Crux Mathematicorum

Volume/tome 48, issue/numéro 1
January/janvier 2022

Canadian Mathematical Society
Société mathématique du Canada

Crux Mathematicorum is a problem-solving journal at the secondary and university undergraduate levels, published online by the Canadian Mathematical Society. Its aim is primarily educational; it is not a research journal. Online submission:

`https://publications.cms.math.ca/cruxbox/`

Crux Mathematicorum est une publication de résolution de problèmes de niveau secondaire et de premier cycle universitaire publiée par la Société mathématique du Canada. Principalement de nature éducative, le Crux n'est pas une revue scientifique. Soumission en ligne:

`https://publications.cms.math.ca/cruxbox/`

ISSN 1496-4309 (Online)

ISSN 1496-4309 (électronique)

Supported by / Soutenu par :

- Intact Financial Corporation
- University of the Fraser Valley

Editorial Board

IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

Happy New Year 2022 and welcome to Volume 48 of ***Crux*** !

As I write this, the pandemic's n-th wave is causing the new wave of restrictions across Canada: only small gatherings are allowed, schools are closed for 2 weeks, universities are moving online for a month. While we somewhat learned to adapt to ever-changing conditions, the persistent emotion is wanting to "go back to normal" despite the fact that said normal no longer exists. So at ***Crux***, we aren't looking back, but rather embracing new audiences and moving forward with more material.

In this Volume, we are introducing a few new features. The new *MathemAttic* column *From the bookshelf of ...* will highlight new and old books, books we have had on our shelves for a while as well as new publications, well-loved books and newly discovered treasures that will interest and inspire our secondary level student and teacher readers. Readers are encouraged to share personal selections. On the other end of the spectrum, the column *From the lecture notes of ...* will feature problems from undergraduate classes that instructors find particularly nice or insightful, with unusual applications of standard techniques. We want to encourage all instructors to "steal" these problems, but reciprocate back through sharing their own in future issues.

Our regular sections of problems, solutions and articles are of course going forward as well. This year we will also see the publication of the *MathemAttic* article writing contest winners.

2022, here we come!

Kseniya Garaschuk

MathemAttic

No. 31

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 30, 2022**.

MA151. *Proposed by Mihaela Berindeanu.*

Solve over real numbers:

$$\sqrt{7+x}+\sqrt{18-x}=x^2-11x+25.$$

MA152. *Proposed by Neculai Stanciu.*

Prove the following cryptarithm, where each letter represents a different digit:

$$DEAD \times REAR < READ \times DEAR.$$

MA153. *Proposed by Roy Barbara.*

Let a, b, x and y be rational numbers so that $x \geq 0$, $a > 0$ and $\sqrt{a}$ is not rational. Suppose further that $\sqrt[3]{\sqrt{a}+b}=\sqrt{x}+y$. Prove that $\sqrt[3]{a-b^2}$ is a rational number.

MA154. Two bags, Bag A and Bag B, each contain 9 balls. The 9 balls in each bag are numbered from 1 to 9. Suppose one ball is removed randomly from Bag A and another ball from Bag B. If X is the sum of the numbers on the balls left in Bag A and Y is the sum of the numbers of the balls remaining in Bag B, what is the probability that X and Y differ by a multiple of 4?

MA155. An arbitrary point is selected inside an equilateral triangle. From this point perpendiculars are dropped to each side of the triangle. Show that the sum of the lengths of these perpendiculars is equal to the length of the altitude of the triangle.

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2022**.

MA151. *Proposeé par Mihaela Berindeanu.*

Dans l'ensemble des nombres reels, résoudre

$$\sqrt{7+x}+\sqrt{18-x}=x^2-11x+25.$$

MA152. *Proposeé par Neculai Stanciu.*

Démontrer le cryptarithme qui suit, où chaque lettre représente un chiffre différent:

$$SOUS \times TOUT < SOUT \times TOUS.$$

MA153. *Proposeé par Roy Barbara.*

Soient a, b, x et y des nombres rationnels tels que $x \geq 0$, $a > 0$, puis tels que $\sqrt{a}$ n'est pas rationnel. De plus, supposer que $\sqrt[3]{\sqrt{a}+b}=\sqrt{x}+y$. Démontrer que $\sqrt[3]{a-b^2}$ est un nombre rationnel.

MA154. Deux sacs, Sac A et Sac B, contiennent chacun 9 billes, les 9 billes de chaque sac étant numérotées de 1 à 9. Supposer qu'on enlève, de façon aléatoire, une bille de chaque sac. Si X dénote la somme des nombres des billes restantes dans le Sac A et Y dénote la somme des nombres des billes restantes dans le Sac B, déterminer la probabilité que X et Y diffèrent par un multiple de 4.

MA155. Un point quelconque se trouve à l'intérieur d'un triangle équilatéral. À partir de ce point, sont tracées les perpendiculaires vers les côtés du triangle. Démontrer que la somme des longueurs de ces perpendiculaires égale la longueur de l'altitude du triangle.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(6), p. 275–276.

MA126. Let A, B, C, X, Y represent distinct, non-zero digits. Consider the following subtraction (and specific example, taking $(A, B, C, X, Y) = (4, 5, 2, 9, 8)$):

$$\begin{array}{cccc} & A & B & C \\ - & C & B & A \\ \hline & 1 & X & Y \end{array} \qquad \text{Example:} \qquad \begin{array}{cccc} & 4 & 5 & 2 \\ - & 2 & 5 & 4 \\ \hline & 1 & 9 & 8 \end{array}$$

How many ordered quintuplets (A, B, C, X, Y) are there that satisfy the subtraction shown above?

Originally from 2013 Manitoba Mathematical Competition, question 8b (slightly reworded).

We received 6 submissions, of which 4 were correct and complete. We present the solution by Alex Bloom.

We can rewrite the given equation to get

$$100A + 10B + C - 100C - 10B - A = 100 + 10X + Y.$$

Simplifying, we get

$$99A - 99C = 100 + 10X + Y,$$

so

$$99(A - C) = 100 + 10X + Y.$$

We know that A, C, X, and Y are all nonzero digits, so $200 > 100+10X+Y > 100$. Given that $(A - C)$ is an integer, it must be 2, as 198 is the only multiple of 99 between 100 and 200. Thus, $100 + 10X + Y = 198$, giving $X = 9$ and $Y = 8$. As we stated earlier, $A - C = 2$, and since the terms with B cancel out, B can be any digit not already taken. We will start by setting A, which can be 7, 6, 5, 4, or 3, as 9 and 8 are taken by X and Y and $A = 2$ would give $C = 0$, and the question specifies that each variable is nonzero. For any of these 5 working cases, C would be determined, and B could be any of the $9 - 4 = 5$ unchosen nonzero digits left. This gives $5 \cdot 5 = 25$ working quintuplets.

MA127. If $\log_{10} 2 = a$ and $\log_{10} 3 = b$, find $\log_5 12$.

Originally from 21st W.J. Blundon Mathematics Contest (2004), problem 8a.

We received 14 submissions, of which 13 were correct and complete. We present the solution by Morgan Orr, Ashley Herbig and Eli Lutz.

Using the change of base formula

$$\log_y x = \frac{\log_c x}{\log_c y},$$

we have

$$\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5}.$$

Now,

$$\log_{10} 12 = \log_{10}(2 \cdot 2 \cdot 3) = \log_{10} 2 + \log_{10} 2 + \log_{10} 3 = 2a + b$$

and

$$\log_{10} 5 = \log_{10} \frac{10}{2} = \log_{10} 10 - \log_{10} 2 = 1 - a,$$

so

$$\log_5 12 = \frac{\log_{10} 12}{\log_{10} 5} = \frac{2a+b}{1-a}.$$

MA128. I invested \$100. Each day, including the 1st day, my investment first increased in value by $p\%$, then decreased in value. The 1st day's decrease was one-quarter of the 1st day's increase. The 2nd day's decrease was two-quarters of the 2nd day's increase. In general, the nth day's decrease was n-quarters of the nth day's increase. (Note that, from day 5 on, the decrease exceeded the increase.) If my investment first became worthless on the 1000th day, what was the value of p?

Originally from Canadian National Mathematics League, Contest 3, January 1994, problem 3-6.

We received four submissions, all of which were correct. We present the solution by Vishak Srikanth, modified by the editor.

On the k^{th} day, the stock first increases by $p\%$ and then decreases by $\frac{kp}{4}\%$ of the value at the beginning of the k^{th} day. Therefore, on the 1000^{th} day, the stock increases $p\%$ then decreases $\frac{1000p}{4}\%$ of the value at the beginning of the 1000^{th} day. Let the stock value at the end of the 999^{th} day be X. At the end of the 1000^{th} day the stock value is zero. Therefore,

$$X\left(1 + \frac{p}{100} - \frac{1000p}{4 \times 100}\right) = 0.$$

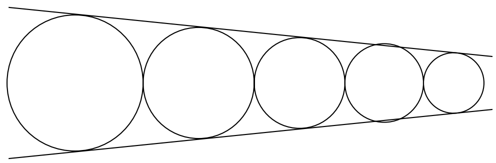
Since the stock price has a non-zero value on the 999^{th} day, $X \neq 0$. Thus

$$1 + \frac{p}{100} - \frac{1000p}{4 \times 100} = 0$$

and so

$$p = \frac{100}{249}.$$

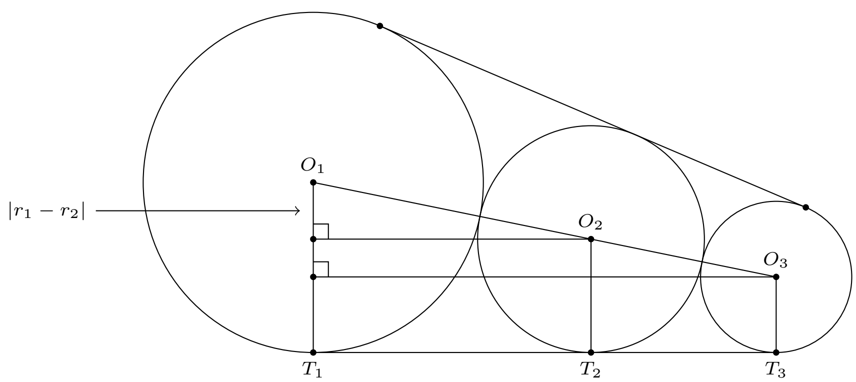
MA129. Five marbles of various sizes are placed in a conical funnel of circular cross section. Each marble is in contact with the adjacent marble(s) and with the funnel wall. The smallest marble has a radius of 8 mm. The largest marble has a radius of 18 mm. Determine the radius, measured in mm, of the middle marble.



Originally from 2012 BC Secondary School Mathematics Contest, Senior Final, Part B, problem 3.

We received 8 solutions, of which 4 were correct. The solutions that were deemed incorrect lacked precision or rigour, such as in arguments about polygon similarity. We present the solution by Miguel Amengual Covas.

Here, and throughout, $O\,(r)$ refers to the circle with centre O and radius r. Suppose $O_1\,(r_1)$, $O_2\,(r_2)$, $O_3\,(r_3)$ are three circles such that they have common external tangents, and $O_2\,(r_2)$ touches $O_1\,(r_1)$ and $O_3\,(r_3)$ externally. We claim that $r_2 = \sqrt{r_1 r_3}$.



Let the points of contact between one of the external tangent and three circles be T_1, T_2, T_3 as shown. Using the theorem that the line joining the centres of two tangent circles passes through the point of contact, and the Pythagorean theorem, we have

$$(T_1T_2)^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2 \implies T_1T_2 = 2\sqrt{r_1 r_2}.$$

Analogously,

$$T_2T_3 = 2\sqrt{r_2 r_3}.$$

Therefore,

$$T_1T_3 = T_1T_2 + T_2T_3 = 2\sqrt{r_2}\left(\sqrt{r_1} + \sqrt{r_3}\right).$$

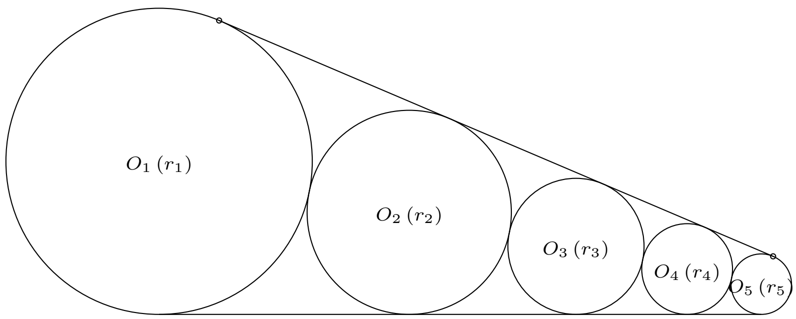
Again, by the Pythagorean theorem,

$$(T_1T_3)^2 + (r_1 - r_3)^2 = (r_1 + 2r_2 + r_3)^2 .$$

Substituting for T_1T_3 from above, and squaring and simplifying repeatedly, we obtain $r_2 = \sqrt{r_1 r_3}$.

Now we apply the lemma to the three triads

$$O_1\,(r_1)\,O_2\,(r_2)\,O_3\,(r_3)\,, \quad O_2\,(r_2)\,O_3\,(r_3)\,O_4\,(r_4)\,, \quad O_3\,(r_3)\,O_4\,(r_4)\,O_5\,(r_5)\,.$$



This yields

$$r_2 = \sqrt{r_1 r_3}, \quad r_3 = \sqrt{r_2 r_4}, \quad r_4 = \sqrt{r_3 r_5}.$$

Multiplying the first and third equations gives $r_2 r_4 = r_3\sqrt{r_1 r_5}$. Substituting r_3^2 for $r_2 r_4$ from the second equation yields

$$r_3\,(r_3 - \sqrt{r_1 r_5}) = 0.$$

Thus,

$$r_3 = \sqrt{r_1 r_5} = \sqrt{8 \cdot 18} = 12 \text{ mm}.$$

MA130. Prove that there are infinitely many positive integers k such that k^k can be expressed as the sum of the cubes of two positive integers.

Originally from 2009 Alberta High School Mathematics Competition, Part II, problem 5.

We received 11 submissions, of which 10 were correct and complete. We present the solution by Richard Hess, slightly edited.

Suppose that $k = a^3 + b^3$ for positive integers a and b and that $k \equiv 1 \pmod 3$. In this case, write $k = 3m + 1$ for m a positive integer, and note that

$$k^k = k^{3m} \cdot k = k^{3m} \cdot (a^3 + b^3) = (ak^m)^3 + (bk^m)^3;$$

that is, k^k is a sum of two cubes.

However, any choice of a and b that satisfies either $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$ or $a \equiv b \equiv 2 \pmod 3$ results in $k \equiv 1 \pmod 3$, so there are infinitely many choices of a and b that will satisfy the desired conditions.

For the case $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$ we get the solutions $k = 28$, 91, 217, 280 and so on. For the case $a \equiv b \equiv 2 \pmod 3$ we get the solutions $k = 16$, 133, 250, 520 and so on.

PROBLEM SOLVING VIGNETTES

No. 20

Shawn Godin

Playing with a Polynomial Problem

In this issue we will look at a problem involving polynomials from the 2021 *Canadian Open Mathematics Challenge* run by the CMS. Below is the last problem, part C number 4, from the contest:

> We call (F, c) a **good pair** if the following three conditions are satisfied:
>
> (1) $F(x) = a_0 + a_1x + \cdots + a_mx^m$, $(m \geq 1)$ is a nonconstant polynomial with integer coefficients.
> (2) c is a real number that is not an integer.
> (3) $F(c)$ is an integer.
>
> For example, both $\left(6x, \frac{1}{3}\right)$ and $\left(1+x^3, 5^{\frac{1}{3}}\right)$ are good pairs, but none of the following pairs $\left(6x, \frac{1}{4}\right)$, $(6x, 2)$, $\left(\frac{x}{6}, \frac{1}{3}\right)$, $\left(\frac{x^2}{6}, 6\right)$ is good.
>
> (a) Let $c = \frac{1}{2}$. Give an example of F such that (F, c) is a good pair but $(F, c+1)$ is not.
> (b) Let $c = \sqrt{2}$. Give an example of F such that both (F, c) and $(F, c+1)$ are good pairs.
> (c) Show that for any good pair (F, c), if c is rational then there exists infinitely many non-zero integers n such that $(F, c+n)$ is also a good pair.
> (d) Show that if $(F, c+n)$ is a good pair for every integer n, then c is rational.

