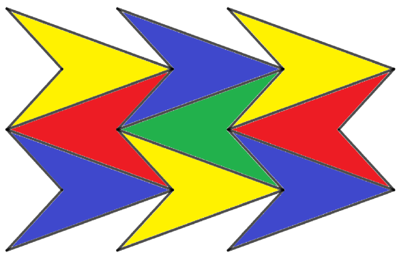
1. What is the smallest number of colours he needs if no two quadrilaterals that meet (even at a point) can have the same colour?
2. Suppose that quadrilaterals that meet along an edge must be coloured differently, but quadrilaterals that meet just at a point may have the same colour. What is the smallest number of colours that Khelen would need in this case?

3. What is the smallest number of colours needed to colour the edges so that edges that meet at a vertex are coloured differently?

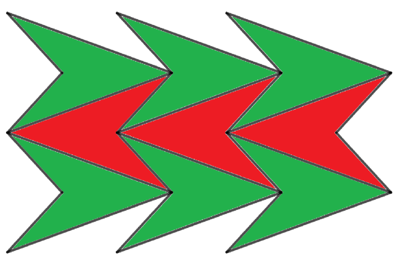
Originally Problem 6 (and its extensions) of the 2019 UK Intermediate Mathematical Challenge.

We received 3 submissions, one of which was complete. We present the solution by the Sigma Problem Solving Group, lightly edited.

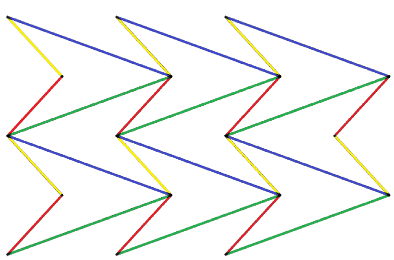
1. Four quadrilaterals meet at each vertex, so Khelen requires at least four colours. To show that four colours suffice, we colour each vertical strip by the colours yellow, red, blue, and green, repeating the colours in this order, with the colourings of two adjacent vertical strips shifted by two colours. A sample of a section is shown.



2. Two quadrilaterals share an edge, so Khelen requires at least two colours. A section of a sample colouring with two colours is shown.



3. Four edges meet at each vertex, so Khelen needs at least four colours. A section of a sample colouring with four colours is shown.



MA53. Find all positive integers m and n which satisfy the equation

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{m^3-1}{m^3+1}=\frac{n^3-1}{n^3+2}.$$

Proposed by John McLoughlin.

We received 4 submissions of which 3 were correct and complete. We present the solution by Derek Dong.

Checking $m = 2, 3, 4$ and $n = 1, 2$, we find of those only $(m, n) = (4, 2)$ works. Now notice $m^2 + m + 1 = (m + 1)^2 - (m + 1) + 1$, so the left side telescopes into

$$\begin{aligned} LHS =& \frac{(3-2)(3^2-3+1)\cdots((m+1)-2)((m+1)^2-(m+1)+1)}{(2^3+1)(3+1)(3^2-3+1)\cdots(m+1)(m^2-m+1)} \\ =& \frac{2(m^2+m+1)}{3(m^2+m)} \\ =& \frac{2}{3}+\frac{2}{3(m^2+m)}. \end{aligned}$$

Now note the right side equals $\dfrac{n^3-1}{n^3+2} = 1 - \dfrac{3}{n^3+2}$. Subtracting each side from 1,

$$\frac{m^2+m-2}{3(m^2+m)}=\frac{3}{n^3+2},$$

or

$$\frac{m^2+m-2}{m^2+m}=\frac{9}{n^3+2}.$$

Now notice that when $m > 4$, the left side is greater than $\frac{9}{10}$, and when $n > 2$, the right side is less than $\frac{9}{10}$, so there are no solutions when $m > 4$ and $n > 2$, so the only solution is $(m, n) = (4, 2)$.

MA54. How many six-digit numbers are there, with leading 0s allowed, such that the sum of the first three digits is equal to the sum of the last three digits, and the sum of the digits in even positions is equal to the sum of the digits in odd positions?

Originally problem 2 from the 1969 Leningrad Math Olympiad, Grade 9.

We received 6 submissions, out of which 5 were correct and complete. We present the solution by Derek Dong, slightly modified by the editor.

The answer is 6700. If we write the six digits as $abcdef$, we are given that

$$\begin{aligned} a+b+c &= d+e+f \\ a+c+e &= b+d+f. \end{aligned}$$

Subtracting the second equation from the first gives

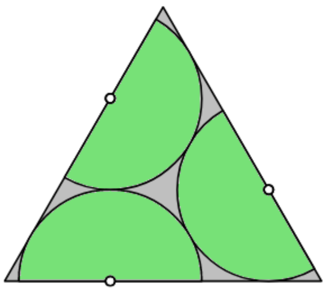
$$b-e=e-b$$

and thus $b = e$ and $a + c = d + f$. The value of $a + c$ can be anything between 0 and 18, thus the number of possibilities for a, c, d, f is

$$1^2 + 2^2 + \cdots + 9^2 + 10^2 + 9^2 + \cdots + 2^2 + 1^2 = \frac{10 \cdot 11 \cdot 21}{6} + \frac{9 \cdot 10 \cdot 19}{6} = 670.$$

For b and e there are ten possibilities, yielding $670 \cdot 10 = 6700$ possibilities in total.

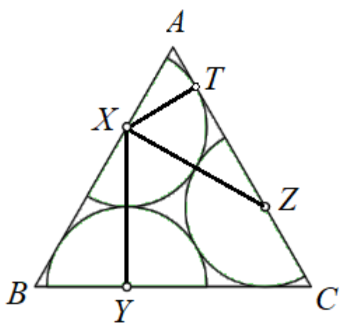
MA55. The diagram shows three touching semicircles with radius 1 inside an equilateral triangle, which each semicircle also touches. The diameter of each semicircle lies along a side of the triangle. What is the length of each side of the equilateral triangle?



Originally Problem 25 of the 2019 UK Intermediate Mathematical Challenge.

We received 7 submissions, all are correct. We present two solutions.

Solution 1, by Missouri State University Problem Solving Group.



Denote the vertices of the triangle by A, B, C and the centers of the semicircles by X, Y, Z as shown in the figure. We have $XY = XZ = 2$. Let T be the foot of the perpendicular from X to AC. Since $XZ = 2$ and $XT = 1$, $\triangle XTZ$ is a $30^\circ - 60^\circ - 90^\circ$ triangle. Therefore $\triangle AXZ$ and $\triangle BYX$ are as well. Consequently

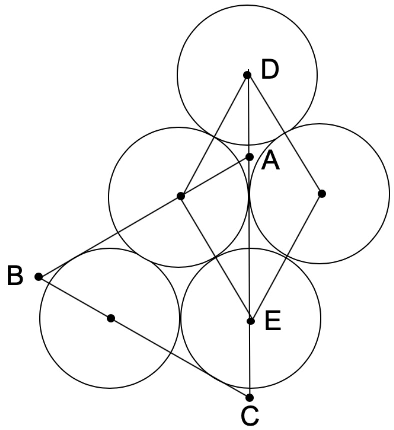
$$\frac{AX}{XZ} = \cot 60^\circ = \frac{\sqrt{3}}{3} \text{ and } \frac{BX}{XY} = \csc 60^\circ = \frac{2\sqrt{3}}{3}.$$

Therefore

$$AX = \frac{2\sqrt{3}}{3}, BX = \frac{4\sqrt{3}}{3}, \text{ and } AB = 2\sqrt{3}.$$

Solution 2, by Richard Hess.

In the figure below, the unit circles define the triangle ABC. It is clear the lengths DA and EC are equal so that the side of triangle ABC is the same length as $DE = 2\sqrt{3}$.



TEACHING PROBLEMS

No. 11

Erick Lee

The Pizza Problem

Alice and Bob are hungry but very polite friends who decide to share a pizza. The pizza is sliced into any number of pieces of various sizes. Each slice is a sector of the circle. They agree on the following terms:

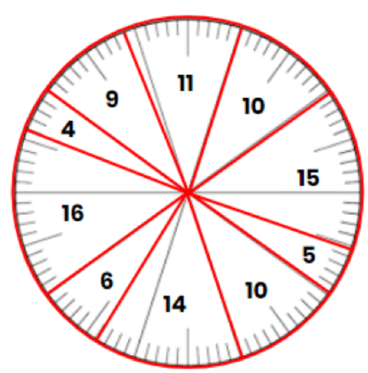
(i) They will alternate turns selecting one slice of pizza;

(ii) Alice will start by selecting any slice she wishes;

(iii) After the first slice, only pieces adjacent to the already-eaten pieces may be selected. This means that on Bobs first turn, he may only select one of the two slices adjacent to the slice that Alice just took.

Alice and Bob continue selecting pieces until none remain. What strategy might Alice and Bob use to get the largest share of the pizza?

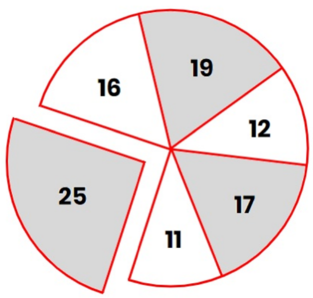
The problem above comes from Dr. Peter Winkler, a professor of mathematics at Dartmouth College in New Hampshire, USA. He originally posed this problem at the Building Bridges mathematics and computer science conference held in Budapest in 2008.