When you are unsure how to attack a problem, it makes sense to dive in and play around with things. In many cases your investigation will trigger an "aha!" that will lead to a solution or, at least, a method of attack.

Having said that, let's start (a) by evaluating $F\left(\frac{1}{2}\right)$ and $F\left(\frac{3}{2}\right) = F\left(\frac{1}{2}+1\right)$.

$$F\left(\frac{1}{2}\right) = a_0 + \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots + \frac{a_m}{2^m}$$
$$F\left(\frac{3}{2}\right) = a_0 + \frac{3a_1}{2} + \frac{9a_2}{4} + \frac{27a_3}{8} + \cdots + \frac{3^m a_m}{2^m}$$

Trivially, we see that if we choose the coefficients so that each term in the evaluation of $F\left(\frac{1}{2}\right)$ will be an integer, then $F\left(\frac{1}{2}\right)$ will be an integer as well. However, the same logic tells us that $F\left(\frac{3}{2}\right)$ will also be an integer, which we do not want for this problem.

Note that since a_0 is an integer, it will have no effect on $F\left(\frac{1}{2}\right)$ being an integer or not, no matter what its value is. As such, to make things easy on ourselves, we will choose $a_0 = 0$. Keep in mind that if we find any solution F, then adding any integer to this polynomial gives another polynomial that satisfies the conditions of the problem.

Again, for simplicity, we will start our investigation with the case $m = 1$. Since $F\left(\frac{1}{2}\right) = \frac{a_1}{2}$, we would need a_1 to be even to make $F\left(\frac{1}{2}\right)$ an integer, but that makes $F\left(\frac{3}{2}\right)$ an integer as well, so we must keep going.

Moving on to $m = 2$, with a_1 an odd integer, we have

$$F\left(\frac{1}{2}\right) = \frac{a_1}{2} + \frac{a_2}{4} = \frac{1}{2}\left(a_1 + \frac{a_2}{2}\right)$$

from which we can see if we pick a_2 to be 2 times an odd integer, then $F\left(\frac{1}{2}\right)$ will be an integer. The simplest case being $a_1 = 1$ and $a_2 = 2$. Unfortunately, this yields

$$\begin{aligned} F\left(\frac{1}{2}\right) &= \frac{1}{2}\left(1 + \frac{2}{2}\right) = 1 \\ F\left(\frac{3}{2}\right) &= \frac{3}{2}\left(1 + \frac{3}{2}(2)\right) = 6 \end{aligned}$$

so we must keep looking.

Continuing with $m = 3$, we can write

$$\begin{aligned} F\left(\frac{1}{2}\right) &= \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} \\ &= \frac{1}{2}\left(a_1 + \frac{a_2}{2} + \frac{a_3}{4}\right) \\ &= \frac{1}{2}\left(a_1 + \frac{1}{2}\left(a_2 + \frac{a_3}{2}\right)\right) \end{aligned}$$

and similarly, we can write

$$F\left(\frac{3}{2}\right) = \frac{3}{2}\left(a_1 + \frac{3}{2}\left(a_2 + \frac{3a_3}{2}\right)\right)$$

Using similar logic to our last case we see that the simplest case that makes $F\left(\frac{1}{2}\right)$ an integer is when $a_3 = 2$ with $a_2 = a_1 = 1$. This yields

$$\begin{aligned} F\left(\frac{1}{2}\right) &= \frac{1}{2}\left(1 + \frac{1}{2}\left(1 + \frac{2}{2}\right)\right) = 1 \\ F\left(\frac{3}{2}\right) &= \frac{3}{2}\left(1 + \frac{3}{2}\left(1 + \frac{3}{2}(2)\right)\right) = \frac{21}{2} \end{aligned}$$

which gives us $F(x) = x + x^2 + x^3$ as one polynomial that satisfies the conditions of the problem.

Reflecting on our method, we see that we have started to develop a family of polynomials:

$$\begin{aligned} P_1(x) &= 2x \\ P_2(x) &= x + 2x^2 = x(1+2x) \\ P_3(x) &= x + x^2 + 2x^3 = x(1 + x(1+2x)) \\ &\vdots \end{aligned}$$

where, from the nested factored form we have chosen, we can see that

$$P_n(x) = x(1 + P_{n-1}(x))$$

Notice that if $P_{n-1}\left(\frac{1}{2}\right) = 1$ then

$$P_n\left(\frac{1}{2}\right) = \frac{1}{2}\left(1 + P_{n-1}\left(\frac{1}{2}\right)\right) = \frac{1}{2}(1+1) = 1$$

Hence, since $P_1\left(\frac{1}{2}\right) = 1$, all polynomials of the form

$$P_m(x) = x + x^2 + x^3 + \cdots + x^{m-1} + 2x^m$$

satisfy $P_m\left(\frac{1}{2}\right) = 1$, an integer, by induction.

However, since $P_n\left(\frac{3}{2}\right) = \frac{3}{2}\left(1 + P_{n-1}\left(\frac{3}{2}\right)\right)$, then if $P_{n-1}\left(\frac{3}{2}\right)$ is not an *odd* integer, $P_n\left(\frac{3}{2}\right)$ will not be an integer. From our work above, $P_2\left(\frac{3}{2}\right) = 6$, hence $P_m\left(\frac{3}{2}\right)$ will not be an integer for all $m > 2$. So by playing around with things a bit, we have generated not only a solution to the problem but an infinite family of solutions.

Those of you that know something about geometric series might want to show that $\left(P_m, \frac{1}{2}\right)$ is a good pair, while $\left(P_m, \frac{3}{2}\right)$ is *not* a good pair, for the family of polynomials described above, with $m > 2$. As a hint, you may want to write

$$P_m(x) = \left(x + x^2 + x^3 + \cdots + x^{m-1} + x^m\right) + x^m$$

I will leave the details to the interested reader.

Moving on to (b), evaluating $F\left(\sqrt{2}\right)$ and $F\left(\sqrt{2}+1\right)$, taking $a_0 = 0$ as in part (a), yields (after a bit of work)

$$\begin{aligned} F\left(\sqrt{2}\right) &= \sqrt{2}a_1 + 2a_2 + 2\sqrt{2}a_3 + \cdots + \left(\sqrt{2}\right)^m a_m \\ F\left(\sqrt{2}+1\right) &= (\sqrt{2}+1)a_1 + (2\sqrt{2}+3)a_2 + (5\sqrt{2}+7)a_3 + \cdots + \left(\sqrt{2}+1\right)^m a_m \end{aligned}$$

It is important to realize that since $\sqrt{2}$ is irrational, then $k\sqrt{2}$ will be irrational for any integer k. Hence, if we want $F(\sqrt{2})$ to be an integer, the only way this is possible is if

$$\sqrt{2}a_1 + 2\sqrt{2}a_3 + 4\sqrt{2}a_5 + \cdots = \sqrt{2}(a_1 + 2a_3 + 4a_5 + \cdots) = 0$$

which means that

$$a_1 + 2a_3 + 4a_5 + \cdots = 0$$

where the sum involves all the coefficients with odd index. As such, we see that if we pick $a_{2k+1} = 0$, then $F(\sqrt{2})$ would be an integer but $F(\sqrt{2}+1)$ would not necessarily have to be an integer. As with part (a), let's try to construct as simple a polynomial as possible that gives us the desired results without all odd coefficients being zero. Since $a_1 \neq 0$ and F is not constant, we would need the degree of F to be at least 3 in order to eliminate the $\sqrt{2}$ terms and end up with an integer. In this case, we need

$$a_0 + \sqrt{2}a_1 + 2a_2 + 2\sqrt{2}a_3$$

to be an integer. Since the coefficients are integers and $\sqrt{2}$ is irrational, this is only possible if $\sqrt{2}a_1 + 2\sqrt{2}a_3 = 0$. In other words $a_1 + 2a_3 = 0$, or $a_1 = -2a_3$.

Moving on to $F\left(\sqrt{2}+1\right)$, we need

$$a_0 + (\sqrt{2}+1)a_1 + (2\sqrt{2}+3)a_2 + (5\sqrt{2}+7)a_3$$

to be an integer. Since the a_i's are already integers, $\sqrt{2}$ is irrational and $a_1 = -2a_3$ this is equivalent to

$$\begin{aligned} \sqrt{2}a_1 + 2\sqrt{2}a_2 + 5\sqrt{2}a_3 &= 0 \\ -2a_3 + 2a_2 + 5a_3 &= 0 \\ a_2 &= -\frac{3a_3}{2} \end{aligned}$$

Hence, if we choose $a_3 = 2$, then choosing $F(x) = -4x - 3x^2 + 2x^3$, makes both $\left(F, \sqrt{2}\right)$ and $\left(F, \sqrt{2}+1\right)$ good pairs. You could also notice that if a_3 was *any* even integer, we could construct a polynomial F that satisfies the conditions of the problem. That is, there is going to be an infinite family of cubic polynomials that work. We could use a similar methodology to construct polynomials of higher degree that satisfy the conditions of part (b).

So we see that we could have solved part (a) and (b) just by playing around and looking for some patterns. That is, we can solve the problem without using too many properties of polynomials. However, if we do know some things about polynomials we may be able to use them to solve the problem more succinctly.

In both part (a) and (b) we are dealing with when a polynomial is an integer or not. If we focus in on the specific integer 0, and knowing the zeros of a polynomial are related to the factors of the polynomial, we may be able to construct solutions from the ground up. For instance in part (a), we want $F\left(\frac{1}{2}\right)$ to be an integer. Looking at $a(x) = 2x - 1$ we see that $a\left(\frac{1}{2}\right) = 0$ and $a\left(\frac{3}{2}\right) = 2$. Thus if we can find another polynomial, b, with integer coefficients, so that $2b\left(\frac{3}{2}\right)$ is not an integer, then $F(x) = a(x)b(x)$ would be a solution to our problem. It is clear that $b(x) = x^2$ satisfies these conditions, so $F(x) = x^2(2x-1)$ is an example for part (a). A little thinking also shows that $F_m(x) = x^m(2x-1)$ is a family of solutions with $m \geq 2$.

In part (b) we want both $\left(F, \sqrt{2}\right)$ and $\left(F, \sqrt{2}+1\right)$ to be good pairs. Hence if we can find polynomials a and b such that $a\left(\sqrt{2}\right) = 0$ and $b\left(\sqrt{2}+1\right) = 0$, then $F(x) = a(x)b(x)$ will clearly be a solution. Since $\left(\sqrt{2}\right)^2 = 2$, then $\sqrt{2}$ is a zero of the polynomial $a(x) = x^2 - 2$. Similarly, since $(\sqrt{2}+1) - 1 = \sqrt{2}$, then $\sqrt{2}+1$ is a zero of the polynomial $b(x) = a(x-1) = (x-1)^2 - 2$. Thus we get

$$\begin{aligned} F(x) &= a(x)b(x) \\ &= (x^2-2)\left((x-1)^2-2\right) \\ &= x^4 - 2x^3 - 3x^2 + 4x + 2 \end{aligned}$$

which gives us our result much more quickly. You may be interested to verify that if we use the method from earlier, we can construct an infinite family of degree 4 polynomials of which the result above is a member.

The following property of integers is also of use to us. Consider the three real numbers x, y, and $x-y$. If x is an integer then either *both* y and $x-y$ are integers or they are both not integers. Hence we could attack parts (a) and (b) by considering $F(c)$ and $F(c+1) - F(c)$. We will use this idea in our solution to part (c), but I will leave the details of the solutions to part (a) and (b) using this method to the interested reader.

To attack part (c), we will need to use the *binomial theorem* for expanding powers of a binomial. That is

$$\begin{aligned} (x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i, \end{aligned}$$

where

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

are called the *binomial coefficients* and are always *integers*.

For the problem at hand, we know we have some nonconstant polynomial $F(x) = a_1x + \cdots + a_mx^m$ with integer coefficients and some rational number c such that $F(c)$ is an integer. Hence, $F(c+n)$ will be an integer if and only if $F(c+n) - F(c)$ is an integer. Looking at this difference we get

$$\begin{array}{rrlll} & F(c+n) = & a_1(c+n) + a_2(c+n)^2 & & + \cdots + a_m(c+n)^m \\ - & F(c) = & a_1c & + a_2c^2 & + \cdots + a_mc^m \\ \hline & F(c+n) - F(c) = & a_1n & + a_2\left((c+n)^2 - c^2\right) & + \cdots + a_m\left((c+n)^m - c^m\right) \end{array}$$

Examining a particular term of this expansion, we see

$$\begin{aligned} & a_k\left((c+n)^k - c^k\right) \\ &= a_k\left(\left(c^k + \binom{k}{1}c^{k-1}n + \binom{k}{2}c^{k-2}n^2 + \cdots + \binom{k}{k-1}cn^{k-1} + n^k\right) - c^k\right) \\ &= a_k\left(\left(\binom{k}{1}c^{k-1}n + \binom{k}{2}c^{k-2}n^2 + \cdots + \binom{k}{k-1}cn^{k-1} + n^k\right)\right). \end{aligned}$$

Thus each term in the expansion of $F(c+n) - F(c)$ will be the product of some integer with a power of c and a power of n. The largest power of c will be c^{m-1}. If we write $c = \frac{p}{q}$ where p and q are relatively prime, then letting $n = \alpha q^\beta$ where $\alpha, \beta \in \mathbb{Z}$ and $\beta \geq m-1$, will result in each term in the expansion of $F(c+n)-F(c)$ being an integer, hence $F(c+n)$ will be an integer. By construction, we see that there are infinitely many n's possible, so the problem is solved.

I will leave part (d) as an exercise to the reader. You can check out the official solution on the CMS website.

Before we finish, let's circle back and play with an interesting pattern that arose from our solution to part (b). You may have noticed that the multipliers of the coefficients in $F\left(\sqrt{2}\right)$ form a predictable sequence:

$$1, \sqrt{2}, 2, 2\sqrt{2}, 4, 4\sqrt{2}, 8, 8\sqrt{2}, \ldots$$

On the other hand, the multipliers of the coefficients in $F\left(\sqrt{2}+1\right)$ may seem a little more obscure:

$$1, \sqrt{2}+1, 2\sqrt{2}+3, 5\sqrt{2}+7, 12\sqrt{2}+17, 29\sqrt{2}+41, 70\sqrt{2}+99, 169\sqrt{2}+239, \ldots$$

However, if we write $(\sqrt{2}+1)^n = a_n\sqrt{2} + b_n$ and look at the sequences a_n and b_n

n	a_n	b_n
0	0	1
1	1	1
2	2	3
3	5	7
4	12	17
5	29	41
6	70	99
7	169	239

some patterns begin to emerge. For one thing, the two sequences seem to be intertwined, that is

$$\begin{aligned} a_n &= a_{n-1} + b_{n-1} \\ b_n &= 2a_{n-1} + b_{n-1} \end{aligned}$$

This can be verified readily using a little algebra. If $(\sqrt{2}+1)^{n-1} = a_{n-1}\sqrt{2}+b_{n-1}$, then $(\sqrt{2}+1)^n = a_n\sqrt{2}+b_n$, but

$$\begin{aligned}(\sqrt{2}+1)^n &= (\sqrt{2}+1)^{n-1}(\sqrt{2}+1)\\ &= (a_{n-1}\sqrt{2}+b_{n-1})(\sqrt{2}+1)\\ &= 2a_{n-1}+a_{n-1}\sqrt{2}+b_{n-1}\sqrt{2}+b_{n-1}\\ &= (a_{n-1}+b_{n-1})\sqrt{2}+(2a_{n-1}+b_{n-1})\end{aligned}$$

which verifies our conjecture.

Looking closer, both sequences seem to satisfy the same recurrence relationships with different starting points. That is, it seems that we have

$$\begin{aligned}&a_0 = 0, a_1 = 1 \text{ and } a_n = 2a_{n-1}+a_{n-2} \text{ for } n > 1,\\ &b_0 = 1, b_1 = 1 \text{ and } b_n = 2b_{n-1}+b_{n-2} \text{ for } n > 1.\end{aligned}$$

This can also be verified using the equations linking a_n and b_n to a_{n-1} and b_{n-1} that we just established. I will leave the details to the readers.

When writing contests, one has to use one's time wisely. However, if we are not in a contest situation, I would urge you to go off on tangents. Ask yourself questions and see if you can answer them. Finally, never be afraid to chase after a pattern. You never know what fun you will have or what beautiful mathematics you will discover!

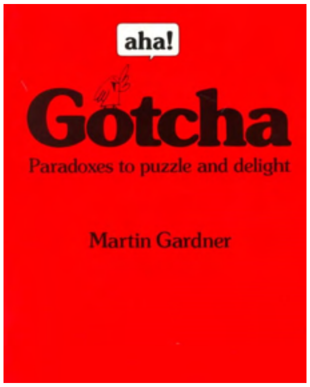
The author would like to thank *Crux* editor Ed Barbeau for his feedback on this column. His input greatly improved the final product.

From the bookshelf of ...

Shawn Godin

This new feature of MathemAttic *brings attention to books of potential interest to the readers. Some of these will be reviews whereas others will be hearty recommendations from the contributors. If you have a book related to mathematics that would be of interest to secondary school students and/or teachers, feel welcome to send along a submission to* `MathemAttic@cms.math.ca`*. Publishers are also welcome to send along books for possible review.*

aha! Gotcha: Paradoxes to puzzle and delight
by Martin Gardner
ISBN 0-7167-1361-6, 164 pages
Published by W. H. Freeman and Company, 1982.



I was introduced to the writing of Martin Gardner early in my teaching career (how his work had escaped me to that point, I do not know). I am sure many *MathemAttic* readers are aware of his work, and if you are not, go out and find something of his to read now! His column in *Scientific American* has been cited by many mathematicians as the reason they took to the subject. All of these articles live on in his books that are dedicated to collections of articles.

The book I am going to talk about is not one of those. *aha! Gotcha – Paradoxes to puzzle and delight*, like its sister book *aha! Insight*, is derived from *The Paradox Box*, a set of filmstrips, cassettes and teacher's guides published by *Scientific American*. The book is divided into six chapters: Logic, Number, Geometry, Probability, Statistics and Time. Each chapter contains a number of very short essays on the chapter topic. The essays start with cartoon panels including text that I suspect were from the filmstrip slides and the accompanying dialogue from the cassettes. The filmstrip material is then followed by a deeper discussion of the topic. Some topics may be known to some *MathemAttic* readers, such as The Barber Paradox (from the chapter on logic) and The Gambler's Fallacy (from the chapter on probability).

I managed to work many of the ideas from this book into my classrooms over the years. Especially fruitful was the chapter on probability. Several of the essays lead to interesting scenarios for the students to explore. An example would be the *Three-Card Swindle*. You are presented with three cards: one has a spade on both sides, one has a diamond on both sides and one has a spade on one side and a diamond on the other. You are allowed to pick one of the cards from a hat at random and place it on the table with one of its faces showing. The person running the game then guesses what symbol is on the other side. Gardner argues, if a spade shows, the other side could either be a spade (with the spade/spade card) or a diamond (with the spade/diamond card), therefore there is a 50% chance that he will guess it correctly. Is this a fair game? If you are not sure, you can readily devise an experiment to test it out.

I have also used the essay entitled *The Deceptive "Average"* in some classes. The filmstrip material outlines an informative special example where the mean, median and mode of a data set are quite different. The situation involves a person getting a job at a company where they are told that the average wage is $600 per week. They are told during training they would earn $150 per week, but would then get a raise. The new worker is unhappy when they get a raise to $200 per week and felt they were misled. The discussion then goes on to show how he wasn't misled, the weekly salaries of the workers were: $4800, $2000, six at $500, five at $400, and ten at $200. Although contrived, it does convey how outliers can skew the mean. It also leads to a nice discussion of how we can be misled by statistics.

No matter if you are a teacher looking for some interesting ideas to incorporate into your classroom, or a student (or teacher!) who just loves mathematics, *aha! Gotcha* will provide you with hours of fun.

This book is a recommendation from the bookshelf of Shawn Godin. Shawn, a retired high school math teacher, is a co-editor of *MathemAttic* and has been involved with ***Crux*** in one form or another for over 20 years. Shawn continues to be involved in mathematical activities in his retirement: helping with mathematics contest creation and marking, writing columns and doing the occasional presentation. He lives in Carleton Place, Ontario with his wife, Julie, and their dog, Daisy.

OLYMPIAD CORNER

No. 399

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **March 30, 2022***.*

OC561. Let $\triangle ABC$ be an arbitrary triangle with area 1. The edge AB is extended past B to a point B' such that $|BB'| = |AB|$. Similarly, the edge BC is extended past C to a point C' such that $|CC'| = 2|BC|$; and CA is extended past A to a point A' such that $|AA'| = 3|CA|$. Find the area of $\triangle A'B'C'$.

OC562. Ruby and Sapphire are celebrating Pi Day by sharing a circular pie. Ruby has two red birthday cake candles, and Sapphire has two blue candles. Ruby starting, they will alternately place one candle on the perimeter of the pie. (Of course, no two candles may be in the same place!) After all the candles are placed, each girl will get the portion of the pie that is closer to one of her candles than to any of the others. The goal is to get strictly more pie than one's opponent; an equal division is a draw.

Either find a winning strategy for one player and show that it is essentially unique, or show that the game, rationally played, is a draw.

OC563. Find, with proof, $\int_0^{\pi/2} \cos^{31416}(x)dx$.

OC564. Define a "Fibonacci-like" sequence as follows: $A_1 = A_2 = 1$, and $A_n = 2A_{n-2} + A_{n-1}$ for $n \geq 3$; so $A_3 = 2 \times 1 + 1 = 3$, $A_4 = 2 \times 1 + 3 = 5$, and so on. Prove that for odd n,

$$\sum_{i=1}^{n-1} A_i = A_n - 1$$

OC565. Given that $\sin(xy) = 1$, find the least upper bound of $\sin(x)\sin(y)$, and show that this is never achieved.

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2022**.

OC561. Soit $\triangle ABC$ un triangle quelconque de surface 1. Le côté AB est prolongé au-delà de B jusqu'à un point B' tel que $|BB'| = |AB|$. De façon similaire, le côté BC est prolongé au-delà de C jusqu'à un point C' tel que $|CC'| = 2|BC|$ et le côté CA est prolongé au-delà de A jusqu'à un point A' tel que $|AA'| = 3|CA|$. Déterminer la surface de $\triangle A'B'C'$.

OC562. Rubis et Saphir célèbrent le Jour Pi en se partageant une tarte de forme circulaire. Au départ, Rubis a deux chandelles rouges de gâteau de fête, tandis que Saphir a deux chandelles bleues. Commençant avec Rubis, elles placent leurs chandelles au périmètre de la tarte, en alternant. (Bien sûr, deux chandelles ne peuvent pas occuper la même place.) Après que toutes les chandelles sont placées, chaque fille obtiendra la portion de la tarte qui se trouve plus près d'une de ses propres chandelles que de toute autre. L'objectif est d'obtenir plus de tarte que l'autre fille ; une division égale résulte en un ex aequo.

Soit déterminer une stratégie gagnante pour une des filles et démontrer qu'elle est unique, soit démontrer que le résultat est un ex aequo, prenant pour acquis un jeu rationnel des deux parts.

OC563. Tout en fournissant la preuve, déterminer la valeur de

$$\int_0^{\pi/2} \cos^{31416}(x)dx.$$

OC564. Déterminer une suite de type Fibonacci de la façon suivante : $A_1 = A_2 = 1$, puis $A_n = 2A_{n-2} + A_{n-1}$ pour $n \geq 3$; ainsi $A_3 = 2 \times 1 + 1 = 3$, $A_4 = 2 \times 1 + 3 = 5$ et ainsi de suite. Démontrer que pour n impair on a

$$\sum_{i=1}^{n-1} A_i = A_n - 1.$$

OC565. Étant donné $\sin(xy) = 1$, déterminer la plus petite borne supérieure pour $\sin(x)\sin(y)$, et démontrer que cette borne n'est jamais atteinte.

OLYMPIAD CORNER SOLUTIONS

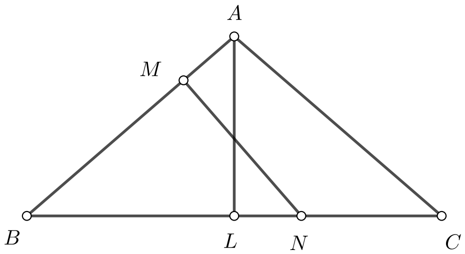
Statements of the problems in this section originally appear in 2021: 47(6), p. 288–289.

OC536. The triangle ABC has $AB = CA$ and BC is its longest side. The point N is on the side BC and $BN = AB$. The line perpendicular to AB which passes through N meets AB at M. Prove that the line MN divides both the area and the perimeter of triangle ABC into equal parts.

Originally from the 2017 British Mathematical Olympiad, Round 1, Problem 3.

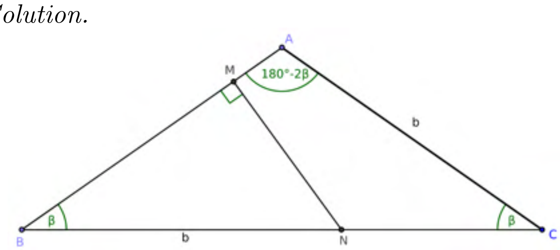
We received 19 submissions, all of which were correct and complete. We present two typical solutions.

Geometric Solutic



Let L be the foot of the perpendicular from A on BC. Since $\triangle ABC$ is isosceles, it follows that AL divides its area and perimeter into equal parts. However, $\triangle ALB$ and $\triangle NMB$ are congruent as they are right-angled triangles sharing $\angle B$ and $AB = NB$. Hence, MN divides the area and perimeter of $\triangle ABC$ into equal parts, like AL.

Trigonometric Solution.



Let $b = AB = AC = BN$, $\beta = \angle CBA = \angle ACB$, and let s denote the semiperimeter of $\triangle ABC$. We obtain

$$\begin{aligned}[ABC] &= \frac{AB \cdot AC \cdot \sin \angle BAC}{2} = \frac{b^2 \sin(180^\circ - 2\beta)}{2} = \frac{b^2 \sin 2\beta}{2} \\ &= b \sin\beta \cdot b\cos\beta = MN \cdot BM = 2[BMN]\end{aligned}$$

and

$$s = \frac{AB + AC + BC}{2} = \frac{b + b + 2b\cos\beta}{2} = b + b\cos\beta = BN + BM.$$

Hence, the line MN divides the area and the perimeter of $\triangle ABC$ into equal parts.

OC537. A, B, C are collinear with B betweeen A and C. K_1 is the circle with diameter AB, and K_2 is the circle with diameter BC. Another circle touches AC at B and meets K_1 again at P and K_2 again at Q. The line PQ meets K_1 again at R and K_2 again at S. Show that the lines AR and CS meet on the perpendicular to AC at B.

Originally from the 2003 Mexican Mathematical Olympiad, problem A2.

We received 8 correct and complete submissions. We present two solutions.

Solution 1, by Oliver Geupel.

Let the lines AR and CS meet at the point T, and let $p(\Gamma)$ denote the power of T with respect to some circle Γ. If the points A, C, R, and S lie on a common circle K, then we have $p(K_1) = TA \cdot TR = p(K) = TC \cdot TS = p(K_2)$, which implies that T lies on the radical axis of K_1 and K_2, that is, on the perpendicular to the line AC at B. It is therefore enough to prove that A, C, R and S are concyclic. Considering the relative positions of the collinear points, we distinguish 3 cases:

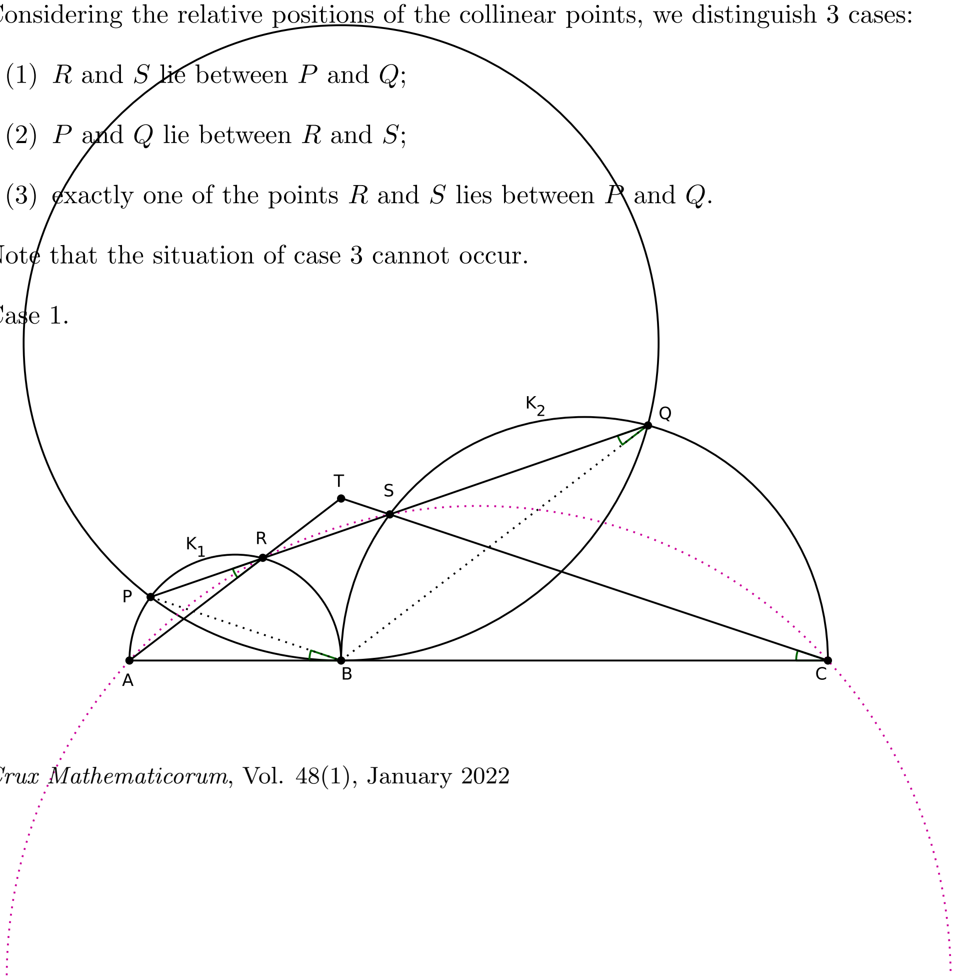
(1) R and S lie between P and Q;

(2) P and Q lie between R and S;

(3) exactly one of the points R and S lies between P and Q.

Note that the situation of case 3 cannot occur.

Case 1.