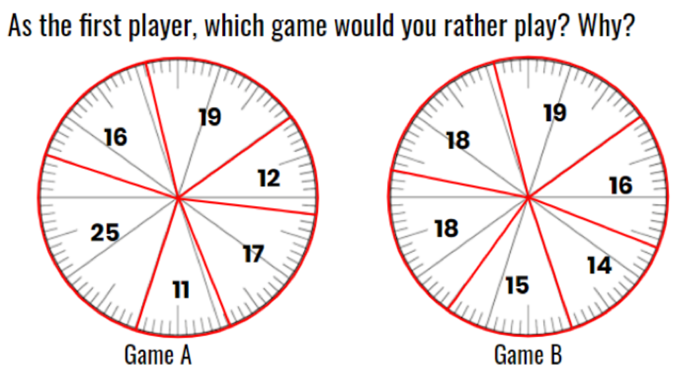
A simple way to introduce this problem is to create a pizza using a blank hundredths circle printed on a sheet of paper. Start by dividing the hundredths circle into an even number of pieces of varying sizes (as in the example shown at right). Using an even number of pieces appeals to students idea of a fair game. Cut the circle into wedges to create the pizza slices. Ask one student to put the wedges together and the other student to be the first player. Students alternate taking slices of pizza until all the slices are gone. Then they can add up the values of their slices to see who has ended up with the most pizza. This is a purposeful way for students to practice addition and logical reasoning skills. To be successful, students will need to use estimation strategies to predict successful courses of action. This game can be enjoyed by even elementary school age students.

This game can be played with pencil and paper, with students crossing off sections of pizza as they are claimed, but cutting up the slices so they can be physically selected adds an extra dimension to the game. It also allows you to quickly create a new pizza with slices in different arrangements.

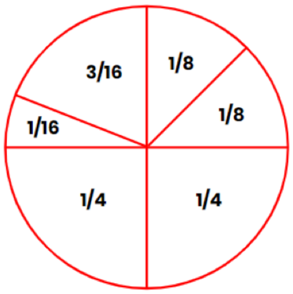
Students should be encouraged to seek a winning strategy. The first player in this game, when there are an even number of slices, can always be assured of getting at least half of the pizza. Think of colouring alternate pieces of the pizza different colours. Sum the total of each different colour to see which one has the largest portion of the pizza. In the example on the left, the grey slices total 61 while the white slices total 39. If the starting player chooses a grey slice, in this case the slice with a value of 25, the opposing player will only be able to select between two different white slices, either the 16 or the 11. The first player then continues to select a grey slice from the same side that the opposing player has picked from. In this manner, the starting player can be assured to select all the slices of the same colour.

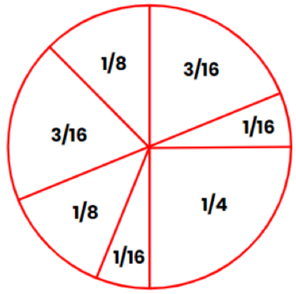


Students may struggle to come up with a strategy like this on their own. You might actually play the game with an even number of alternately coloured slices. Ask students to notice which colour slices each player has available to them on their turn. They should notice that the second player will always have only white slices to choose from provided that the starting player always chooses grey slices. To prompt their thinking, or to start a class discussion once students have had time to explore this game, you might pose a question like the one below.



Instead of using whole number value slices, older students can be challenged to play the same game with the entire pizza having a value of 1 and each slice having a fractional value. Start with an already created pizza for students and later challenge them to create their own game board. Creating a list of fractions that sum to one is an open question that can be a challenge in itself for some junior high students. Pizzas may have an even or an odd number of slices. Once students have discovered the strategy discussed above for an even number of slices, encourage them to explore pizzas with an odd number of slices.





Kolja Knauer, Piotr Micek and Torsten Ueckerdt collaborated on an article titled "How to Eat 4/9th of a Pizza" in the journal *Discrete Mathematics* (Volume 311, Issue 16, 2011, Pages 1635-1645) and at `https://arxiv.org/abs/0812.2870`. The authors describe several strategies for the first player to eat as large a portion of the pizza as possible. They start with the proposition that the first player can eat at least 1/2 of a pizza with an even number of slices as was described above. Using the same strategy, the first player can also eat at least 1/3 of a pizza with an odd number of slices. Using a modified version of this strategy, they show that the first player can eat at least 3/7 of any pizza. They further refine this strategy to show that the first player can eat at least 4/9 of any pizza. The authors prove that 4/9 of any pizza is the best that the first player can be guaranteed to achieve in a pizza with an odd number of slices.

Dr. Winkler is a prolific creator of mathematical puzzles. For additional problems, I highly recommend his book *Mathematical Puzzles: A Connoisseur's Collection* published in 2003. Dr. Peter Winkler is a professor of mathematics at Dartmouth College in New Hampshire, USA. His puzzle book was reviewed by Peter Hardy in *Crux Mathematicorum*, Vol. 30 Number 7.

A Follow Up Problem – The Pizza Race Problem

Keyue Gao, proposed a variation on the Pizza Problem – The Pizza Race Problem (see `https://arxiv.org/abs/1212.2525`). In this variation, time is included as a factor. Players eat pizza at the same rate and can only select their next slice once they have finished eating their current slice. A player might opt to take a smaller slice in order to get two selections in a row while the other player is still finishing their slice. Give this version of the game a try. Do you think this change helps the starting player or makes it harder for them? Once you have given it a try, check out Gao's paper to see the strategy that he describes and how much of the pizza that the starting player can achieve.

Erick Lee is a Mathematics Support Consultant for the Halifax Regional Centre for Education in Dartmouth, NS. Erick blogs at `https://pbbmath.weebly.com/` and can be reached via email at `elee@hrce.ca` and on Twitter at `@TheErickLee`.

OLYMPIAD CORNER

No. 384

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **September 15, 2020**.

OC486. There are 2017 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1009 of the given points.

OC487. Let a, b, c be real numbers such that $1 < b \leq c^2 \leq a^{10}$ and

$$\log_a b + 2\log_b c + 5\log_c a = 12.$$

Show that

$$2\log_a c + 5\log_c b + 10\log_b a \geq 21.$$

OC488. Prove that the equation

$$(x^2 + 2y^2)^2 - 2(z^2 + 2t^2)^2 = 1$$

has infinitely many integer solutions.

OC489. The incircle of a triangle ABC touches AB and AC at points D and E, respectively. Point J is the center of the excircle of triangle ABC tangent to side BC. Points M and N are midpoints of segments JD and JE, respectively. Lines BM and CN intersect at point P. Prove that P lies on the circumcircle of triangle ABC.

OC490. Find the smallest prime number that cannot be written in the form $|2^a - 3^b|$ with nonnegative integers a, b.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 septembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC486. Il y a 2017 points dans le plan avec la propriété que pour trois quelconques d'entre eux, on peut en choisir deux qui sont à distance inférieure à 1. Démontrer qu'il existe un cercle de rayon 1, contenant au moins 1009 des points donnés.

OC487. Soient a, b, c des nombres réels tels que $1 < b \leq c^2 \leq a^{10}$ et

$$\log_a b + 2\log_b c + 5\log_c a = 12.$$

Démontrer que

$$2\log_a c + 5\log_c b + 10\log_b a \geq 21.$$

OC488. Démontrer que l'équation

$$(x^2 + 2y^2)^2 - 2(z^2 + 2t^2)^2 = 1$$

possède un nombre infini de solutions entières.

OC489. Le cercle inscrit du triangle ABC touche AB et AC en D et E respectivement. Le point J est le centre du cercle exinscrit du triangle ABC, tangent au côté BC. Les points M et N sont les mi points des segments JD et JE respectivement. Enfin, les lignes BM et CN intersectent en P. Démontrer que P se trouve sur le cercle circonscrit du triangle ABC.

OC490. Déterminer le plus petit nombre premier ne pouvant pas être représenté sous la forme $\left|2^a - 3^b\right|$, où a et b sont des entiers non négatifs.

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(1), p. 18–19.

OC461. Let A and B be two finite sets. Determine the number of functions $f : A \to A$ with the property that there exist two functions $g : A \to B$ and $h : B \to A$ such that $g(h(x)) = x\ \forall x \in B$ and $h(g(x)) = f(x)\ \forall x \in A$.

Originally from 2017 Romanian Math Olympiad, 4th Problem, Grade 10, Final Round.

We received 5 *correct submissions. We present the solution by Missouri State University Problem Solving Group.*

Let $|A| = m$ and $|B| = n$. Since $g(h(x)) = x$, for all $x \in B$, h is injective, so $|B| = |h(B)|$. Let $h(b) \in h(B)$. Then $f(h(b)) = h(g(h(b)) = h(b)$, i.e., f restricted to $h(B)$ is the identity function. There are $\binom{m}{n}$ ways of choosing this subset. The elements of $A - h(B)$ must be mapped to $h(B)$ and there are n^{m-n} such functions. This gives an upper bound of

$$\binom{m}{n} n^{m-n}$$

on the number of such f.