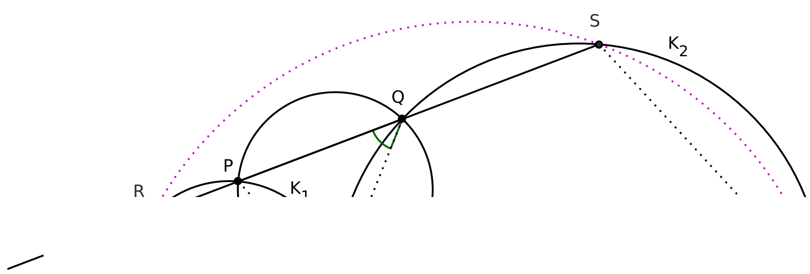


We have

$$\begin{aligned} 180^\circ - \angle ARS \quad = \angle PRA &= \angle PBA \quad \text{- inscribed angles in } K_1 \\ &= \angle PQB \quad \text{- since } AB \text{ is tangent to circle } (PBQ) \\ &= \angle SQB \\ &= \angle SCB \quad \text{- inscribed angles in } K_2 \\ &= \angle SCA, \end{aligned}$$

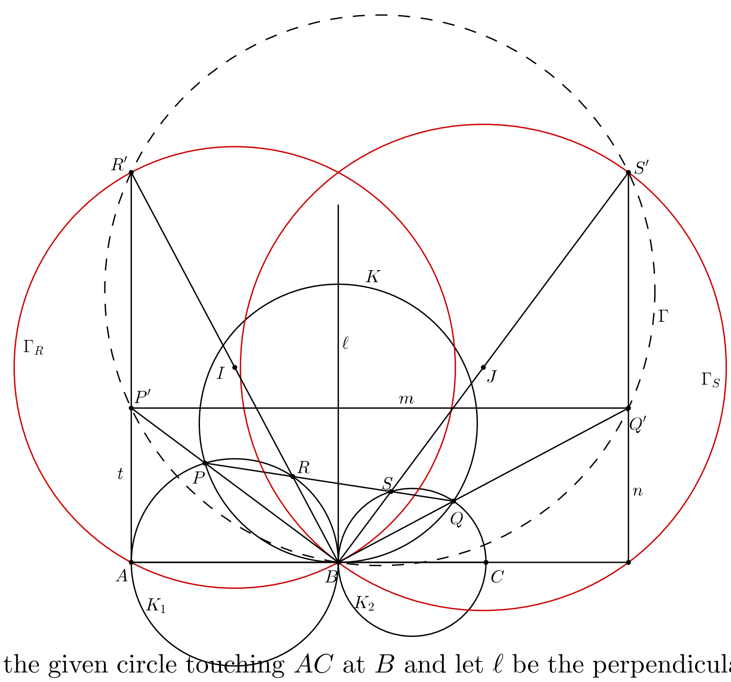
which proves that the points A, C, R and S are concyclic.

Case 2.



Similarl
A, C, R

Solution



Let K be the given circle touching AC at B and let ℓ be the perpendicular to AC at B. We denote by $\mathbf{I}$ the inversion in the circle with center B and radius BA.

The circle K_1 inverts into its tangent line t at A and $P' = \mathbf{I}(P)$ is the point of intersection of t and BP. The circle K inverts into the line m orthogonal to ℓ through P'. Then $Q' = \mathbf{I}(Q)$ is the point of intersection of m and BQ and $\mathbf{I}(K_2)$ is the perpendicular n to AC through Q'. Also, SR intersects t at $R' = \mathbf{I}(R)$ and BS intersects n at $S' = \mathbf{I}(S)$.

The line AR (resp. CS) inverts into the circle Γ_R (resp. Γ_S) with diameter BR' (resp. BS'). It clearly suffices to prove that the point of intersection of Γ_R and Γ_S other than B is on ℓ.

As P, Q, R, S are collinear, the points P', Q', R', S' are on a circle Γ passing through B. Since $P'Q'$ is orthogonal to $Q'S'$ and to $P'R'$, the segments $P'S'$ and $Q'R'$ are diameters of Γ. It follows that $S'R'$ is perpendicular to $R'P' = t$ and so $R'S'$ is parallel to AC. In consequence, the line IJ through the respective centers I and J of Γ_R and Γ_S is parallel to AC. The radical axis of Γ_R and Γ_S, which passes through B and is perpendicular to IJ, is the line ℓ. Thus, the second point of intersection of Γ_R and Γ_S is on ℓ, as desired.

OC538. Let us consider a polynomial $P(x)$ with integer coefficients satisfying $P(-1) = -4$, $P(-3) = -40$, and $P(-5) = -156$. What is the largest possible number of integers x satisfying $P(P(x)) = x^2$?

Originally from the 2019, Baltic Way, Szczecin, Poland, Problem 20.

We received 5 submissions, all of which were correct and complete. We present the solution by Theo Koupelis.

Start with

$$P(x) = (x+1)(x+3)(x+5)Q(x) + ax^2 + bx + c,$$

where a, b, c are integers and $Q(x)$ is a polynomial with integer coefficients. The given conditions lead to

$$P(x) = (x+1)(x+3)(x+5)Q(x) - 10x^2 - 22x - 16.$$

Equivalently,

$$P(x) = (x+1)(x+3)(x+5)Q(x) - 3(3x^2 + 7x + 5) - (x^2 + x + 1).$$

Because $(x+1)(x+3)(x+5) \equiv 0 \pmod 3$ for all integers x, we have

$$P(x) \equiv -(x^2 + x + 1) \pmod 3.$$

We distinguish the following cases.

(i) If $x \equiv 0 \pmod 3$, or $x \equiv -1 \pmod 3$, then

$$P(x) \equiv -1 \pmod 3 \quad \text{and} \quad P(P(x)) \equiv P(-1) \pmod 3 \equiv -1 \pmod 3.$$

Thus the equation $P(P(x)) = x^2$ has no solution.

(ii) If $x \equiv 1 \pmod 3$, then

$$P(x) \equiv 0 \pmod 3 \quad \text{and} \quad P(P(x)) \equiv P(0) \pmod 3 \equiv -1 \pmod 3.$$

Thus the equation $P(P(x)) = x^2$ has no solution.

Therefore there are no integers x satisfying $P(P(x)) = x^2$.

OC539. A pair of real numbers (a, b) with $a^2 + b^2 \leq \frac{1}{4}$ is chosen at random. If p is the probability that the curves with equations $y = ax^2 + 2bx - a$ and $y = x^2$ intersect, then identify the integer that is closest to $100p$.

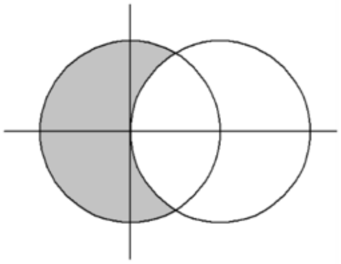
Originally from the 2021 Fermat Contest, Grade 11, Problem 24.

We received 9 submissions, of which 7 were correct and complete. We present a typical solution.

The two curves intersect if and only if the quadratic equation $ax^2 + 2bx - a = x^2$ has a real solution. Therefore the discriminant, $4b^2 + 4a(a-1)$, of this equation must be non-negative. This is equivalent to

$$b^2 + \left(a - \frac{1}{2}\right)^2 \geq \frac{1}{4}.$$

The points (a, b) satisfying this inequality correspond to those points on or outside the circle of radius $1/2$ centered at the point $(1/2, 0)$. Combining this with the fact that the points must satisfy $a^2 + b^2 \leq 1/4$ implies that the points lie in the shaded region shown below.



We can calculate the area of the shaded region using geometric or analytic arguments as:

$$\frac{\pi}{4} - 4\int_{1/4}^{1/2} \sqrt{\frac{1}{4} - x^2}\, dx = \frac{3\sqrt{3} + 2\pi}{24}.$$

As the area of a circle of radius $1/2$ is $\pi/4$, it follows that the requested probability is

$$p = \frac{3\sqrt{3} + 2\pi}{24} \times \frac{4}{\pi} = \frac{1}{3} + \frac{\sqrt{3}}{2\pi}.$$

Using a calculator we can find that the integer closest to $100p$ is 61.

OC540. Let $S_r(n) = 1^r + 2^r + \cdots + n^r$ where r is a rational number and n a positive integer. Find all triplets $(a, b, c) \in \mathbb{Q}_+ \times \mathbb{Q}_+ \times \mathbb{N}$ for which there exist infinitely many positive integers n satisfying $S_a(n) = (S_b(n))^c$

Originally from the 2012 Turkish IMO Team Selection Test, Day 3, Problem 7.

We received 5 correct and complete submissions. We present a solution submitted independently by two problem solving groups: UCLan Cyprus and Missouri State University.

We first observe that

$$\frac{S_r(n)}{n^{r+1}} = \frac{1}{n}\left(\left(\frac{1}{n}\right)^r + \left(\frac{2}{n}\right)^r + \cdots + \left(\frac{n}{n}\right)^r\right)$$

is a Riemann Sum of $f(x) = x^r$ over $[0, 1]$ with respect to the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n} \ldots, 1\}$. Since the mesh of the partition converges to 0 as n tends to infinity, we get that

$$\frac{S_r(n)}{n^{r+1}} \to \int_0^1 f(x)\,\mathrm{d}x = \frac{1}{r+1}\,.$$

Suppose now that $S_a(n) = S_b(n)^c$ for infinitely many values of n. Then

$$\frac{S_a(n)}{n^{a+1}} = \frac{S_b(n)^c}{n^{a+1}} = \frac{S_b(n)^c}{n^{c(b+1)}} n^{c(b+1)-(a+1)}$$

for infinitely many values of n. The left hand side converges to $1/(a+1)$. However, the right hand side converges to 0 if $a+1 > c(b+1)$, or $+\infty$ if $a+1 < c(b+1)$. So we must have $a+1 = c(b+1)$. In this case, the right hand side tends to $\frac{1}{(b+1)^c}$ so we must also have $a+1 = (b+1)^c$.

Now $c(b+1) = (b+1)^c$ gives $(b+1)^{c-1} = c$. For $c > 1$, in order for $(b+1)^{c-1}$ to be an integer, we must have that b is also an integer. But then $b \geqslant 1$ and so by Bernoulli's inequality $c = (b+1)^{c-1} \geqslant (1+1)^{c-1} \geqslant 1+(c-1) = c$. The inequality is strict if $c-1 \geqslant 2$ so the only possibilities are $c = 1$ and $c = 2$.

For $c = 1$ we must have $a = b$ and for all such choices we have $S_a(n) = S_b(n)^c$ for all values of n. For $c = 2$ we must have $b+1 = 2$ and therefore $b = 1$. Then we must also have $a+1 = c(b+1) = 4$ and therefore $a = 3$. In this case

$$S_a(n) = \left(\frac{n(n+1)}{2}\right)^2 = S_b(n)^2$$

for all values of n.

Therefore the only possible triples are $(a, a, 1)$ for $a \in \mathbb{Q}_+$ and $(3, 1, 2)$.

From the lecture notes of . . .

Vanessa Radzimski

In this new feature of ***Crux****, we share some of our favourite problems from first and second year undergraduate courses. These problems are a bit non-standard, elegant or unexpected. If you have a problem you would like to share (and it fits on one page), please send it along with its solution and a description of the course/audience it is intended for to* `crux.eic@gmail.com`*.*

This month's column is brought to you by Vanessa Radzimski. Vanessa is an Assistant Professor in Mathematics at the University of the Fraser Valley, where she researches the role of advanced mathematics coursework for future secondary math teachers. She lives on the lands of the Kwantlen First Nation with her husband Paul and their dog Poppy. In her spare time she enjoys skiing, macrame, and training with Poppy for competitive dog agility.

These problems were developed for a first-year course in Integral Calculus for students majoring in the sciences. The problems can be given as-is for homework assignments or extra practice, but can be easily scaffolded for inclusion in tests or exams.

Problems

1. Let $f(t)$ be a function such that $f''(t)$ is continuous and positive for all t. Prove that $\displaystyle\int_0^{2\pi} f(t)\cos(t)\,dt \geq 0$.
2. You walk 81 metres in one direction, turn ninety degrees to the right and walk 27 metres, turn ninety degrees to the right and walk 9 metres, turn ninety degrees to the right and walk 3 metres, and so on. This pattern continues indefinitely. As the number of turns you take approaches infinity, you approach a point T. Find the distance between your original position and T.

Solutions

1. Let's use integration by parts. Taking $u = f(t)$ and $dv = \cos(t)\,dt$, we have

$$\int_0^{2\pi} f(t)\cos(t)\,dt = f(t)\sin(t)\Big|_0^{2\pi} - \int_0^{2\pi} f'(t)\sin(t)\,dt = -\int_0^{2\pi} f'(t)\sin(t)\,dt$$

We can use integration by parts again on the resulting integral, taking $u = f'(t)$ and $dv = \sin(t)\,dt$, yielding:

$$\begin{aligned} -\int_0^{2\pi} f'(t)\sin(t)\,dt &= f'(t)\cos(t)\Big|_0^{2\pi} - \int_0^{2\pi} \cos(t)f''(t)\,dt \\ &= f'(2\pi) - f'(0) - \int_0^{2\pi} \cos(t)f''(t)\,dt. \end{aligned}$$

Now, we know that $f''(t)$ is positive for all t and $-1 \leq \cos(t) \leq 1$, so

$$\int_0^{2\pi} \cos(t)f''(t)\,dt \leq \int_0^{2\pi} f''(t)\,dt = f'(t)\Big|_0^{2\pi} = f'(2\pi) - f'(0).$$

Bringing this all together, we have

$$\int_0^{2\pi} f(t)\cos(t)\,dt = f'(2\pi) - f'(0) - \int_0^{2\pi} \cos(t)f''(t)\,dt \geq 0$$

as desired.

2. The distance, D, from our original position to the point T will depend on our horizontal and vertical displacement. If we call our horizontal displacement H and vertical displacement V, we have $D = \sqrt{H^2 + V^2}$. For finding H, we have that

$$H = 81 - 9 + 1 - \frac{1}{9} + \frac{1}{81} - \frac{1}{729} + \cdots = \sum_{n\geq 0} 81\left(\frac{-1}{9}\right)^n = \frac{81}{1+\frac{1}{9}}.$$

Similarly for V, we have that

$$V = 27 - 3 + \frac{1}{3} - \frac{1}{27} + \frac{1}{243} - \cdots = \sum_{n\geq 0} 27\left(\frac{-1}{9}\right)^n = \frac{27}{1+\frac{1}{9}}.$$

So, we have that

$$D = \sqrt{H^2 + V^2} = \sqrt{(\frac{243}{10})^2 + (\frac{729}{10})^2} = \frac{1}{10}\sqrt{590490}.$$

A Problem in Combinatorial Geometry

Andy Liu

Here is a problem to which the complete solution is not given. Reader participation is invited.

> *A county consists of five villages at the vertices of a regular pentagon. How should the total population of 2018 villagers be distributed among the villages so as to maximize the sum of the squares of the pairwise distances between all pairs of villagers?*

Let the regular pentagon be $ABCDE$ with circumradius 1. Let $\lambda = \frac{\sqrt{5}-1}{2}$, which is 1 less than the golden ratio $\phi = \frac{\sqrt{5}+1}{2}$. Then $\lambda^2 + \lambda - 1 = 0$. By the Cosine Law, the square of the length of a side of the pentagon is $2 - \lambda$ while the square of the length of a diagonal of the pentagon is $3 + \lambda$. It follows that the square of each pairwise distance is one of 0, $2 - \lambda$ and $3 + \lambda$.

Let a, b, c, d and e be the respective numbers of villagers in A, B, C, D and E respectively. At least three of these numbers are of the same parity. We may have all five having the same parity. If only four of them have the same parity, let them be b, c, d and e. Suppose exactly three of them have the same parity. We may let them be either e, a and b or a, c and d. In all cases, b has the same parity as e and c has the same parity as d.

We wish to maximize

$$S = \phi^2(ac + bd + ce + da + eb) + (ab + bc + cd + de + ea)$$

with the constraint $a + b + c + d + e = 2018$. We replace b and e by $b' = e' = \frac{b+e}{2}$ and simultaneously c and d by $c' = d' = \frac{c+d}{2}$. Let

$$S' = \phi^2(ac' + b'd' + c'e' + d'a + e'b') + (ab' + b'c' + c'd' + d'e' + e'a).$$

Then

$$\begin{aligned}
&S' - S\\
&= (2-\lambda)\left(\left(\frac{c+d}{2}\right)^2 + 2\left(\frac{b+e}{2}\right)\left(\frac{c+d}{2}\right)\right) + (3+\lambda)\left(\left(\frac{b+e}{2}\right)^2 + 2\left(\frac{b+e}{2}\right)\left(\frac{c+d}{2}\right)\right)\\
&\quad - (2-\lambda)(cd + bc + de) - (3+\lambda)(be + bd + ce)\\
&= (2-\lambda)\left(\left(\frac{c-d}{2}\right)^2 + \frac{(b-e)(c-d)}{2}\right) + (3+\lambda)\left(\left(\frac{b-e}{2}\right)^2 + \frac{(b-e)(c-d)}{2}\right)\\
&= \left(\sqrt{2-\lambda}\left(\frac{c-d}{2}\right) + \sqrt{3+\lambda}\left(\frac{b-e}{2}\right)\right)^2 + \left(1 + 2\lambda - \sqrt{(2-\lambda)(3+\lambda)}\right)\frac{(b-e)(c-d)}{2}\\
&\geq 0.
\end{aligned}$$

This is because $(1+2\lambda)^2 = 5 = (2-\lambda)(3+\lambda)$. It follows that we may make $b = e$ and $c = d$ without diminishing S, thereby reducing the number of variables from five to three.

More generally, let the total population be n. Since $2018 \equiv 3 \pmod 5$, we focus on the numbers of the form $n = 5k+3$. The empirical data leading to the optimal distribution for $n = 2018$ are given in the following table.

n	k	a	b	c
3	0	$F_2 = 1$	0	$F_1 = 1$
8	1	$2F_2 = 2$	$F_2 = 1$	$F_3 = 2$
		0	$F_4 = 3$	$F_2 = 1$
13	2	1	4	2
18	3	$F_3 = 2$	$F_5 = 5$	$F_4 = 3$
		$F_6 = 8$	0	$F_5 = 5$
23–53	4–10	9–15	1–7	6–12
58	11	$2F_6 = 16$	$F_6 = 8$	$F_7 = 13$
		0	$F_8 = 21$	$F_6 = 8$
63–118	12–23	1–12	22–33	9–20
123	24	$F_7 = 13$	$F_9 = 34$	$F_8 = 21$
		$F_{10} = 55$	0	$F_9 = 34$
128–343	25–68	56–109	1–54	35–88
348	69	$2F_{10} = 110$	$F_{10} = 55$	$F_{11} = 89$
		0	$F_{12} = 144$	$F_{10} = 55$
353–838	70–167	1–88	145–232	56–143
843	168	$F_{11} = 89$	$F_{13} = 233$	$F_{12} = 144$
		$F_{14} = 377$	0	$F_{13} = 233$
848–2013	169–402	378–611	1–234	234–467
2018	**403**	**612**	**235**	**468**

For $k = 0, 1, 2, 3$, the respective distributions $(a, b, c) = (1, 0, 1)$, (0,3,1), (1,4,2) and (8,0,5) are indeed optimal. Note that apart from (1,4,2), the nonzero numbers so far are all Fibonacci numbers: $F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, $\ldots$. That they play an important role in our argument is hardly surprising as they are also strongly associated with ϕ. In fact, $F_n = \frac{1}{\sqrt{5}}(\phi^n - (-\frac{1}{\phi})^n)$.

In going from one value of k to the next, **Operation 1** is to add 1 to each of a, b and c, as in the case going from $k = 1$ to 2. Suppose the optimal distribution for a certain value of k is (a, b, c). For the value $k+1$, if we go to $(a+1, b+1, c+1)$, the increase in S is

$$\begin{aligned}&\phi^2(2(a+1)(c+1) + 2(b+1)(c+1) + (b+1)^2 - 2ac - 2bc - b^2)\\ &\quad + 2(a+1)(b+1) + 2(b+1)(c+1) + (c+1)^2 - 2ab - 2bc - c^2.\end{aligned}$$

Since $a+2b+2c = n$, this simplifies to $(\phi^2+1)(2n+5)$, an expression independent of the distribution.

Suppose we do not use Operation 1 and there are no unoccupied villages in the resulting distribution. We can express it as $(x+1, y+1, z+1)$. The total distance is $(\phi^2+1)(2n+5)$ more than that of the distribution (x, y, z). Since (x, y, z) is inferior to (a, b, c), $(x+1, y+1, z+1)$ will be inferior to $(a+1, b+1, c+1)$. In this case, there is no reason why we should not use the Operation 1.

It follows that when we deviate from using Operation 1 and obtain a distribution (a, b, c), at least one of a, b and c must be 0. It is easy to see that we cannot have $c = 0$ or $a = b = 0$. Hence we either have $a = 0$, as in the case going from $k = 0$ to $k = 1$, or $b = 0$, as in the case going from $k = 2$ to $k = 3$. We call these Operation 2 and Operation 3 respectively.

Operation 2.
From the empirical data, Operation 2 occurs in $k = 1$, 11 and 69. We switch from $(a, b, c) = (2F_{4t-2}, F_{4t-2}, F_{4t-1})$ to a distribution of the form $(0, b', c')$. Since $c' \neq 0$, we take $c' = b = F_{4t-2}$. Then $b' = \frac{a}{2} + c$, which is equal to $F_{4t-2} + F_{4t-1} = F_{4t}$.

Before the switch, we have

$$S = \phi^2(6F_{4t-1}F_{4t-2} + F_{4t-1}^2) + 2F_{4t-1}F_{4t-2} + 4F_{4t-2}^2 + F_{4t-1}^2.$$

After the switch, we have

$$S' = \phi^2(2F_{4t}F_{4t-2} + F_{4t}^2) + 2F_{4t}F_{4t-2} + F_{4t-2}^2.$$

Comparing the terms multiplied by ϕ^2, we have

$$\begin{aligned}
& (2F_{4t}F_{4t-2} + F_{4t}^2) - (6F_{4t-1}F_{4t-2} + F_{4t-1}^2) \\
= \; & 2F_{4t-2}(F_{4t-1} + F_{4t-2}) + (F_{4t-1} + F_{4t-2})^2 - 6F_{4t-1}F_{4t-2} - F_{4t-1}^2 \\
= \; & F_{4t-1}^2 + 2F_{4t-2}^2 - 2F_{4t-1}F_{4t-2} \\
= \; & (F_{4t-2} + F_{4t-3})^2 + 2F_{4t-2}^2 - 2F_{4t-2}(F_{4t-2} + F_{4t-3}) \\
= \; & F_{4t-2}^2 + F_{4t-3}^2 \\
= \; & F_{8t-5}.
\end{aligned}$$

Comparing the other terms, we have

$$\begin{aligned}
& (2F_{4t-1}F_{4t-2} + 4F_{4t-2}^2 + F_{4t-1}^2) - (2F_{4t}F_{4t-2} + F_{4t-2}^2) \\
= \; & 2F_{4t-1}F_{4t-2} + 4F_{4t-2}^2 + F_{4t-1}^2 - 2F_{4t-2}(F_{4t-1} + F_{4t-2}) - F_{4t-2}^2 \\
= \; & F_{4t-1}^2 + F_{4t-2}^2 \\
= \; & F_{8t-3}.
\end{aligned}$$

It follows that $S' - S = \phi^2 F_{8t-5} - F_{8t-3} > 0$ since $\dfrac{\phi^2 F_{2m-1}}{F_{2m+1}} = \dfrac{\phi^{2m+1} + \phi^{-2m+3}}{\phi^{2m+1} + \phi^{-2m-1}} > 1$.

Operation 3.
From the empirical data, Operation 3 occurs in $k = 3$, 24 and 168. We switch from $(a, b, c) = (F_{4t-1}, F_{4t+1}, F_{4t})$ to a distribution of the form $(a', 0, c')$. Since $c' \neq 0$, we take $c' = b = F_{4t+1}$. This time, we have $a' = a + 2c$, which is equal to $F_{4t-1} + 2F_{4t} = F_{4t+1} + F_{4t} = F_{4t+2}$.

Before the switch, we have

$$S = \phi^2(2F_{4t}F_{4t-1} + 2F_{4t+1}F_{4t-1} + F_{4t+1}^2) + 2F_{4t+1}F_{4t-1} + 2F_{4t+1}F_{4t} + F_{4t}^2.$$

After the switch, we have

$$S' = \phi^2(2F_{4t+2}F_{4t+1}) + F_{4t+1}^2.$$

Comparing the terms multiplied by ϕ^2, we have

$$\begin{aligned}
& 2F_{4t+2}F_{4t+1} - (2F_{4t}F_{4t-1} + 2F_{4t+1}F_{4t-1} + F_{4t+1}^2) \\
= \;& 2F_{4t+1}(F_{4t+1} + F_{4t}) - 2F_{4t}F_{4t-1} - 2F_{4t+1}F_{4t} - F_{4t+1}^2 \\
= \;& F_{4t+1}^2 - 2F_{4t}F_{4t-1} \\
= \;& (F_{4t} + F_{4t-1})^2 - 2F_{4t}F_{4t-1} \\
= \;& F_{4t}^2 + F_{4t-1}^2 \\
= \;& F_{8t-1}.
\end{aligned}$$

Comparing the other terms, we have

$$\begin{aligned}
& (2F_{4t+1}F_{4t-1} + 2F_{4t+1}F_{4t} + F_{4t}^2) - F_{4t+1}^2 \\
= \;& 2F_{4t+1}(F_{4t+1} - F_{4t}) + 2F_{4t+1}F_{4t} + F_{4t}^2 - F_{4t+1}^2 \\
= \;& F_{4t+1}^2 + F_{4t}^2 \\
= \;& F_{8t+1}.
\end{aligned}$$

It follows that $S' - S = \phi^2 F_{8t-1} - F_{8t+1} > 0$ as in Operation 2.

We have now come to the crucial point in the argument. The key questions are:

1. *When do we deviate from using Operation 1?*
2. *If we deviate from using Operation 1, when will we use Operation 2 and when will we use Operation 3?*
3. *If we use Operation 2 or Operation 3, why is the given modification of the distribution optimal?*

These questions need to be answered with justifications. Please send your ideas by email to `acfliu@gmail.com`. We hope to be able to report progress in a follow-up paper.

FOCUS ON...

No. 49

Michel Bataille

The Gamma, Beta, and Digamma Functions

Introduction

Among the so-called *special functions*, the Gamma function and the closely connected Beta and Digamma functions are those most frequently met in problem corners. The purpose of this number is to illustrate simple results about them. The reader will find a very clear and accessible exposition of these results in the delightful, forty-page book [**1**]. Following this book, we will only consider the case when the variable is real. To the interested reader, we point out that Chapter 7 of [**2**] presents the case of a complex variable in detail.

Gamma

The first time a student comes across the Gamma function is likely during a lecture on integrals: for positive x, the number $\Gamma(x)$ is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

This defines a function, introduced by Euler in a desire to generalize the factorial, as shown by $\Gamma(n+1) = n!$ and the relation $\Gamma(x+1) = x\Gamma(x)$. Of course, Γ will appear in the evaluation of integrals. Here are two examples taken out of an exercise proposed in [**2**]:

$$\text{Evaluate } I(a) = \int_0^\infty e^{-x^a}\,dx \text{ and } J(a,b) = \int_0^1 \left(\ln\frac{1}{x}\right)^{a-1} x^{b-1}, \quad (a,b>0).$$

The substitution $x = u^{1/a}$ gives

$$I(a) = \int_0^\infty e^{-u}\cdot\frac{u^{\frac{1}{a}-1}}{a}\,du = \frac{\Gamma(1/a)}{a}$$

while $x = e^{-u}$ yields

$$J(a,b) = \int_0^\infty u^{a-1}e^{-bu}\,du = \int_0^\infty \frac{t^{a-1}}{b^{a-1}}\cdot e^{-t}\cdot\frac{dt}{b} = \frac{\Gamma(a)}{b^a}.$$

Note in passing that $I(2) = \frac{\sqrt{\pi}}{2}$ leads to $\Gamma(1/2) = \sqrt{\pi}$.

A natural extension of the Gamma function is given by

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!\cdot n^x}{x(x+1)(x+2)\cdots(x+n)}, \tag{1}$$

a useful formula that makes Gamma a function defined on a larger set, namely $D = \mathbb{R} - \mathbb{Z}^-$, where $\mathbb{Z}^- = \{0, -1, -2, \ldots\}$ denotes the set of nonpositive integers. As a direct application, we consider the following problem adapted from problem 906 proposed in *The College Mathematics Journal* in 2009:

For $x > 0$, find the value of

$$\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right)^{(-1)^{n-1}}.$$

For $N \in \mathbb{N}$, let $Q_N = \prod_{n=1}^{N} \left(1 + \frac{x}{n}\right)^{(-1)^{n-1}}$. Since $Q_{2N-1} = Q_{2N}\left(1 + \frac{x}{2N}\right) \sim Q_{2N}$ as $N \to \infty$, it suffices to determine $\lim_{N\to\infty} Q_{2N}$. Now,

$$\begin{aligned} Q_{2N} &= \frac{2 \cdot 4 \cdots (2N)}{1 \cdot 3 \cdot 5 \cdots (2N-1)} \cdot \frac{(x+1)\cdots(x+2N-1)}{(x+2)\cdots(x+2N)} \\ &= \frac{(N!)^2}{(2N)!} \cdot \frac{(x+1)(x+2)\cdots(x+2N-1)(x+2N)}{[(x/2+1)\cdots(x/2+N)]^2}, \end{aligned}$$

and since from (1)

$$(x+1)(x+2)\cdots(x+N) \sim \frac{N!N^x}{x\Gamma(x)}$$

as $N \to \infty$, we obtain

$$Q_{2N} \sim \frac{(N!)^2}{(2N)!} \cdot \frac{(2N)!2^x N^x}{x\Gamma(x)} \cdot \frac{\left(\frac{x}{2}\Gamma\left(\frac{x}{2}\right)\right)^2}{(N!)^2 N^x} \sim x2^{x-2}\frac{(\Gamma(x/2))^2}{\Gamma(x)}$$

and the required product is equal to $x2^{x-2}\frac{(\Gamma(x/2))^2}{\Gamma(x)}$.

We conclude this section with two formulas that deserve to be known

$$\begin{aligned} \Gamma(x)\Gamma(1-x) &= \frac{\pi}{\sin \pi x} \quad (x \in (0,1)), \\ \sqrt{\pi}\,\Gamma(x) &= 2^{x-1}\Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x+1}{2}\right) \quad (x \in D) \end{aligned} \tag{2}$$

and two easy applications:

Evaluate

$$\int_0^1 \ln(\Gamma(t))\,dt \qquad \text{and} \qquad \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\Gamma(\frac{m+n}{2})\Gamma(\frac{m+n+1}{2})}.$$

[the double sum is inspired by problem 868 of *The College Mathematics Journal* (January 2008).] Clearly, we have

$$\int_0^1 \ln(\Gamma(1-t))\,dt = \int_0^1 \ln(\Gamma(t))\,dt$$

and therefore

$$\begin{aligned} 2\int_0^1 \ln\Gamma(t)\,dt &= \int_0^1 \ln\left(\Gamma(t)\cdot\Gamma(1-t)\right)\,dt \\ &= \int_0^1 \ln\left(\frac{\pi}{\sin\pi t}\right)\,dt \\ &= \ln(\pi) - \frac{1}{\pi}\int_0^{\pi} \ln(\sin u)\,du \\ &= \ln(\pi) - \frac{1}{\pi}\cdot(-\pi\ln 2) = \ln(2\pi). \end{aligned}$$

Thus, $\int_0^1 \ln(\Gamma(t))\,dt = \ln(\sqrt{2\pi})$.