For any $C \subseteq A$ with $|C| = |B|$ and any function $\phi : A - C \to C$, define $f_{C,\phi}$ as follows

$$f_{C,\phi}(x) = \begin{cases} x \text{ if } x \in C \\ \phi(x) \text{ if } x \notin C \end{cases}.$$

We will first show that $f_{C,\phi}$ satisfies the hypotheses. Without loss of generality, we may assume that $B = \{1, 2, \ldots, m\}$ and $C = \{c_1, \ldots, c_m\}$. Let

$$h_C(i) = c_i, i = 1, \ldots, m$$
$$g_{C,\phi}(x) = i \text{ if } x = c_i \text{ or } \phi(x) = c_i.$$

It is immediate that $g_{C,\phi}(h_C(i)) = i$ for all $i \in B$ and $h_C(g_{C,\phi}(x)) = f_{C,\phi}(x)$ for all $x \in A$.

We claim that if $(C_1, \phi_1) \neq (C_2, \phi_2)$, then $f_{C_1,\phi_1} \neq f_{C_2,\phi_2}$. If $C_1 \neq C_2$, then there is an $x \in C_1 - C_2$. Therefore $f_{C_1,\phi_1}(x) = x \in C_1$ cannot equal $f_{C_2,\phi_2}(x) \in C_2$. If $C_1 = C_2$, but $\phi_1 \neq \phi_2$, there is an $x \in A - C_1 = A - C_2$ such that

$$f_{C_1,\phi_1}(x) = \phi_1(x) \neq \phi_2(x) = f_{C_2,\phi_2}(x).$$

This shows that $\binom{m}{n} n^{m-n}$ is a lower bound for the number of f satisfying the conditions of the problem. Therefore the number of such f is exactly

$$\binom{m}{n} n^{m-n}.$$

OC462. The integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \cdot \ldots \cdot a_n$, then prove that

$$(a_1 a_2 \cdot \ldots \cdot a_n)^{m-1} \geq (n!)^m.$$

Originally from 2017 Poland Math Olympiad, 3rd Problem, Final Round.

We received 3 *submissions, all correct. We present the solution by Oliver Geupel.*

Let p be any prime divisor of $a_1 a_2 \cdot \ldots \cdot a_n$, let $\nu_p(a)$ denote the exponent of p in the prime factorisation of the natural number a, and let $b_k = a_k / p^{\nu_p(a_k)}$ where $1 \leq k \leq n$. If j and k are indices such that $1 \leq j < k \leq n$ and $b_j = b_k$, then we have $p^{\nu_p(a_j)} = a_j/b_j < a_k/b_k = p^{\nu_p(a_k)}$ and thus

$$p \leq \frac{p^{\nu_p(a_k)}}{p^{\nu_p(a_j)}} = \frac{b_k p^{\nu_p(a_k)}}{b_j p^{\nu_p(a_j)}} = \frac{a_k}{a_j} < 2,$$

which is impossible. Hence, the numbers b_1, b_2, …, b_n are distinct.

We obtain $b_1 b_2 \cdot \ldots \cdot b_n \geq n!$; whence

$$a_1 a_2 \cdot \ldots \cdot a_n = \prod_{k=1}^{n} b_k p^{\nu_p(a_k)} = \left(\prod_{k=1}^{n} b_k\right)\left(\prod_{k=1}^{n} p^{\nu_p(a_k)}\right) \geq n! p^{\nu_p(a_1 a_2 \cdot \ldots \cdot a_n)}$$

If p_1, p_2, …, p_m are the distinct prime divisors of $a_1 a_2 \cdot \ldots \cdot a_n$, we have

$$\prod_{k=1}^{m} p_k^{\nu_{p_k}(a_1 a_2 \cdot \ldots \cdot a_n)} = a_1 a_2 \cdot \ldots \cdot a_n.$$

Consequently,

$$(a_1 a_2 \cdot \ldots \cdot a_n)^m \geq (n!)^m \cdot \prod_{k=1}^{m} p_k^{\nu_{p_k}(a_1 a_2 \cdot \ldots \cdot a_n)} = (n!)^m \cdot a_1 a_2 \cdot \ldots \cdot a_n.$$

Hence the result.

OC463. A 6×6 table is filled with the integers from 1 to 36.

(a) Give an example of such a fill of the table so that the sum of every two numbers in the same row or column is greater than 11.

(b) Prove that in some row or column, no matter how you fill the table, you will always find two numbers whose sum does not exceed 12.

Originally from 2017 Czech-Slovakia Math Olympiad, 2nd Problem, Category C, Second Round.

We received 7 *correct submissions. We present the solution by Oliver Geupel.*

The following table gives an example for part (a):

1	12	13	14	15	16
17	**2**	18	19	20	21
22	23	**3**	24	25	26
27	28	29	**4**	30	31
32	33	34	35	**5**	36
11	**10**	**9**	**8**	**7**	**6**

It is enough to check the pairs of numbers from $1, 2, \ldots, 10$ that are in the same row or column, which is straightforward.

Moving on to part (b), let us suppose that we have a table where all sums of pairs of numbers in the same row or column are at least 13. Then, the six numbers $1, 2, \ldots, 6$ are in distinct rows and in distinct columns. Hence, every row and every colum of the table contains exactly one of the numbers $1, 2, \ldots, 6$. Then, the number 7 shares a common row with one of these numbers and also a common column with another one. That is, the number 7 is in a common row or column with a number from $1, 2, \ldots, 5$, a contradiction. The result (b) follows.

OC464. Given an acute triangle ABC with orthocenter H. The angle bisector of $\angle BHC$ intersects side BC at D. Let E and F be the symmetric points of D with respect to lines AB and AC, respectively. Prove that the circumcircle of triangle AEF passes through the midpoint G of arc BAC.

Originally from 2017 Czech-Slovakia Math Olympiad, 5th Problem, Category A, Final Round.

We received 9 *submissions. We present* 2 *solutions.*

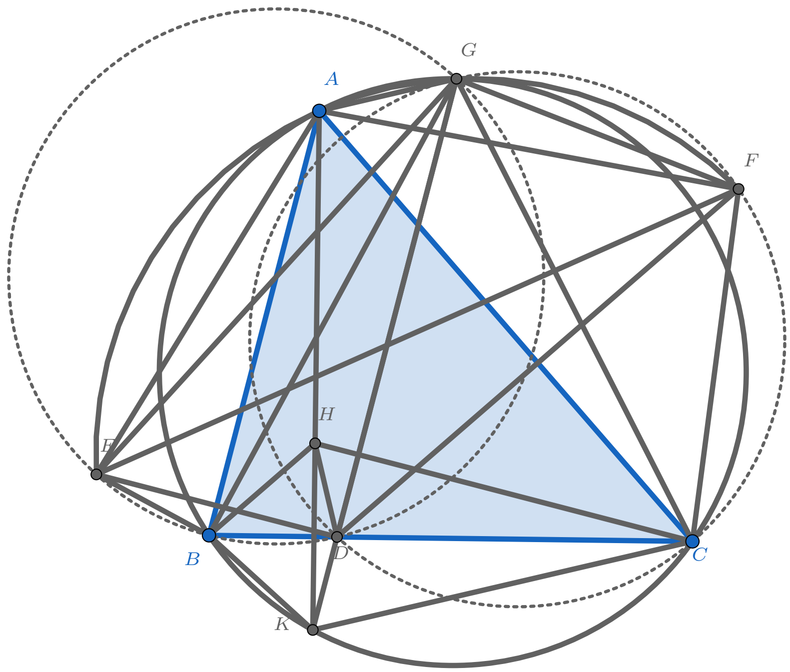
Solution 1, by Bui Nguyen Huu An.

See the picture on the next page.

Let (O) be the circumcircle of triangle ABC. Let G be the midpoint of arc BAC of (O). Let AH intersect (O) at K. It is clear that we only need to prove $EAGF$ is cyclic.

Observe that $\angle HBC = \angle HAC = \angle KBC$. Therefore, K is the reflection of H across BC. It follows that

$$\frac{DB}{DC} = \frac{HB}{HC} = \frac{KB}{KC}.$$



We conclude that KD is the angle bisector of $\angle BKC$ so KD passes through the midpoint G of arc BAC of (O). Hence

$$\Rightarrow \angle DFC = \angle FDC = 90^0 - \angle DCA = \angle KAC = \angle KGC = \angle DGC.$$

Therefore, $GFCD$ is cyclic. Analogously, $GEBD$ is cyclic.

We obtain,

$$\begin{aligned}
\angle EGF &= \angle EGD + \angle DGF \\
&= 180^0 - \angle EBD + 180^0 - \angle FCD \\
&= 360^0 - 2\angle ABC - 2\angle ACB \\
&= 360^0 - 2(180^0 - \angle BAC) \\
&= 2\angle BAC \\
&= 2(\angle BAD + \angle DAC) \\
&= \angle EAD + \angle FAD \\
&= \angle EAF
\end{aligned}$$

$\Rightarrow EAGF$ is cyclic.

Solution 2, by Oliver Geupel.

Let Γ and Γ' denote the circumcircles of triangles ABC and BCH, respectively, and let A' denote the second point of intersection of Γ and the line AH. The perpendicular bisector of BC intersects the longer arc $\widehat{BC}$ of circle Γ' at a point G' which lies on the line DH. Hence, the point D is the intersection of the chords BC and $G'H$ of Γ'. Since $\angle BHC = 180^\circ - \hat{A}$, the circles Γ and Γ' are symmetric with respect to reflection in the axis BC. Thus, the point D is also the intersection of the chords BC and $A'G$ of Γ.