As for the double sum, letting $u(m,n) = \frac{1}{\Gamma(\frac{m+n}{2})\Gamma(\frac{m+n+1}{2})}$, we have $u(m,n) \geq 0$ for all positive integers m,n, hence we can evaluate $S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u(m,n)$ as follows:

$$S = \sum_{k=2}^{\infty}\sum_{m+n=k} u(m,n) = \sum_{k=2}^{\infty}(k-1)u(k-1,1) = \sum_{k=1}^{\infty} ku(k,1).$$

From (2), we obtain

$$\Gamma\left(\frac{k+1}{2}\right)\cdot\Gamma\left(\frac{k+2}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(k+1)}{2^{(k+1)-1}} = \frac{\sqrt{\pi}\,k!}{2^k}$$

so that $u(k,1) = \frac{2^k}{k!\sqrt{\pi}}$ for each positive integer k. As a result,

$$S = \sum_{k=1}^{\infty}\frac{2^k}{(k-1)!\sqrt{\pi}} = \frac{2}{\sqrt{\pi}}\sum_{\ell=0}^{\infty}\frac{2^{\ell}}{\ell!} = \frac{2e^2}{\sqrt{\pi}}.$$

Beta

The Beta function, denoted by B (or sometimes β), is also defined by an integral: for positive x,y,

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt.$$

The following formula links the Beta and Gamma functions

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

and readily shows that

$$B(y,x) = B(x,y), \quad B(x,n+1) = \frac{n!}{x(x+1)\cdots(x+n)}, \quad B(m,n) = \frac{1}{m\binom{m+n-1}{m}},$$

where m,n are positive integers. Here are two examples in which the Beta function allows a direct solution. The first one is a recent problem set in *Mathematics Magazine* in April 2019:

Prove that the series

$$\sum_{n=1}^{\infty} \frac{3 \cdot 6 \cdots (3n)}{7 \cdot 10 \cdots (3n+4)} \cdot \frac{1}{3n+7}$$

converges, and find its sum.

Let $u_n = \dfrac{3 \cdot 6 \cdots (3n)}{7 \cdot 10 \cdots (3n+4)} \cdot \dfrac{1}{3n+7}$. Then

$$u_n = \frac{3^n n!}{3^{n+1}(7/3)(7/3+1)\cdots(7/3+n)} = \frac{1}{3} \cdot \frac{\Gamma_n(7/3)}{n^{7/3}}$$

where $\Gamma_n(x) = \frac{n! \cdot n^x}{x(x+1)(x+2)\cdots(x+n)}$. From (1), we see that $u_n \sim \frac{\Gamma(7/3)}{3} \cdot \frac{1}{n^{7/3}} > 0$ as $n \to \infty$ and, since $7/3 > 1$, the series $\sum_{n \geq 1} u_n$ is convergent.

Now, we have $u_n = \frac{1}{3} B(7/3, n+1) = \frac{1}{3} \int_0^1 t^n (1-t)^{4/3}\, dt$, hence for any integer $N > 1$, we have:

$$\begin{aligned}
\sum_{n=1}^{N} u_n &= \frac{1}{3} \int_0^1 (1-t)^{4/3} \left(\sum_{n=1}^{N} t^n \right) dt \\
&= \frac{1}{3} \int_0^1 (1-t)^{1/3} (t - t^{N+1})\, dt \\
&= \frac{1}{3} (B(4/3, 2) - B(4/3, N+2)),
\end{aligned}$$

that is,

$$\sum_{n=1}^{N} u_n = \frac{1}{3}\Gamma_1(4/3) - \frac{\Gamma_{N+1}(4/3)}{3(N+1)^{4/3}} = \frac{1}{3} \cdot \frac{1}{4/3(4/3+1)} - \frac{\Gamma_{N+1}(4/3)}{3(N+1)^{4/3}} = \frac{3}{28} - \frac{\Gamma_{N+1}(4/3)}{3(N+1)^{4/3}}.$$

Since $\displaystyle\lim_{N \to \infty} \frac{\Gamma_{N+1}(4/3)}{(N+1)^{4/3}} = 0$ we can conclude that $\sum\limits_{n=1}^{\infty} u_n = \lim\limits_{N \to \infty} \sum\limits_{n=1}^{N} u_n = \frac{3}{28}$.

Our second example, the evaluation of a combinatorial sum, is provided by problem 11509 posed in *The American Mathematical Monthly* June-July 2010 issue:

Let m be a positive integer. Prove that

$$\sum_{k=m}^{m^2-m+1} \frac{\binom{m^2-2m+1}{k-m}}{k\binom{m^2}{k}} = \frac{1}{m\binom{2m-1}{m}}.$$

We observe that the right-hand side of the proposed identity is just $B(m, m)$. This prompts us to use the Beta function in the computation of the left side L:

$$\begin{aligned}
L &= \sum_{k=m}^{m^2-m+1} \binom{(m-1)^2}{k-m} B(k, m^2-k+1)\\
&= \sum_{j=0}^{(m-1)^2} \binom{(m-1)^2}{j} B(m+j, m^2-m-j+1)\\
&= \sum_{j=0}^{(m-1)^2} \int_0^1 \binom{(m-1)^2}{j} x^{m+j-1}(1-x)^{m^2-m-j}\,dx\\
&= \int_0^1 x^{m-1}(1-x)^{m-1}\left(\sum_{j=0}^{(m-1)^2} \binom{(m-1)^2}{j} x^j(1-x)^{(m-1)^2-j}\right)dx\\
&= \int_0^1 x^{m-1}(1-x)^{m-1}(x+(1-x))^{(m-1)^2}\,dx\\
&= \int_0^1 x^{m-1}(1-x)^{m-1}\,dx
\end{aligned}$$

so that $L = B(m,m)$, as desired.

Other integral presentations of the Beta function are possible. For example, the natural change of variables defined by $t = (\cos\theta)^2$ easily leads to

$$B(x,y) = 2\int_0^{\pi/2} (\cos\theta)^{2x-1}(\sin\theta)^{2y-1}\,d\theta.$$

Less obvious is the substitution $t = \frac{u}{1+u}$, which leads to

$$B(x,y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}}\,du.$$

The latter can be used in the solution to problem 1879 from the October issue of *Mathematics Magazine*:

> Let m and n be positive integers such that $m < n$ and let a and b be positive real numbers. Evaluate
>
> $$\int_0^\infty \frac{x^{2(n-m)}(x^2-1)^{2m}}{ax^{2n}+b(x^2-1)^{2n}}\,dx.$$

The given integral I is equal to

$$\int_0^1 \left(x-\frac{1}{x}\right)^{2m} \frac{1}{a+b\left(x-\frac{1}{x}\right)^{2n}}\,dx + \int_1^\infty \left(x-\frac{1}{x}\right)^{2m} \frac{1}{a+b\left(x-\frac{1}{x}\right)^{2n}}\,dx.$$

The changes of variable $x = e^{-t}$ in the first integral and $x = e^t$ in the second integral yield

$$I = \int_0^\infty \frac{(e^t - e^{-t})^{2m}(e^t + e^{-t})}{a + b(e^t - e^{-t})^{2n}}\, dt = \int_0^\infty \frac{v^{2m}}{a + bv^{2n}}\, dv.$$

Then, the change of variable $v = \left(\frac{a}{b}\right)^{\frac{1}{2n}} X^{\frac{1}{2n}}$ gives

$$I = \frac{1}{2nb^\alpha a^{1-\alpha}} \int_0^\infty \frac{X^{\alpha-1}}{X+1}\, dX$$

where $\alpha = \dfrac{2m+1}{2n}$.

Since

$$\int_0^\infty \frac{X^{\alpha-1}}{X+1}\, dX = B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \pi \csc(\pi\alpha)$$

we obtain

$$I = \frac{\pi \csc(\pi\alpha)}{2nb^\alpha a^{1-\alpha}}.$$

Digamma

It is also worth giving some results about the Digamma function. This function, denoted by ψ, is defined as $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ for positive x. Note that a formula seen in (2) gives

$$\psi(1-x) - \psi(x) = \pi \cot(\pi x) \qquad (x \in (0,1)).$$

Using (1), an alternative expression is obtained as

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{x+n}\right)$$

where γ denotes the Euler constant. A useful consequence is that for $p, q > 0$

$$\psi(q) - \psi(p) = \sum_{n=0}^{\infty} \left(\frac{1}{n+p} - \frac{1}{n+q}\right),$$

a relation that leads to an immediate solution of **4511** [2020 : 77 ; 2021 : 320]:

Evaluate the following sum in closed form:

$$\sum_{n=1}^{\infty} \left(\frac{1}{8n-7} - \frac{1}{8n-1}\right).$$

The given sum equals

$$\frac{1}{8}\sum_{n=0}^{\infty} \left(\frac{1}{n+\frac{1}{8}} - \frac{1}{n+\frac{7}{8}}\right) = \frac{1}{8}\left(\psi\left(1 - \frac{1}{8}\right) - \psi\left(\frac{1}{8}\right)\right) = \frac{\pi \cot(\pi/8)}{8} = \frac{\pi}{8(\sqrt{2}-1)}$$

and we conclude
$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-1}\right)=\frac{\pi(1+\sqrt{2})}{8}.$$

Before our last example, let us show that if a and $a+b$ are positive, then
$$\psi(a+b)-\psi(a)=\int_0^1\frac{t^{a-1}(1-t^b)}{1-t}\,dt. \tag{3}$$

We remark that
$$\frac{t^{a-1}(1-t^b)}{1-t}=\sum_{n=0}^{\infty}(t^{n+a-1}-t^{n+a+b-1})$$

and that $t^{n+a-1}-t^{n+a+b-1}$ has the same sign as b when $t\in(0,1)$. From
$$\begin{aligned}\sum_{n=0}^{\infty}\int_0^1|t^{n+a-1}-t^{n+a+b-1}|\,dt&=\sum_{n=0}^{\infty}\left|\int_0^1(t^{n+a-1}-t^{n+a+b-1})\,dt\right|\\&=\sum_{n=0}^{\infty}\frac{|b|}{(n+a)(n+a+b)}\\&<\infty,\end{aligned}$$

we can interchange sum and integral and get
$$\begin{aligned}\int_0^1\frac{t^{a-1}(1-t^b)}{1-t}\,dt&=\sum_{n=0}^{\infty}\int_0^1(t^{n+a-1}-t^{n+a+b-1})\,dt\\&=\sum_{n=0}^{\infty}\left(\frac{1}{n+a}-\frac{1}{n+a+b}\right)\\&=\psi(a+b)-\psi(a).\end{aligned}$$

We make use of (3) in the following solution to problem 11564 proposed in the April 2011 issue of *The American Mathematical Monthly*:

Prove that $\displaystyle\int_0^{\infty}\frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}\,dx=\ln\left(\frac{3+\sqrt{5}}{2}\right).$

First, the substitution $e^{-10x}=t$ shows that the integral is equal to
$$I=\int_0^1\frac{t^{\frac{1}{10}-1}(1-t^{\frac{3}{5}})(1-t^{\frac{1}{5}})}{(1-t)(-\ln t)}\,dt.$$

Second, suppose that $a, a+b, a+c$, and $a+b+c$ are positive. Then, for $t\in(0,1)$ we have
$$\frac{t^{a-1}(1-t^b)}{-\ln t}=\int_a^{a+b}t^{u-1}\,du$$

so that

$$\begin{aligned}\int_0^1 \frac{t^{a-1}(1-t^b)(1-t^c)}{(1-t)(-\ln t)}\,dt &= \int_0^1 \left(\int_a^{a+b} \frac{t^{u-1}(1-t^c)}{1-t}\,du\right)\,dt \\ &= \int_a^{a+b} \left(\int_0^1 \frac{t^{u-1}(1-t^c)}{1-t}\,dt\right)\,du \\ &= \int_a^{a+b} (\psi(u+c)-\psi(u))\,du = [\ln(\Gamma(u+c)-\ln(\Gamma(u)]_a^{a+b} \\ &= \ln\left(\frac{\Gamma(a+b+c)\Gamma(a)}{\Gamma(a+b)\Gamma(a+c)}\right).\end{aligned}$$

Taking $a=\frac{1}{10}, b=\frac{3}{5}, c=\frac{1}{5}$ leads to

$$I = \ln\left(\frac{\Gamma(9/10)\Gamma(1/10)}{\Gamma(3/10)\Gamma(7/10)}\right)$$

and the requested result follows since

$$\frac{\Gamma(9/10)\Gamma(1/10)}{\Gamma(3/10)\Gamma(7/10)} = \frac{\sin(3\pi/10)}{\sin(\pi/10)} = 3-4\sin^2(\pi/10) = 1+2\cos(\pi/5) = 1+\frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}.$$

Exercises

1. (From problem 91.J posed in *The Mathematical Gazette* in November 2007). For $\alpha,\beta>0$, prove that

$$\int_0^1 (1-x^\alpha)^{\frac{1}{\beta}}\,dx = \int_0^1 (1-x^\beta)^{\frac{1}{\alpha}}\,dx.$$

2. Let $a\in(0,1)$. For $n\in\mathbb{N}$, let $I_n(a)=\displaystyle\int_0^\infty \frac{dt}{(1+t^{1/a})^n}$. Find $\displaystyle\lim_{n\to\infty} n^a I_n(a)$.
[*Hint*: substitution $t=(1-u)^a u^{-a}$]

3. Let a,b,c,d,k be positive real numbers such that $k=a+b=c+d$. Prove that

$$\prod_{n=0}^{\infty} \frac{(kn+a)(kn+b)}{(kn+c)(kn+d)} = \frac{\sin(\pi a/k)}{\sin(\pi c/k)}.$$

4. (Problem **3567** [2010 : 396, 398 ; 2011 : 400]) Prove that

$$\int_0^\infty \frac{e^{-x}(1-e^{-2x})(1-e^{-4x})(1-e^{-6x})}{x(1-e^{-14x})}\,dx = \ln 2.$$

References

[1] E. Artin, *The Gamma Function*, Holt, Rinehart and Winston, 1964

[2] K.R. Stromberg, *An Introduction to Classical Real Analysis*, Wadsworth, 1981

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 30, 2022**.

4701. *Proposed by Michel Bataille.*

Let AD, BE, CF be the internal angle bisectors of ΔABC (with D on BC, E on CA, F on AB). Let the perpendicular to BC through D intersect the perpendicular bisector of AD at A' and let B', C' be similarly constructed. Prove that the lines AA', BB', CC' are concurrent and that

$$AA' \cdot BB' \cdot CC' \leq \left(\frac{3R}{4}\right)^3,$$

where R is the circumradius of ΔABC.

4702. *Proposed by S. Chandrasekhar.*

Let p be a prime which is congruent to 3 (mod 4). Let S denote the set of square elements in the field of integers modulo p. Then show that

$$\prod_{\substack{a\,<\,b \\ a,\,b\,\in\,S}} (a+b) = \pm 1 \pmod{p}.$$

4703. *Proposed by Jiahao Chen.*

Given a triangle ABC with circumcenter O, denote by DEF the triangle formed by the tangents to the circumcircle at A, B, C with A on EF, B on FD, and C on DE. If D', E', F' are the reflections of D, E, F in the lines BC, CA, AB, respectively, prove that $D'E' || OB$ if and only if $D'F' || OC$.

4704. *Proposed by Daniel Sitaru.*

For $a, b, c, d \in [0, 1)$, prove that

$$\frac{1}{1-a^6} + \frac{1}{1-b^6} + \frac{1}{1-c^6} + \frac{1}{1-d^2} \geq \frac{2}{1-(abc)^2} + \frac{2}{1-abcd}.$$

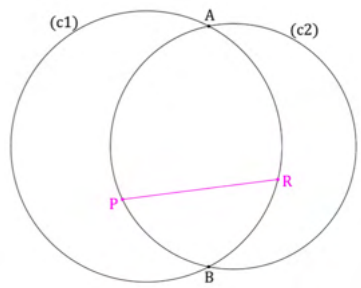
4705. *Proposed by Nguyen Viet Hung.*

Find the following limit

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}.$$

4706. *Proposed by Thanos Kalogerakis.*

In the figure below, find the midpoint of segment PR using the straightedge alone and prove that your construction works.



4707. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$. Prove that

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \frac{n^2-1}{3} \quad \text{and} \quad \sum_{k=1}^{n-1} \csc^4\left(\frac{k\pi}{n}\right) = \frac{n^4+10n^2-11}{45}.$$

4708. *Proposed by Conar Goran.*

Let α, β, γ be angles of an arbitrary triangle. Prove that the following inequality holds

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{3} \leq \cot\left(\frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}\right).$$

When does the equality occur?

4709. *Proposed by Ion Patrascu.*

Let ABC be an acute triangle and O the center of its circumcircle. We denote by D, E and F the intersections of the lines AO and BC, BO and CA, CO and AB, respectively. If $BD\cos A = CE\cos B = AF\cos C$, prove that ABC is an equilateral triangle.

4710$\star$. *Proposed by Omar Sonebi, modified by the Editorial Board.*

Show that there exist 2021 consecutive natural numbers none of which is the sum of a perfect square and a perfect cube.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2022**.

4701. *Proposeé par Michel Bataille.*

Soient AD, BE et CF les bissectrices internes des angles de ΔABC, où D se trouve sur BC, E sur CA et F sur AB. La perpendiculaire vers BC passant par D rencontre la bissectrice perpendiculaire de AD en A' ; B' et C' sont définis de façon similaire. Démontrer que les lignes AA', BB' et CC' sont concourantes et que

$$AA' \cdot BB' \cdot CC' \leq \left(\frac{3R}{4}\right)^3,$$

où R est le rayon du cercle circonscrit de ΔABC.

4702. *Proposeé par S. Chandrasekhar.*

Soit p un nombre premier congru à 3 (mod 4) et soit S l'ensemble des éléments carrés dans le corps des entiers modulo p. Démontrer que

$$\prod_{\substack{a\,<\,b\\ a,\,b\,\in\,S}} (a+b) = \pm 1 \pmod{p}.$$

4703. *Proposeé par Jiahao Chen.*

Soit O le centre du cercle circonscrit de ABC. Les tangentes à ce cercle, en A, B et C, forment un triangle DEF, où A se trouve sur EF, B sur FD, puis C sur DE. Soient alors D', E' et F' les reflexions de D, E et F par rapport aux lignes BC, CA et AB, respectivement. Démontrer que $D'E'||OB$ si et seulement si $D'F'||OC$.

4704. *Proposeé par Daniel Sitaru.*

Si $a, b, c, d \in [0, 1)$, démontrer que

$$\frac{1}{1-a^6}+\frac{1}{1-b^6}+\frac{1}{1-c^6}+\frac{1}{1-d^2} \geq \frac{2}{1-(abc)^2}+\frac{2}{1-abcd}.$$

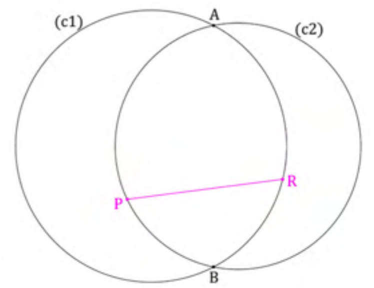
4705. *Proposeé par Nguyen Viet Hung.*

Déterminer la limite suivante

$$\lim_{n\to\infty} \frac{1}{\sqrt[3]{n^2}} \sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}}.$$

4706. *Proposeé par Thanos Kalogerakis.*

Déterminer le point milieux du segment PR au schéma ci-bas, seulement à l'aide d'une règle non graduée; démontrer que votre construction marche.



4707. *Proposeé par Michel Bataille.*

Soit n un entier tel que $n \geq 2$. Démontrer que

$$\sum_{k=1}^{n-1} \csc^2\left(\frac{k\pi}{n}\right) = \frac{n^2-1}{3} \quad \text{et} \quad \sum_{k=1}^{n-1} \csc^4\left(\frac{k\pi}{n}\right) = \frac{n^4+10n^2-11}{45}.$$

4708. *Proposeé par Conar Goran.*

Soient α, β, γ les angles d'un triangle quelconque. Démontrer l'inégalité qui suit et déterminer toute condition pour que l'égalité tienne:

$$\frac{\cot\alpha + \cot\beta + \cot\gamma}{3} \leq \cot\left(\frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}\right).$$

4709. *Proposeé par Ion Patrascu.*

Soit ABC un triangle acutangle et soit O le centre de son cercle circonscrit. Dénotons par D, E et F les points d'intersection des lignes AO et BC, puis BO et AC et enfin, CO et AB, respectivement. Si $BD\cos A = CE\cos B = AF\cos C$, démontrer que ABC est équilatéral.

4710$\star$. *Proposeé par Omar Sonebi, avec modification venant de l'éditeur.*

Démontrer l'existence de 2021 entiers naturels consécutifs, dont aucun est la somme d'un carré d'un entier naturel et d'un cube d'un entier naturel.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(6), p. 300–305.

4651. *Proposed by Michel Bataille.*

The complex numbers z_1 and z_2 represent points on or inside the unit circle of the Euclidean plane such that both $\mathrm{Re}(z_1 + z_2) \geq 1$ and $\mathrm{Im}(z_1 + z_2) \geq 1$. Find the extremal values of $\mathrm{Re}(z_1 z_2)$ and the pairs (z_1, z_2) at which they are attained.

We received 10 solutions, 8 of which were correct, 1 was incomplete and 1 was incorrect.

The minimum value is $-\frac{1}{2}$, attained when

$$z_1 = z_2 = \cos\pi/3 + i\sin\pi/3 = \frac{1}{2}[1 + i\sqrt{3}];$$

the maximum value is $\frac{1}{2}$, attained when

$$z_1 = z_2 = \cos\pi/6 + i\sin\pi/6 = \frac{1}{2}[\sqrt{3} + i].$$

Solution 1, by Oliver Geupel.

Note that since the real and imaginary parts of z_1 and z_2 do not exceed 1, while $\mathrm{Re}(z_1 + z_2) \geq 1$ and $\mathrm{Im}(z_1 + z_2) \geq 1$, it follows that z_1 and z_2 lie in the first quadrant. Therefore $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$ where $0 \leq r_1, r_2 \leq 1$ and $0 \leq \phi_1, \phi_2 \leq \pi/2$.

Recall that the sine and cosine are both concave functions on $[0, \pi/2]$, so that

$$1 \leq \mathrm{Re}\ (z_1 + z_2) = r_1\cos\phi_1 + r_2\cos\phi_2 \leq \cos\phi_1 + \cos\phi_2 \leq 2\cos\tfrac{1}{2}[\phi_1 + \phi_2],$$

and

$$1 \leq \mathrm{Im}\ (z_1 + z_2) = r_1\sin\phi_1 + r_2\sin\phi_2 \leq \sin\phi_1 + \sin\phi_2 \leq 2\sin\tfrac{1}{2}[\phi_1 + \phi_2].$$

From these two inequalities, we find that $\pi/3 \leq \phi_1 + \phi_2 \leq 2\pi/3$, whence

$$-\tfrac{1}{2} = \cos\tfrac{2\pi}{3} \leq r_1 r_2\cos(\phi_1 + \phi_2) = \mathrm{Re}\ (z_1 z_2) \leq \cos\tfrac{\pi}{3} = \tfrac{1}{2}.$$

Equality on the left occurs when $r_1 = r_2 = 1$, $\phi_1 = \phi_2 = \pi/3$, and on the right when $r_1 = r_2 = 1$, $\phi_1 = \phi_2 = \pi/6$.

Solution 2, by Borche Joshevski.

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, so that $x_1 + x_2 \geq 1$, $y_1 + y_2 \geq 1$, $x_1^2 + y_1^2 \leq 1$, and $x_2^2 + y_2^2 \leq 1$. Then $0 \leq x_1, x_2, y_1, y_2 \leq 1$ and

$$\begin{aligned}\mathrm{Re}(z_1 z_2) &= x_1 x_2 - y_1 y_2 \leq \sqrt{(1-y_1^2)(1-y_2^2)} - y_1 y_2 \\ &\leq \tfrac{1}{2}(1 - y_1^2 + 1 - y_2^2) - y_1 y_2 = 1 - \tfrac{1}{2}(y_1 + y_2)^2 \leq \tfrac{1}{2},\end{aligned}$$

with equality if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$ and $y_1 = y_2 = \frac{1}{2}(y_1 + y_2) = \frac{1}{2}$, *i.e.* when $x_1 = x_2 = \sqrt{3}/2$ and $y_1 = y_2 = 1/2$.

Similarly

$$\begin{aligned}-\mathrm{Re}(z_1 z_2) &= y_1 y_2 - x_1 x_2 \leq \sqrt{(1-x_1^2)(1-x_2^2)} - x_1 x_2 \\ &\leq \tfrac{1}{2}(1 - x_1^2 + 1 - x_2^2) - x_1 x_2 = 1 - \tfrac{1}{2}(x_1 + x_2)^2 \leq \tfrac{1}{2},\end{aligned}$$

with equality if and only if $x_1 = x_2 = 1/2$ and $y_1 = y_2 = \sqrt{3}/2$.

Solution 3, by Theo Koupelis.

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, so that $x_1 + x_2 \geq 1$, $y_1 + y_2 \geq 1$, $x_1^2 + y_1^2 \leq 1$, and $x_2^2 + y_2^2 \leq 1$. Since $2x_1 x_2 \leq x_1^2 + x_2^2$ and $2y_1 y_2 \leq y_1^2 + y_2^2$, we have that

$$\begin{aligned}\mathrm{Re}\ (z_1 z_2) &= x_1 x_2 - y_1 y_2 \geq \tfrac{1}{2}[(x_1 + x_2)^2 - (x_1^2 + x_2^2) - (y_1^2 + y_2^2)] \\ &\geq \tfrac{1}{2}[1 - (x_1^2 + y_1^2) - (x_2^2 + y_2^2)] \geq -\tfrac{1}{2},\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\ (z_1 z_2) &= x_1 x_2 - y_1 y_2 \leq \tfrac{1}{2}[(x_1^2 + x_2^2) - (y_1 + y_2)^2 + (y_1^2 + y_2)^2] \\ &\leq \tfrac{1}{2}[(x_1^2 + y_1^2) + (x_2^2 + y_2^2) - 1] \leq \tfrac{1}{2}.\end{aligned}$$

Equality occurs in the first instance when we have $y_1 = y_2$, $x_1 + x_2 = 1$ and $x_1^2 + y_1^2 = x_2^2 + y_2^2 = 1$. Since $0 \leq y_1, y_2 \leq 1$, then we get $y_1 = y_2 = 1/2$ and $x_1 = x_2 = \sqrt{3}/2$.

Equality occurs in the second instance when $x_1 = x_2$, $y_1 + y_2 = 1$, which leads to $y_1 = y_2 = 1/2$ and $x_1 = x_2 = \sqrt{3}/2$.

4652. *Proposed by Nguyen Viet Hung.*

Let ABC be an equilateral triangle with centroid O and let M be any point inside of the triangle. D, E, F are feet of altitudes from M onto the sides BC, CA, AB respectively. Prove that

$$(MD - ME)^4 + (ME - MF)^4 + (MF - MD)^4 = \frac{81}{8} MO^4.$$

We received 15 submissions, all of which were correct, and present a composite of the similar solutions submitted independently by Kee-Wai Lau, Didier Pinchon, and Sorin Rubinescu.

We fix a coordinate system so that the given equilateral triangle has its centroid at the origin, and its vertices are $A = (2, 0), B = (-1, \sqrt{3})$, and $C = (-1, -\sqrt{3})$. The equations of BC, CA, and AB are respectively $x + 1 = 0, x - \sqrt{3}y - 2 = 0$, and $x + \sqrt{3}y - 2 = 0$. We set $M = (s, t)$; because we assume that M is in the interior of ΔABC, we have $s + 1 > 0, s - \sqrt{3}t - 2 < 0$, and $s + \sqrt{3}t - 2 < 0$. It follows that the distance from M to the sides (using the formula for distance from a point to a line) are

$$MD = s + 1, \quad ME = \frac{-s + \sqrt{3}t + 2}{2}, \quad MF = \frac{-s - \sqrt{3}t + 2}{2} \ ,$$

whence

$$MD - ME = \frac{3s - \sqrt{3}t}{2}, \quad ME - MF = \sqrt{3}t, \quad MF - MD = \frac{-3s - \sqrt{3}t}{2}. \quad (1)$$

Finally, by direct expansion of the expressions in (1), we conclude that

$$(MD - ME)^4 + (ME - MF)^4 + (MF - MD)^4 = \frac{81}{8}(s^2 + t^2)^2 = \frac{81}{8} MO^4 \ ,$$

as required.

Editor's comments. Several readers observed that the analogous result with the exponent "4" replaced by "2" also has an attractive form, namely

$$(MD - ME)^2 + (ME - MF)^2 + (MF - MD)^2 = \frac{9}{2} MO^2 \ ;$$

this result also follows immediately from the equations in (1). Giuseppe Fera went a step further and proved the lovely generalization,

For the points defined in the original statement of the problem together with a fixed positive integer n, there exists a constant K_n for which

$$(MD - ME)^n + (ME - MF)^n + (MF - MD)^n = K_n \cdot MO^n$$

if and only if n equals 1, 2, or 4.

It is easily seen that $K_1 = 0$, while we already know that $K_2 = \frac{9}{2}$ and $K_4 = \frac{81}{8}$. For his proof that no other value of n will lead to a constant, he considered the following function of the variable point $M = (s, t)$:

$$f(M) = \frac{(MD - ME)^n + (ME - MF)^n + (MF - MD)^n}{MO^n}.$$

He then found two explicit points M that produced different values of $f(M)$, and thereby concluded that $f(M)$ could not be constant. He used barycentric coordinates for his calculations, but you might wish to try it for yourselves using the points $M = (s, 0)$ and $M = (0, t)$ in the above equations in (1), and observe that you get different results for all n except, miraculously, for $n = 1$, $n = 2$, and $n = 4$.

4653. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with inradius r and circumradius R. It is known (e.g. Item 2.48 on page 31 of *"Geometric Inequalities"* by Bottema et al.) that

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\geq 4.$$

Prove that

$$\sec^2\frac{A}{2}+\sec^2\frac{B}{2}+\sec^2\frac{C}{2}\leq\frac{2R}{r}.$$

We received 41 solutions, all of which were correct. This included 22 solutions by Mehra Vivek. We present the solution by C. R. Pranesachar.

Let $a=BC,\ b=CA,\ c=AB$ be the side lengths of triangle ABC; let s be its semiperimeter; and F its area. We have

$$\sec^2\frac{A}{2}=\frac{1}{\cos^2\frac{A}{2}}=\frac{bc}{s(s-a)},$$

with similar expressions for $\sec^2\frac{B}{2}$ and $\sec^2\frac{C}{2}$. Also

$$R=\frac{abc}{4F},\ r=\frac{F}{s},\ F^2=s(s-a)(s-b(s-c),$$

so

$$\frac{2R}{r}=\frac{2\cdot\frac{abc}{4F}}{\frac{F}{s}}=\frac{abcs}{2F^2}\ .$$

Hence in the given inequality

$$\begin{aligned}
\text{rhs}-\text{lhs}\ =&\frac{abcs}{2F^2}-\left(\frac{bc}{s(s-a)}+\frac{ca}{s(s-b)}+\frac{ab}{s(s-c)}\right)\\
=&\frac{abcs}{2F^2}-\frac{bc(s-b)(s-c)+ca(s-c)(s-a)+ab(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\\
=&\frac{1}{4F^2}\big(abc(a+b+c)-bc(c+a-b)(a+b-c)\\
&-ca(a+b-c)(b+c-a)-ab(b+c-a)(c+a-b)\big)\\
=&\frac{1}{4F^2}\big(abc(a+b+c)-bc(a^2-(b-c)^2)\\
&-ca(b^2-(c-a)^2)-ab(c^2-(a-b)^2)\big)\\
=&\frac{1}{4F^2}(bc(b-c)^2+ca(c-a)^2+ab(a-b)^2)\geq 0.
\end{aligned}$$

This proves the inequality. Also we see that equality holds good if and only if $a=b=c$, that is, if and only if triangle ABC is equilateral. This completes the proof.

4654. *Proposed by Andrei Eckstein and Leonard Giugiuc.*

Consider positive real numbers $a_1, a_2, \dots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

where $n \geq 3$. Prove that

$$\sum_{i<j} a_i a_j \geq \frac{n(n-1)}{2}.$$

We received 13 solutions, all correct. Many of the submitted solutions used Maclaurin's inequality or Newton's inequalities, but we have chosen to include a solution that uses no heavy machinery. We present the solution by Mehra Vivek.