Consider the problem in the plane of complex numbers such that Γ is the unit circle. We will denote by the lower-case letter m the affix of any point M. There is no loss of generality in assuming that $g = 1$. Then, $bc = 1$, $aa' = -1$, and

$$d = \frac{a'g(b+c) - bc(a'+g)}{a'g - bc} = \frac{-\frac{1}{a}(b+c) - \left(-\frac{1}{a} + 1\right)}{-\frac{1}{a} - 1} = \frac{a+b+c-1}{a+1}.$$

If P is the foot of the perpendicular from D to AB, then

$$p = \left(a + b + d - ab\overline{d}\right)/2$$

and $e - d = 2(p - d)$. Therefore,

$$e = a + b - ab\overline{d}$$

and similarly

$$f = a + c - ac\overline{d}.$$

Hence

$$\frac{f-a}{e-a} = \frac{c}{b}.$$

A tedious but straightforward calculation gives

$$\frac{c(e-g)}{b(f-g)} = \frac{c(a+b-1)(a+1) - a(ab+ac+1-a)}{b(a+c-1)(a+1) - a(ab+ac+1-a)} = \frac{1+a^2-a^2b-c}{1+a^2-a^2c-b},$$

and the conjugate of

$$z = (1+a^2-a^2b-c)/(1+a^2-a^2c-b)$$

is

$$\frac{1+\frac{1}{a^2}-\frac{c}{a^2}-b}{1+\frac{1}{a^2}-\frac{b}{a^2}-c} = \frac{a^2+1-c-a^2b}{a^2+1-b-a^2c} = z,$$

so that $z = \overline{z}$. Hence, the complex number

$$\frac{(f-a)(e-g)}{(e-a)(f-g)}$$

is real. This proves that the points A, E, F, and G are concyclic.

OC465. The sequence (a_n) is defined by

$$a_1 = 1, \qquad a_n = \lfloor \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1} \rfloor \qquad \text{if } n > 1.$$

Find a_{2017}.

Originally from 2017 Bulgaria Math Olympiad, 3rd Problem, Grade 10, Second Round.

We received 10 *submissions. We present the solution by UCLan Cyprus Problem Solving Group.*

We will prove, by induction on n, that $a_{2n} = a_{2n+1} = n$ for every $n \geqslant 1$. In particular we get that $a_{2017} = 1008$.

There is some ambiguity on whether $a_2 = \left\lfloor \sqrt{1} \right\rfloor$ or $a_2 = \left\lfloor \sqrt{2} \right\rfloor$ but in both cases we get $a_2 = 1$ and thus $a_3 = \left\lfloor \sqrt{2+1} \right\rfloor = 1$. So the claim is true for $n = 1$.

Assume the claim is true for $n = k$. Then

$$\begin{aligned} a_{2n+2} &= \left\lfloor \sqrt{2a_{2n+1} + a_{2n} + \cdots + a_1} \right\rfloor \\ &= \left\lfloor \sqrt{2n + n + 2((n-1) + (n-2) + \cdots + 1) + 1} \right\rfloor \\ &= \left\lfloor \sqrt{n + n(n+1) + 1} \right\rfloor = n + 1 \end{aligned}$$

and therefore

$$\begin{aligned} a_{2n+3} &= \left\lfloor \sqrt{2a_{2n+2} + a_{2n+1} + \cdots + a_1} \right\rfloor \\ &= \left\lfloor \sqrt{2(n+1) + 2(n + (n-1) + \cdots + 1) + 1} \right\rfloor \\ &= \left\lfloor \sqrt{(n+2)(n+1) + 1} \right\rfloor = n + 1 \end{aligned}$$

as $(n+1)^2 = n^2 + 2n + 1 < n^2 + 3n + 3 < n^2 + 4n + 4 = (n+2)^2$.

So the claim is true for every $n \geqslant 1$.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **September 15, 2020**.

4551. *Proposed by Michel Bataille.*

Let ABC be a triangle with sides $BC = a, CA = b$ and $AB = c$. Suppose $b > c$ and let A_1, A_2 be the two points such that ΔA_1BC and ΔA_2BC are equilateral. Express the circumradius of ΔAA_1A_2 as a function of a, b, c.

4552. *Proposed by Anupam Datta.*

Given positive integers a, b and n, prove that the following are equivalent:

1. $b \equiv ax \pmod{n}$ has a solution with $\gcd(x, n) = 1$;
2. $b \equiv ax \pmod{n}$ and $a \equiv by \pmod{n}$ have solutions $x, y \in \mathbb{Z}$;
3. $\gcd(a, n) = \gcd(b, n)$.

4553. *Proposed by Daniel Sitaru.*

Find

$$\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)$$

4554. *Proposed by George Stoica.*

Let ϵ be a given constant with $0 < \epsilon < 1$, and let (a_n) be a sequence with $0 \le a_n < \epsilon$ for all $n \ge 1$. Prove that $(1 - a_n)^n \to 1$ as $n \to \infty$ if and only if $na_n \to 0$ as $n \to \infty$.

4555. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Prove that if a, b, c and d are positive numbers that satisfy

$$ab + bc + cd + da + ac + bd = 6,$$

then

$$a + b + c + d \ge 2\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.$$

When does the equality hold?

4556. *Proposed by Marian Cucoanes and Lorean Saceanu.*

Let $x \geq 1$ be a real number and consider a triangle ABC. Prove that

$$\frac{(x-\cos A)(x-\cos B)(x-\cos C)}{(x+\cos A)(x+\cos B)(x+\cos C)} \leq \left[\frac{(3x-1)R-r}{(3x+1)R+r}\right]^3.$$

When does the equality hold?

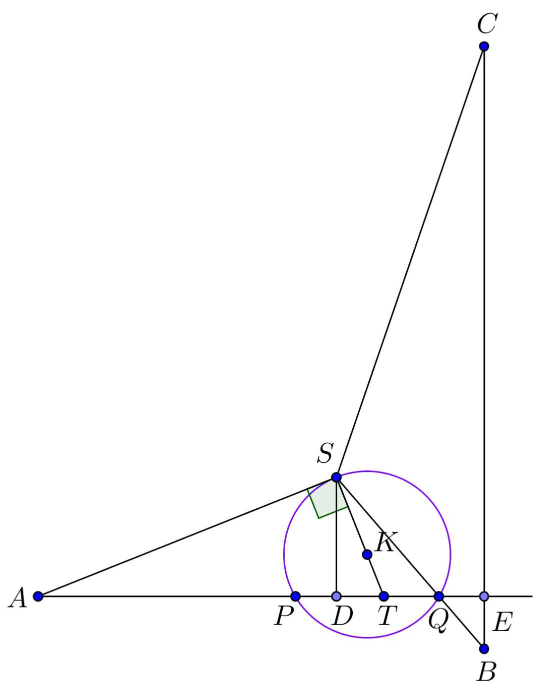
4557. *Proposed by George Apostolopoulos.*

Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let a, b and c be the lengths of the sides of ABC. Prove that

$$\frac{24r^2}{R} \leq \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq \frac{4r^2-2Rr}{r}.$$

4558. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

Consider a diagram below, where triangle SAT is right-angled and $\tan T > 2$. The point K lies on the segment ST such that $SK = 2KT$. The circle centered at K with radius KS intersects the line AT at P and Q. Point D is the projection of S on AT and E is a point on AT such that D lies on AE and $AD = 2DE$. Finally, suppose that SQ and SP intersect the perpendicular at E on AT at B and C respectively. Prove that S is the incenter of the triangle ABC.



4559. *Proposed by Nho Nguyen Van.*

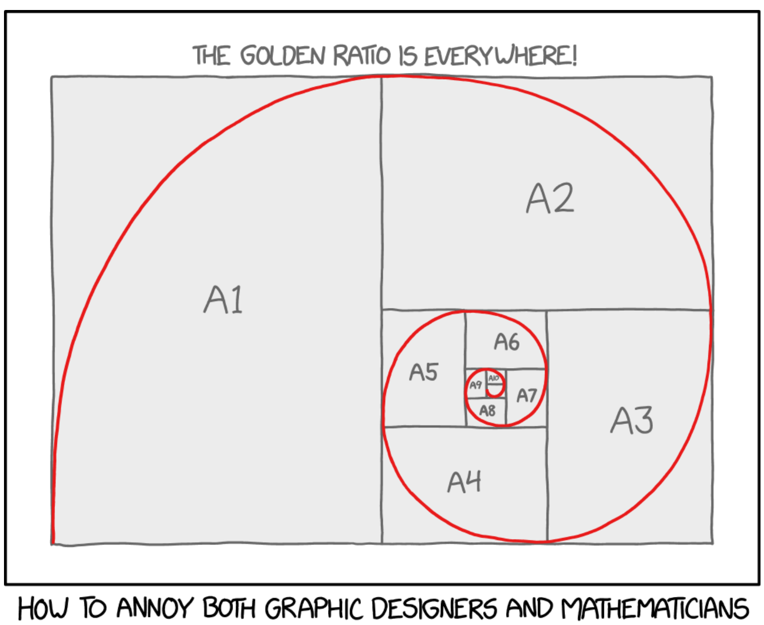
Let x_k be positive real numbers. Prove that for every natural number $n \geq 2$, we have

$$\left(\sum_{k=1}^{n} x_k^{10}\right)^3 \geq \left(\sum_{k=1}^{n} x_k^{15}\right)^2$$

4560. *Proposed by Mihaela Berindeanu.*

Let E and F be midpoints on the respective sides CA and AB of triangle ABC, and let P be the second point of intersection of the circles ABE and ACF. Prove that the circle AEF intersects the line AP again in the point X for which $AX = 2XP$.

. . . .



https://xkcd.com/2322/

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 septembre 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4551. *Proposé par Michel Bataille.*

Soit ABC un triangle où $BC = a, CA = b$ et $AB = c$. Si $b > c$ et si A_1, A_2 sont les deux points tels que $\triangle A_1BC$ et $\triangle A_2BC$ sont équilatéraux, exprimer le rayon du cercle circonscrit de $\triangle AA_1A_2$ en termes de a, b et c.

4552. *Proposé par Anupam Datta.*

Soient a, b et n des entiers positifs. Démontrer que les affirmations suivantes sont équivalentes:

1. $b \equiv ax \pmod{n}$ a une solution telle que $\mathrm{pgcd}(x, n) = 1$;
2. $b \equiv ax \pmod{n}$ et $a \equiv by \pmod{n}$ ont des solutions $x, y \in \mathbb{Z}$;
3. $\mathrm{pgcd}(a, n) = \mathrm{pgcd}(b, n)$.

4553. *Proposé par Daniel Sitaru.*

Déterminer

$$\lim_{n\to\infty} \left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n} \right)$$

4554. *Proposé par George Stoica.*

Soit ϵ une constante telle que $0 < \epsilon < 1$, et soit (a_n) une suite telle que $0 \leq a_n < \epsilon$ pour tout $n \geq 1$. Démontrer que $(1 - a_n)^n \to 1$ lorsque $n \to \infty$ si et seulement si $na_n \to 0$ lorsque $n \to \infty$.

4555. *Proposé par Michael Rozenberg et Leonard Giugiuc.*

Démontrer que si a, b, c et d des nombres positifs tels que

$$ab + bc + cd + da + ac + bd = 6,$$

alors

$$a + b + c + d \geq 2\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.$$

Quand l'égalité tient-elle?

4556. *Proposé par Marian Cucoanes et Lorean Saceanu.*

Considérer le triangle ABC et $x \geq 1$ un nombre réel. Démontrer que

$$\frac{(x-\cos A)(x-\cos B)(x-\cos C)}{(x+\cos A)(x+\cos B)(x+\cos C)} \leq \left[\frac{(3x-1)R-r}{(3x+1)R+r}\right]^3.$$

Quand l'égalité tient-elle?

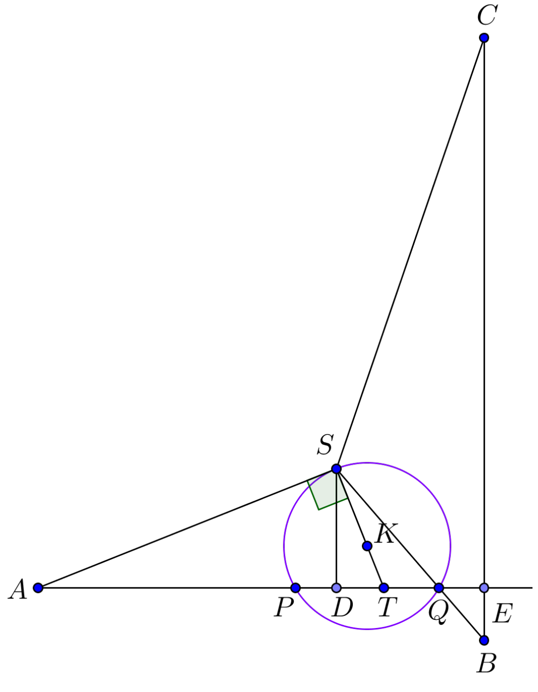
4557. *Proposé par George Apostolopoulos.*

Soient m_a, m_b et m_c les longueurs des médianes d'un triangle ABC dont les longueurs des côtés sont a, b et c, et dont les rayons des cercles circonscrit et inscrit sont R et r respectivement. Démontrer que

$$\frac{24r^2}{R} \leq \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \leq \frac{4r^2 - 2Rr}{r}.$$

4558. *Proposé par Thanos Kalogerakis, Leonard Giugiuc et Kadir Altintas.*

Le diagramme ci-bas montre un triangle rectangle SAT tel que $\tan T > 2$. Le point K se trouve sur le segment ST de façon à ce que $SK = 2KT$. Le cercle centré à K de rayon KS intersecte la ligne AT en P et Q. Le point D est la projection de S vers AT et E est un point sur AT tel que D se trouve sur AE et $AD = 2DE$. Enfin, supposer que SQ et SP intersectent la perpendiculaire, vers AT au point E, dans les points B et C respectivement. Démontrer que S est le centre du cercle inscrit du triangle ABC.



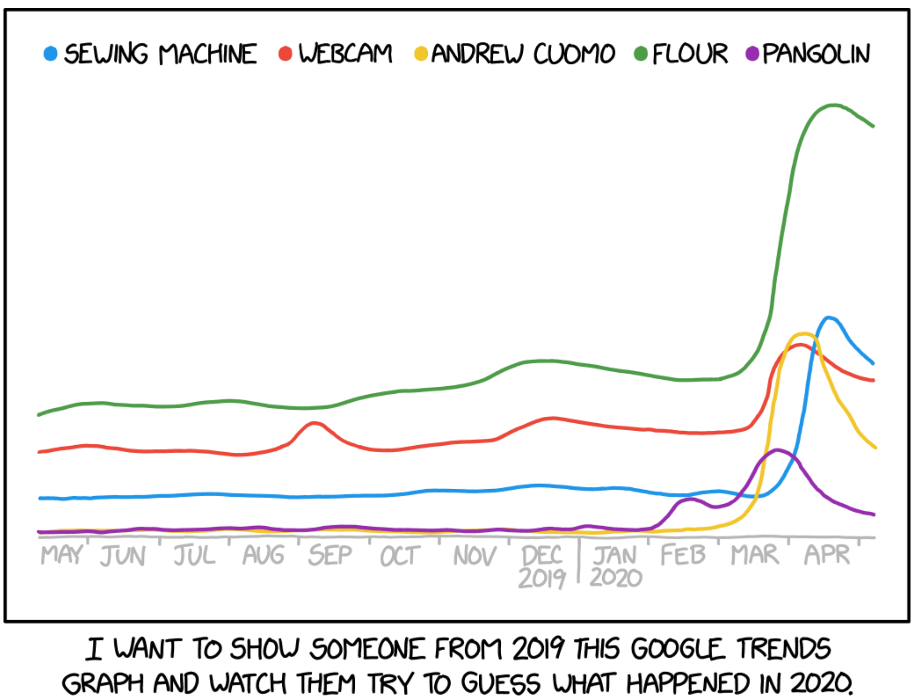
4559. *Proposé par Nho Nguyen Van.*

Soient x_k des nombres réels positifs. Démontrer, pour chaque nombre naturel $n \geq 2$, que

$$\left(\sum_{k=1}^{n} x_k^{10}\right)^3 \geq \left(\sum_{k=1}^{n} x_k^{15}\right)^2$$

4560. *Proposé par Mihaela Berindeanu.*

Soient E et F les mi points des côtés CA et AB du triangle ABC, respectivement, et soit P le deuxième point d'intersection des cercles ABE et ACF. Démontrer que le cercle AEF intersecte la ligne AP, de nouveau, dans le point P pour lequel $AX = 2XP$.



https://xkcd.com/2302/

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

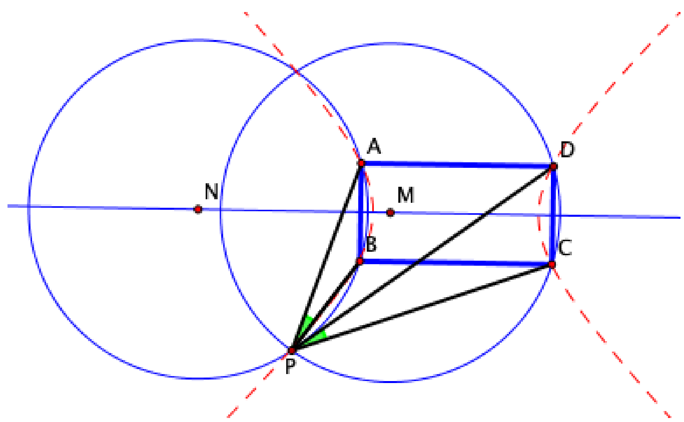
Statements of the problems in this section originally appear in 2020: 46(1), p. 32–35.

4501. *Proposed by Václav Konečný, modified by the Board.*

Given the rectangle whose vertices have Cartesian coordinates $A(0, b)$, $B(0, 0)$, $C(a, 0)$, $D(a, b)$, find the equation of the locus of points $P(x, y)$ in the third quadrant (with $x, y < 0$) for which $\angle BPA = \angle CPD$.

Comment from the proposer: this problem was inspired by problem #4301 in Crux 44(1) proposed by Bill Sands.

We received 14 submissions, all complete and correct. Our featured solution is by Ivko Dimitrić.



Since $\angle BPA = \angle CPD$, equal-length segments AB and CD subtend equal angles at P. It follows that the circumcircles of triangles BPA and CPD must have the same radius. The sides AB and CD of the rectangle are parallel chords of these two circumcircles, so their respective centers N and M lie on the common perpendicular bisector of AB and CD (the line $y = b/2$). Then NB and MC have the same length, namely the common radius of the two circumcircles. Since $MN \parallel BC$, the quadrilateral $BCNM$ is either a parallelogram or an equilateral trapezoid. Because P is in the third quadrant while the rectangle is in the first, triangles PBA and PCD are obtuse, so N and B lie on the opposite sides of line AP; hence $\angle NBC$ is obtuse. Likewise, M and C are separated by the line PD and thus $\angle MCB$ is acute. Therefore, $BCMN$ is a parallelogram. Let coordinates of M be $M(t, b/2)$ for some real number t. Since $MN = CB$, then N has coordinates $N(t-a, b/2)$. Point P in the third quadrant $\mathcal{Q}$ is one of the two points of intersection of the two congruent circumcircles, so its distance to M and N is the same as the radius of either circle, say NB. Moreover, since $PN = PM$, P lies on the perpendicular bisector of MN. Since the midpoint S of MN is $S(t - \frac{a}{2}, \frac{b}{2})$, point P lies on the line $x = t - \frac{a}{2}$ (where the circles centered at N and M, both of squared radius equal to $NB^2 = (t-a)^2 + \frac{b^2}{4}$, meet). The intersection of the two circles and the bisector of MN is obtained by solving

$$(x-t)^2 + \left(y - \frac{b}{2}\right)^2 = (t-a)^2 + \frac{b^2}{4}, \qquad x = t - \frac{a}{2}. \tag{1}$$

Eliminating $t = x + \frac{a}{2}$ from (1) gives

$$\frac{a^2}{4} + \left(y - \frac{b}{2}\right)^2 = \left(x - \frac{a}{2}\right)^2 + \frac{b^2}{4} \tag{2}$$

or

$$\left(x - \frac{a}{2}\right)^2 - \left(y - \frac{b}{2}\right)^2 = \frac{a^2 - b^2}{4}. \tag{3}$$

If $a = b$ (so that $ABCD$ is a square), from (3) we get

$$y - \frac{a}{2} = \pm\left(x - \frac{a}{2}\right), \quad \text{so that} \quad y = x \quad \text{or} \quad y = -x + a.$$

The line $y = -x + a$ has no points in $\mathcal{Q}$; the locus of P in this case, therefore, is contained in the portion of the line $y = x,\ x < 0$.

If $a > b$, the equation (3) defines an equilateral hyperbola centered at $(a/2, b/2)$ with real axis given by $y = \frac{b}{2}$; we take only that portion of the left branch of the hyperbola given by equation (3) which is in $\mathcal{Q}$.

If $a < b$, the equation (3) gives the hyperbola (conjugate to the one above) with real axis $x = \frac{a}{2}$; we take that portion of its lower branch which lies in $\mathcal{Q}$.

In each of the three cases, the indicated portions of the curve (3) contain the locus of points P.

Conversely, if P is a point in $\mathcal{Q}$ that belongs to the curve defined by (3), by reversing the argument we see that $P(x, y)$ satisfies (2) and (1) with $t = x + \frac{a}{2}$. Thus, P is equidistant from points $M(t, b/2)$ and $N(t-a, b/2)$ (since P belongs to the perpendicular bisector of MN). Because the right-hand side of (1) is NB^2 and $BCMN$ is a parallelogram, we have $MC = NB$, so the circle given in (1) centered at M has radius MC, passes through C and D and is hence the circumcircle of $\triangle PCD$. Further, since $PN = PM = MC = NB$, the circle centered at N with radius NB is the circumcircle of $\triangle PBA$ and the two circumcircles have equal radii. So, equal-length chords AB and CD in these two congruent circles have equal inscribed angles at their intersection point P so $\angle BPA = \angle CPD$.

This shows that the locus of points P is that portion which is in $\mathcal{Q}$ of the hyperbola or line defined by equation (3).

Editor's comments. Only one reader observed that the featured solution to problem 4301 [2019: 38-39] included the solution to the current problem (although the notation was different). The argument there equated the tangents of the relevant angles, an approach used by all but three solvers of this problem. Our featured solution above, as well as all but one of the other solutions we received, assumes tacitly that a and b are positive (so that the rectangle sits in the first quadrant in agreement with problem 4301). Only Marie-Nicole Gras considered undirected angles: this allows $\angle BPA = \angle APB$ so that the given angles, namely $\angle BPA$ and $\angle CPD$, are equal or supplementary. She proved that any point P that satisfied the angle requirement could lie on the line $x = \frac{a}{2}$ (the perpendicular bisector of the horizontal sides of the rectangle) and $x^2 + y^2 - ax - by = 0$ (the circumcircle of the rectangle) as well as on the hyperbola $x^2 - y^2 - ax + by = 0$ that everybody else found. When it is assumed that $a > 0$ (as most readers assumed), neither of these two further possibilities have points in the third quadrant; however, when $a < 0$ (so that the given rectangle sits in the second or third quadrant), the locus of P consists of those parts of the hyperbola, circle, and line that lie in the third quadrant.

4502. *Proposed by George Apostolopoulos.*

Let a, b, c be the side lengths of triangle ABC with inradius r and circumradius R. Prove that

$$\frac{3}{2} \cdot \frac{r}{R} \leq \sum_{\text{cyc}} \frac{a}{2a+b+c} \leq \frac{3}{8} \cdot \frac{R}{r}.$$

We received 21 submissions, of which 20 were correct. We present a composite of solutions by UCLan Cyprus Problem Solving Group (for the right inequality) and Subhankar Gayen (for the left inequality).

To prove the right inequality, note first that

$$\frac{1}{a+b+c} - \frac{1}{2a+b+c} = \frac{a}{(2a+b+c)(a+b+c)}.$$

So

$$\begin{aligned}\sum_{\text{cyc}} \frac{a}{2a+b+c} &= \sum_{\text{cyc}} \left(\frac{a}{a+b+c} - \frac{a^2}{(2a+b+c)(a+b+c)} \right) \\ &= 1 - \frac{1}{a+b+c} \sum_{\text{cyc}} \frac{a^2}{(2a+b+c)}. \qquad (1)\end{aligned}$$

Now by Titu's Lemma, a special case of the Cauchy-Schwarz Inequality, we have

$$\sum_{\text{cyc}} \frac{a^2}{2a+b+c} \geqslant \frac{(a+b+c)^2}{4(a+b+c)} = \frac{a+b+c}{4},$$

so

$$\frac{1}{a+b+c}\sum_{\text{cyc}}\frac{a^2}{2a+b+c} \geq \frac{1}{4}. \tag{2}$$

From (1) and (2), we obtain

$$\sum_{\text{cyc}}\frac{a}{2a+b+c} \leqslant 1-\frac{1}{4}=\frac{3}{4}\leqslant\frac{3R}{8r}$$

since $2r \leq R$ by well-known Euler's Inequality.

To prove the left inequality, let s denote the semiperimeter of $\triangle ABC$. By Titu's inequality again, we have

$$\begin{aligned}\sum_{\text{cyc}}\frac{a}{2a+b+c} &= \sum_{\text{cyc}}\frac{a^2}{2s^2+ab+ac}\\ &\geq \frac{(a+b+c)^2}{2(a^2+b^2+c^2+ab+bc+ca)}=\frac{2s^2}{a^2+b^2+c^2+ab+bc+ca}.\end{aligned} \tag{3}$$

Next, it is well-known that

$$2s^2 \geq 27Rr \tag{4}$$

and

$$a^2+b^2+c^2 \leq 9R^2 \tag{5}$$

(see Items 5.12 and 5.13 on p.52 of *Geometric Inequalities* by O. Bottema et al.). Using (3), (4) and (5) together with the obvious fact that $a^2+b^2+c^2 \geq ab+bc+ca$, we then obtain

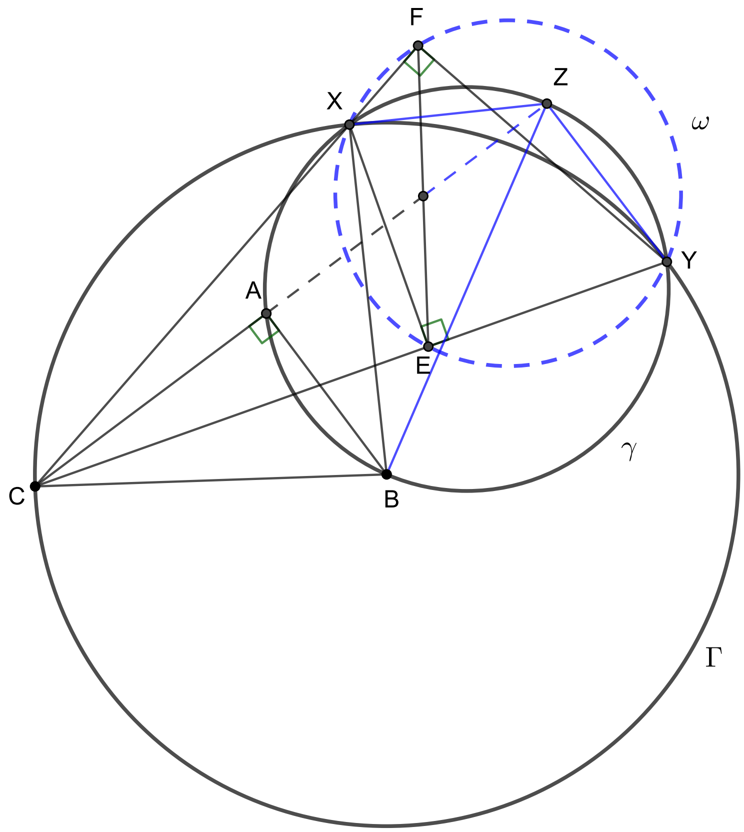
$$\sum_{\text{cyc}}\frac{a}{2a+b+c} \geq \frac{s^2}{a^2+b^2+c^2} \geq \frac{s^2}{9R^2} \geq \frac{1}{9R^2}\cdot\frac{27Rr}{2}=\frac{3r}{2R}$$

completing the proof.

4503. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90^\circ$ and let Γ be the circle with centre B and radius BC. A circle γ passing through B and A intersects Γ at X, Y with $X \neq Y$. Let E and F be the orthogonal projections of X and Y onto CY and CX, respectively. Prove that the line CA bisects EF.

We received 10 submissions, of which 9 were correct and complete. We present the solution by UCLan Cyprus Problem Solving Group.



Let Z be the antipodal point of B on the circle γ. Then $\angle BAZ = 90^\circ$. Since also $\angle BAC = 90^\circ$ then C, A, Z are collinear and it is therefore enough to prove that CZ bisects EF.

Now BZ is a diameter of γ, so $\angle BXZ = \angle BYZ = 90^\circ$; that is, ZX and ZY are tangents of Γ at X and Y respectively. Since Γ is the circumcircle of $\triangle CXY$, it follows that CZ is the C-symmedian of $\triangle CXY$.

Since $XE \perp CY$ and $YF \perp CX$, then X, Y, E, F are concyclic on a circle ω of diameter XY. By the power of the point C with respect to ω we get $(CE)(CY) = (CF)(CX)$ which gives that the triangles CEF and CXY are similar.

We can map $\triangle CXY$ to $\triangle CEF$ by first reflecting on their common bisector of angle C and then applying a homothety with center C. Under this transformation, the midpoint M of XY is mapped on the C-symmedian CZ of CXY and then stays on CZ. Therefore, since XY is mapped to EF, M is mapped on the intersection of CZ with EF. Thus, this intersection is the midpoint of EF.

4504. *Proposed by Warut Suksompong.*

Find all positive integers (a, b, c, x, y, z), $a \le b \le c$ and $x \le y \le z$, for which the following two equations hold:

$$a + b + c = xy + yz + zx,$$
$$x + y + z = abc.$$

We received 16 submissions, all of which were correct though some of them so condensed that it was difficult to justify considering them completely valid. We present the solution by Vasile Teodorovici, modified and enhanced by the editor.

Note first that from the given conditions, we have

$$3c \ge a + b + c \ge x^2 + xy + yz = x(x + y + z) = abcx,$$

so

$$ab \le \frac{3}{x} \quad \text{and} \quad x \le \frac{3}{ab} \le 3 \qquad \Longrightarrow x = 1, 2, 3.$$

Hence, there are 3 cases to be considered.

Case 1: $x = 3$. Then $ab \le 1 \implies a = b = 1 \implies c + 2 = yz + 3(y + z)$ and $y + z = c - 3 \implies yz = c + 2 - 3(c - 3) = -2c + 11$, which is impossible since $yz \ge 9 \implies 2c \le 2 \implies c = 1 \implies y + z = -2$. So there are no solutions.

Case 2: $x = 2$. Then $ab \le \frac{3}{2} \implies a = b = 1 \implies c + 2 = yz + 2(y + z)$ and $y + z = c - 2 \implies yz = c + 2 - 2(c - 2) = -c + 6 < 6$. But since $c - 2 = y + z \ge 4$, then $c \ge 6$, a contradiction. So there are no solutions.

Case 3: $x = 1$. Then $ab \le 3 \implies (a, b) = (1, 1), (1, 2)$ or $(1, 3)$, so we have 3 subcases to consider.

Case 3i: $(a, b) = (1, 1)$. Then $c + 2 = yz + y + z$ and $y + z = c - 1$, hence

$$yz = c + 2 - (c - 1) = 3 \implies (y, z) = (1, 3) \implies c + 2 = 7 \implies c = 5,$$

so we have the solution $(a, b, c, x, y, z) = (1, 1, 5, 1, 1, 3)$.

Case 3ii: $(a, b) = (1, 2)$. Then $c + 3 = yz + y + z$ and $y + z = 2c - 1$, hence

$$yz = c + 3 - (2c - 1) = -c + 4 \implies -c + 4 \ge 1 \implies c \le 3 \implies c = 2 \text{ or } 3.$$

But $c = 3 \implies yz = 1 \implies y = z = 1 \implies c + 3 = 3$, a contradiction. Then $c = 2 \implies y + z = 3 \implies (y, z) = (1, 2)$ and we get the solution $(a, b, c, x, y, z) = (1, 2, 2, 1, 1, 2)$.

Case 3ii: $(a, b) = (1, 3)$. Then $c + 4 = yz + y + z$ and $y + z = 3c - 1$, hence

$$yz = c + 4 - (3c - 1) = -2c + 5 < 0,$$

since $c \ge b \implies c \ge 3$, a contradiction. Hence there are no solutions.

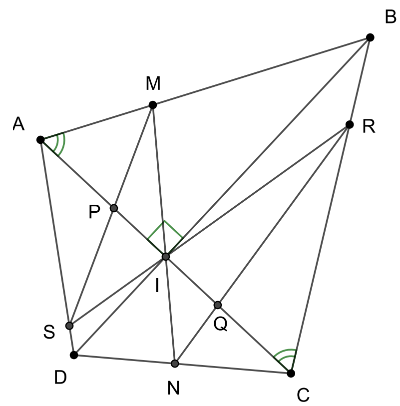
To summarize, there are exactly two solutions given by

$$(a, b, c, x, y, z) = (1, 1, 5, 1, 1, 3), (1, 2, 2, 1, 1, 2).$$

4505. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral such that $AC \perp BD$ and $AB = BC$. Let I denote the point of intersection of AC and BD. A straight line l_1 passes through I and intersects BC and AD in R and S, respectively. Similarly, straight line l_2 passes through I and intersects AB and CD in M and N, respectively. The lines MS and RN intersect AC at P and Q, respectively. Prove that $IP = IQ$.

We received 13 submissions, of which 12 were complete and correct. We present the solution by the UCLan Cyprus Problem Solving Group.



Consider the self-intersecting hexagon $AMNCRS$. Note that $AM \cap CR = B$, $MN \cap RS = I$ and $NC \cap SA = D$. By the converse of Pascal's Theorem, since B, I and D are collinear, the points A, M, N, C, R and S lie on a conic Γ. Since I is the midpoint of the chord AC in Γ, and the chords MN and RS pass through I, we can apply the Butterfly Theorem for conics to Γ and the degenerate conic given by the pair of lines MS and RN to conclude that I is also the midpoint of PQ.

Editor's Comments. The UCLan Cyprus Problem Solving Group cited *A survey of Geometry* by H. Eves (1972) for the Butterfly Theorem (p. 255). One of the early editions of *Crux*, Vol. 2 (1) from 1976 (when it was still called *Eureka*), has an article *The Celebrated Butterfly Problem* by Léo Sauvé, which also lists this theorem; it can be found in the online archive.

4506. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} \dfrac{a_{n+1}}{na_n} = a$, where $a \in \mathbb{R}_+^*$. Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}}.$$

We received 15 submissions, of which 14 were complete and correct. We present the solution by Florentin Visescu.

Let $x_n = \sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}}$ and $y_n = n$. Note that y_n is strictly increasing and $y_n \to \infty$. Moreover,

$$\lim_{n\to\infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \lim_{n\to\infty} \frac{n+1}{\sqrt[n+1]{a_{n+1}}} = \lim_{n\to\infty} \frac{n}{\sqrt[n]{a_n}} = \lim_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}}.$$

By the Cezàro-Stolz Theorem, if this limit exists and is equal to L, then $\lim\limits_{n\to\infty} \frac{x_n}{y_n}$ is equal to L as well. We will show below that this limit is $\frac{e}{a}$.

Let $z_n = \frac{n^n}{a_n}$. By Cauchy-d'Alembert, if $\lim\limits_{n\to\infty} \frac{z_{n+1}}{z_n}$ exists, then $\lim\limits_{n\to\infty} \sqrt[n]{z_n}$ also exists and is equal to it. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{a_{n+1}} \cdot \frac{a_n}{n^n} &= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \frac{na_n}{a_{n+1}} \cdot \frac{n+1}{n} \\ &= e \cdot \lim_{n\to\infty} \frac{na_n}{a_{n+1}} = \frac{e}{a}.\end{aligned}$$

We conclude that

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^n}{a_n}} = \frac{e}{a}$$

as well; that is, $L = \frac{e}{a}$. Therefore,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}} = \frac{e}{a}.$$

4507. *Proposed by Eduardo Silva.*

Suppose that $a_0 < \cdots < a_r$ are integers. If $\{b_i\}$ are distinct integers with $a_i \leq b_i$, for each i, and σ is a permutation so that $b_{\sigma(0)} < \cdots < b_{\sigma(r)}$, prove that $a_i \leq b_{\sigma(i)}$ for each i. Further, if $a_j = b_{\sigma(j)}$ for some j, then $\sigma(j) = j$, so that $a_j = b_j$.

We received 6 solutions, all correct. We present the solution by the UCLan Cyprus Problem Solving Group.

Assume that $a_i \geq b_{\sigma(i)}$ for some i. We have $b_j \geq a_j > a_i \geq b_{\sigma(i)}$ for each $j > i$, therefore $\sigma(i) \leq i$.

Similarly, we have $b_j \geq a_j \geq a_i \geq b_{\sigma(i)} > b_{\sigma(k)}$ for each $k < i \leq j$. Thus $\sigma(k) \neq j$ for each $k < i \leq j$. In particular, we must have that

$$\{\sigma(0), \ldots, \sigma(i-1)\}\} \subseteq \{0, 1, \ldots, i-1\}$$

and therefore $\{\sigma(0), \ldots, \sigma(i-1)\} = \{0, 1, \ldots, i-1\}$.

From the above we get that if $a_i \geq b_{\sigma(i)}$ for some i, then $\sigma(i) = i$ and therefore $a_i = b_i = b_{\sigma(i)}$. This proves both results.

4508. *Proposed by Hung Nguyen Viet.*

Let x, y, z be nonzero real numbers such that $x + y + z = 0$. Find the minimum possible value of

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

We received 27 correct solutions and 3 incorrect solutions. There was great variation in the amount of trouble encountered by the solvers in arriving at the result, and several resorted to calculus. We present three solutions.

For convenience, let

$$f(x, y, z) = (x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

Solution 1, by the proposer.

At least two variables have the same sign; suppose them to be x and y. Since

$$0 < 4xy \leq z^2 = (x + y)^2 \leq 2(x^2 + y^2),$$

we have that

$$f(x, y, z) \geq \frac{3z^2}{2}\left[\frac{2}{xy} + \frac{1}{z^2}\right] \geq \frac{3z^2}{2}\left[\frac{8}{z^2} + \frac{1}{z^2}\right] = \frac{27}{2},$$

with equality if and only if $(x, y, z) = (t, t, -2t)$ for some nonzero t.

Solution 2, by Brian Bradie.

Let x and y have the same sign and replace z^2 by $(x + y)^2$. Then

$$f(x, y, z) = 6 + 2\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + 2\left(\frac{x}{y} + \frac{y}{x}\right) - \frac{2xy}{(x + y)^2} \geq 6 + 4 + 4 - \frac{1}{2} = \frac{27}{2}$$

with equality if and only if $x = y = t$ and $z = -2t$ for some nonzero t.

Solution 3, by Oliver Geupel and Joel Schlosberg (independently).

Replacing z^2 by $(x + y)^2$, we find that

$$f(x, y, z) = \frac{2(x^2 + xy + y^2)^3}{x^2y^2(x + y)^2} = \frac{27}{2} + \frac{1}{2}\left[\frac{(x - y)(2x + y)(x + 2y)}{xy(x + y)}\right]^2.$$

Therefore, the minimum value of $f(x, y, z)$ is $27/2$, which occurs exactly when $x = y$, $y = -2x$ or $x = -2y$.

Editor's Comments. Several solvers made the substitution $t = \frac{x}{y} + \frac{y}{x}$, so that

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} = t^2 - 2, \qquad \frac{(x + y)^2}{xy} = t + 2,$$

and

$$\frac{x^2+y^2}{(x+y)^2} = 1 - \frac{2xy}{(x+y)^2} = 1 - \frac{2}{t+2}.$$

Then

$$f(x,y,z) = 2\left(t^2+t+1-\frac{1}{t+2}\right) = \frac{2(t+1)^3}{t+2},$$

which can be conveniently analyzed. Another way of reducing the problem to a single variable situation is to take advantage of the homogeneity, assign a numerical value $-c$ to z (say -1) and let $y = c - x$.

4509. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let B and C be two distinct fixed points that lie in the plane α and let M be the midpoint of BC. Find the locus of points $A \in \alpha$, $A \notin BC$, for which $4R \cdot AM = AB^2 + AC^2$, where R is the circumradius of ABC.

We received 15 solutions, 11 of which were correct and complete. We present two solutions.

Solution 1, by Theo Koupelis.

Because AM is a median, we have

$$4 \cdot AM^2 = 2(AB^2 + AC^2) - BC^2 = 8R \cdot AM - BC^2.$$

This is a quadratic in AM with solutions

$$AM = R \pm \sqrt{R^2 - BM^2},$$

or, if O is the center of the circumcircle,

$$AM = AO \pm OM.$$

- When $O \equiv M$, then the locus includes all points on the circle (M, R), where $R = BC/2$, except the points B and C.
- When $O \not\equiv M$, then by the triangle inequality for triangle AOM we find that the points A, O, M are colinear, and therefore the locus includes all the points on the perpendicular bisector to BC, excluding the point M.

Solution 2, by Marie-Nicole Gras.

Let $a = BC$, $b = CA$ and $c = AB$ be the side lengths of $\triangle ABC$. We recall the well known formulas:

$$a = 2R\sin A,\ 4AM^2 = b^2 + c^2 + 2bc\cos A,\ BC^2 = b^2 + c^2 - 2bc\cos A.$$

We obtain

$$\begin{aligned}
&4R\cdot AM = AB^2+BC^2\\
&\quad\iff 4R^2\cdot 4(AM)^2=(b^2+c^2)^2\\
&\quad\iff \frac{a^2}{\sin^2 A}\big(b^2+c^2+2bc\cos A\big)=\big(b^2+c^2\big)^2\\
&\quad\iff (b^2+c^2-2bc\cos A)(b^2+c^2+2bc\cos A)=(b^2+c^2)^2\sin^2 A\\
&\quad\iff (b^2+c^2)^2-4b^2c^2cos^2A=(b^2+c^2)^2\sin^2 A\\
&\quad\iff (b^2+c^2)^2\cos^2 A=4b^2c^2\cos^2 A\\
&\quad\iff \cos^2 A\cdot(b^2-c^2)=0.
\end{aligned}$$

It follows that the locus of points A, $A\notin BC$, for which $4R\cdot AM=AB^2+BC^2$ is the union of two curves:

- the circle of diameter BC minus points B and C;
- the perpendicular bissector of BC, minus point M.

4510. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a non-obtuse triangle. Prove that

$$\cos A\cos B+\cos A\cos C+\cos B\cos C>2\sqrt{\cos A\cos B\cos C}.$$

We received 14 submissions, all correct. We present the solution by Subhankar Gayen, modified slightly by the editor.

Note first that if one of A,B or C equals $\pi/2$, then $\cos A\cos B\cos C=0$ and $\sum_{\text{cyc}}\cos A\cos B>0$, so the given inequality is obvious. Hence we may assume that $\cos A\cos B\cos C\neq 0$.

Now the given inequality is clearly equivalent to

$$\sqrt{\frac{\cos A\cos B}{\cos C}}+\sqrt{\frac{\cos B\cos C}{\cos A}}+\sqrt{\frac{\cos C\cos A}{\cos B}}>2. \tag{1}$$

Next, note that

$$\begin{aligned}
&\cos B\cos C(\tan B+\tan C)=\tan A\cos A \iff\\
&\sin B\cos C+\cos B\sin C=\sin A \iff\\
&\sin(B+C)=\sin A \iff\\
&\sin(\pi-A)=\sin A,
\end{aligned}$$

which is true, so we get

$$\frac{\cos B\cos C}{\cos A}=\frac{\tan A}{\tan B+\tan C}. \tag{2}$$

(Note: identity (2) was used in the featured solution to ***Crux*** problem 4053, p.273, Vol. 42(6), 2016.)

Using (2) and two other similar identities, we see by setting $\alpha = \tan A, \beta = \tan B$ and $\gamma = \tan C$, that it suffices to show that

$$\sum_{\text{cyc}} \sqrt{\frac{\alpha}{\beta + \gamma}} > 2. \tag{3}$$

Note that for all $x, y, z > 0$, we have

$$\sqrt{\frac{x}{y+z}} \geq \frac{2x}{x+y+z} \iff (x+y+z)^2 \geq 4x(y+z) \iff (y+z-x)^2 \geq 0,$$

which is true and equality holds if and only if $x = y + z, y = z + x$ and $z = x + y$.

Therefore,

$$\sum_{\text{cyc}} \sqrt{\frac{\alpha}{\beta+\gamma}} \geq \sum_{\text{cyc}} \frac{2\alpha}{\alpha+\beta+\gamma} = \frac{2(\alpha+\beta+\gamma)}{\alpha+\beta+\gamma} = 2$$

with equality if and only if $\alpha = \beta + \gamma, \beta = \alpha + \gamma, \gamma = \alpha + \beta$ or $\alpha = \beta = \gamma = 0$, contradiction.

Hence (3) follows, completing the proof.