Let

$$S = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \frac{1}{a_i}.$$

For each index i, we find that

$$(S - a_i)\left(S - \frac{1}{a_i}\right) \geq (n-1)^2$$

by the AM-HM inequality because each of the two multiplicands have $n-1$ terms, and the terms of the left one are the reciprocals of the terms of the right one. We can manipulate this to get

$$\begin{aligned}(S - a_i)(a_i S - 1) &\geq a_i (n-1)^2\\ a_i S^2 - a_i^2 S + a_i - S &\geq a_i (n-1)^2.\end{aligned}$$

Adding copies of this inequality for $i = 1, 2, \dots, n$, we find that

$$\begin{aligned}S^3 - S \cdot \sum_{i=1}^{n} a_i^2 + S - nS &\geq (n-1)^2 S\\ S^2 - \sum_{i=1}^{n} a_i^2 + 1 - n &\geq (n-1)^2\\ S^2 - \sum_{i=1}^{n} a_i^2 &\geq (n-1)^2 + (n-1) = n(n-1)\\ \frac{1}{2} \cdot \left(S^2 - \sum_{i=1}^{n} a_i^2\right) &\geq \frac{n(n-1)}{2}.\end{aligned}$$

But this is what we wanted to prove because, by multinomial expansion,

$$\frac{1}{2} \cdot \left(S^2 - \sum_{i=1}^{n} a_i^2\right) = \sum_{1 \leq i < j \leq n} a_i a_j.$$

4655. *Proposed by Daniel Brin.*

Let $A = (a_{ij})$ be a matrix of order n where $n > 1$ is odd. Let $C = (-1)^{i+j} M_{ij}$ denote the cofactor matrix of A where M_{ij} are the minors of A. If X is an $n \times n$ matrix such that $XMX = C$, find the sum of all the entries of X.

The problem turned out to be ill posed, and attracted one submission from UC Lan Cyprus Problem Solving Group, upon which the following comments are based.

The matrix X that satisfies $XMX = C$ is not unique; replacing X by $-X$ will do the trick as well. The proposer had in mind the solution

$$X = \text{diag}\,(-1, 1, -1, 1, \ldots, -1, 1, -1),$$

for which the entry sum is -1; but $X = \text{diag}\,(1, -1, 1, -1, \ldots, 1)$ with entry sum 1 will work as well.

In certain situations, the set of matrix solutions is infinite. For example, when $A = I$, then $M = C = I$ and all we require is $X^2 = I$. This is satisfied, for example, by the matrix X whose diagonal entries are all 1 except for the last entry which is -1, all of whose other entries vanish except possibly for the offdiagonal entry r in position $(n-1, n)$; this has entry sum $n - 2 + r$ for any number r.

If the rank of A does not exceed $n-2$, then $M = C = O$ and there is no restriction on X at all. This leaves open the question as to whether anything interesting or useful can be said for the number of solutions for other matrices A. A related investigation can treat the solutions of the equation $X^{-1}MX = C$.

4656. *Proposed by Abdollah Zohrabi.*

If a, b, c and d are positive real numbers such that $abcd = 1$, prove that

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq 2(ab + cd)(bd + ac)(cb + da).$$

We received 22 submissions of which 19 were correct and complete. We present two solutions.

Solution 1, by Brian Bradie.

By Hölder's inequality,

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq (1 + abcd)^4 = 2^4.$$

Also by Hölder's inequality,

$$\begin{aligned}
(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) &\geq (ab + cd)^4, \\
(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) &\geq (bd + ac)^4, \text{ and} \\
(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) &\geq (cb + da)^4.
\end{aligned}$$

Multiplying these four inequalities together and then taking the fourth root yields

$$(1 + a^4)(1 + b^4)(1 + c^4)(1 + d^4) \geq 2(ab + cd)(bd + ac)(cb + da).$$

Solution 2, by Emil Khalilov.

Let x, y, z, t be permutation of a, b, c, d such that $xy + zt$ is maximum. We have:

$$xy + zt \geq 2\sqrt{xyzt} = 2\sqrt{abcd} = 2.$$

So,

$$\begin{aligned} P = (1+a^4)(1+b^4)(1+c^4)(1+d^4) &= (1+x^4)(z^4+1)(1+y^4)(t^4+1) \\ \text{(by Cauchy-Schwarz)} \quad &\geq ((x^2+z^2)(y^2+t^2))^2 \geq (xy+zt)^4 \\ &= (xy+yz)(xy+yz)(xy+yz)(xy+yz) \\ &\geq 2(ab+cd)(ac+bd)(ad+bc). \end{aligned}$$

Equality occurs when $a = b = c = d = 1$.

4657. *Proposed by George Stoica.*

Let us consider the equation $f(x) + f(2x) = 0$, $x \in \mathbb{R}$.

(i) Prove that, if f is continuous at 0, then $f(x) = 0$ for all $x \in \mathbb{R}$.

(ii) Construct a function f, discontinuous at every $x \in \mathbb{R}$, that solves the given equation.

We received 14 submissions, 13 of them were complete and correct. We present the solution by the majority of solvers.

(i) Let $x \in \mathbb{R}$. Using induction and the given functional equation, we deduce that

$$f(2^{-n}x) = (-1)^n f(x), \quad \forall n \in \mathbb{N}.$$

Letting $n \to \infty$ and using the continuity of f at 0, the left hand side converges to $f(0) = 0$. Hence the right hand side must converge to 0 as well, that is, $f(x) = 0$.

(ii) Given a nonzero integer n, write $v_2(n)$ for the maximum power of 2 dividing n. Given a nonzero rational number x, we can write $x = m/n$ for some relatively prime integers m, n, and define $v_2(x) = v_2(m) - v_2(n)$. Then,

$$f(x) = \begin{cases} (-1)^{v_2(x)}, & \text{if } x \in \mathbb{Q} \setminus \{0\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that $f(x) + f(2x) = 0$ holds for each $x \in \mathbb{R}$, and that f is discontinuous everywhere since both rational numbers and irrational numbers are dense in $\mathbb{R}$.

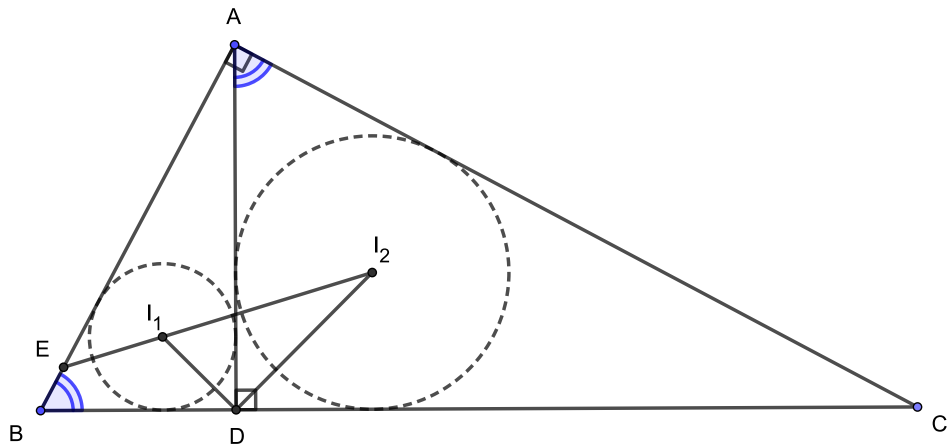
Editor's Comment. Raymond Mortini, C. R. Pranesachar, and Rudolf Rupp pointed out such a function f is uniquely determined by its restriction to $[-2,-1) \cup \{0\} \cup [1,2)$: for an arbitrary function g defined on $[-2,-1) \cup [1,2)$, we may extend g to $\mathbb{R}$ by setting

$$f(x) = (-1)^n g(x/2^n)$$

for each $x \neq 0$, where n is the unique integer such that $|x| \in [2^n, 2^{n+1})$; and setting $f(0)$ arbitrarily. In particular, to find a nowhere continuous f, it suffices to find an arbitrary nowhere continuous function g defined on $[-2,1) \cup [1,2)$ and use the above extension of g to $\mathbb{R}$.

4658. *Proposed by Mihaela Berindeanu.*

In the right triangle ABC, let D be the foot of the altitude on the hypotenuse BC, and let I_1 and I_2 be the incenters of triangles ABD and ADC, respectively.



Denote by E the point of intersection of I_1I_2 with AB. Since AD is the height of the right-angled $\triangle ABC$, we have $\triangle ADC \sim \triangle BDA$, and we can conclude that $\dfrac{DI_2}{DI_1} = \dfrac{AC}{BA}$. Note moreover that $\angle ADI_1 = \angle ADI_2 = 45°$, and so $\angle I_1DI_2 = 90°$. It follows that $\triangle I_1DI_2 \sim \triangle BAC$; in particular, $\angle DI_1I_2 = \angle ABC$. We have shown that for the quadrilateral I_1DBE the exterior angle $\angle DI_1I_2$ is equal to its opposite interior angle $\angle DBA$, so the quadrilateral is cyclic. Therefore, E belongs to the circumcircle of BDI_1, as required.

4659. *Proposed by Tien Nguyen.*

For each positive integer n, find $\gcd(a_n, b_n)$ such that

$$(4+\sqrt{5})^n = a_n + b_n\sqrt{5},$$

where a_n and b_n are positive integers.

We received 21 submissions, 20 of which were correct and complete. Presented is the solution by the UCLan Cyprus Problem Solving Group, slightly edited.

We have $(a_n+b_n\sqrt{5})(4+\sqrt{5}) = (4a_n+5b_n)+(a_n+4b_n)\sqrt{5}$. So we get the recurrence relations $a_{n+1} = 4a_n + 5b_n$ and $b_{n+1} = 4b_n + a_n$ with initial conditions $a_1 = 4$, $b_1 = 1$.

We will show by induction that $\gcd(a_n, b_n) = 1$ for every integer n. For the inductive step assume that a prime p divides both a_{n+1} and b_{n+1}. Then p divides $4a_{n+1} - 5b_{n+1} = 11a_n$ and $4b_{n+1} - a_{n+1} = 11b_n$. Since by the induction hypothesis $\gcd(a_n, b_n) = 1$, then $p = 11$. We will obtain a contradiction by showing that $11 \nmid a_n$ for every $n \in \mathbb{N}$. To this end, oberve that

$$\begin{aligned} a_{n+2} = 4a_{n+1} + 5b_{n+1} &= 4a_{n+1} + 20b_n + 5a_n \\ &= 4a_{n+1} + 4(a_{n+1} - 4a_n) + 5a_n \\ &= 8a_{n+1} - 11a_n. \end{aligned}$$

Thus $a_{n+2} \equiv 8a_{n+1} \pmod{11}$ for each $n \in \mathbb{N}$ and since $a_1 = 4, a_2 = 21$, then $a_n \not\equiv 0 \pmod{11}$ for every $n \in \mathbb{N}$ as required.

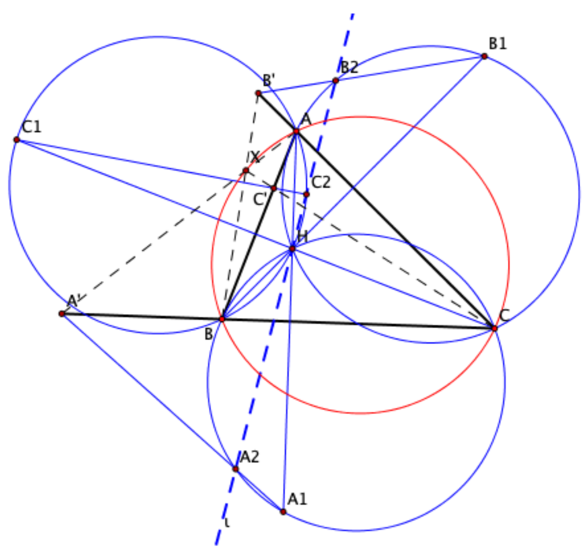
4660. *Proposed by Thanh Tung Vu, modified by the Editorial Board.*

a) Given a triangle ABC with its orthocenter H, define the three circles

$$\alpha = (HBC), \quad \beta = (HCA), \quad \text{and} \quad \gamma = (HAB).$$

For a fixed line ℓ through H let
A_1 and A_2 be the points where α again meets ℓ and AH,
B_1 and B_2 be the points where β again meets ℓ and BH,
C_1 and C_2 be the points where γ again meets ℓ and CH.

Finally, define $A' = BC \cap A_1A_2$, $B' = CA \cap B_1B_2$, $C' = AB \cap C_1C_2$. Prove that the cevians AA', BB', CC' are concurrent at some point X of the circumcircle of ΔABC.



b)* Establish the corresponding result with the orthocenter H replaced by an arbitrary point P not on a side of ΔABC; prove that the locus of resulting point X as ℓ turns about P is an ellipse that circumscribes ΔABC.

We received three submissions. All were correct, but the calculations in two of them required a computer. We present the solution by the UCLan Cyprus Problem Solving Group, with one detail expanded by the editor.

We work with a typical arrangement of the configuration as depicted in the given diagram, except that we replace the orthocenter H there with an arbitrary point P. In particular, we will not use directed angles; later on we shall add a few comments together with a simpler, alternative argument for part (a) (where P is the orthocenter). In triangles $A'A_2B$ and $A'A_2C$ we have

$$\frac{A'B}{BA_2} = \frac{\sin(\angle A'A_2B)}{\sin(\angle A_2A'B)} \quad \text{and} \quad \frac{A'C}{CA_2} = \frac{\sin(\angle A'A_2C)}{\sin(\angle A_2A'C)} = \frac{\sin(\angle A'A_2C)}{\sin(\angle A_2A'B)}\,.$$

Since B, P, C, A_1, A_2 are concyclic, then

$$\angle A'A_2B = 180^\circ - \angle BA_2A_1 = \angle BPA_1 = 180^\circ - \angle APB,$$

and

$$\angle A'A_2C = 180^\circ - \angle A_1A_2C = \angle 180^\circ - \angle A_1PC = \angle APC.$$

Therefore

$$\frac{A'B}{A'C} = \frac{BA_2}{CA_2} \cdot \frac{\sin(\angle A'A_2B)}{\sin(\angle A'A_2C)} = \frac{BA_2}{CA_2} \cdot \frac{\sin(\angle APB)}{\sin(\angle APC)}\,.$$

Furthermore, in ΔBA_2C

$$\frac{BA_2}{CA_2} = \frac{\sin(\angle BCA_2)}{\sin(\angle CBA_2)} = \frac{\sin(\angle BPA_2)}{\sin(\angle CPA_2)}\,.$$

Thus

$$\frac{A'B}{A'C} = \frac{\sin(\angle BPA_2)}{\sin(\angle CPA_2)} \cdot \frac{\sin(\angle APB)}{\sin(\angle APC)}\,.$$

We also have the analogous expressions

$$\frac{B'C}{B'A} = \frac{\sin(\angle CPB_2)}{\sin(\angle APB_2)} \cdot \frac{\sin(\angle BPC)}{\sin(\angle BPA)} \quad \text{and} \quad \frac{C'A}{C'B} = \frac{\sin(\angle APC_2)}{\sin(\angle BPC_2)} \cdot \frac{\sin(\angle CPA)}{\sin(\angle CPB)}\,.$$

Since

$$\angle APB_2 = \angle APC_2, \quad \angle BPA_2 + \angle BPC_2 = 180^\circ, \quad \angle CPA_2 + \angle CPB_2 = 180^\circ,$$

then we get

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1\,.$$

This equality will imply that the cevians AA', BB', CC' are concurrent (as required) if we can show further that exactly one of A', B', or C' lies in the interior of a side of the triangle while the other two lie on extensions of the sides. We note, for example, that C' belongs on the segment AB if and only if C_1 and C_2 are in opposite arcs of BC on the circle γ. Let us first examine the situation where the line ℓ intersects the sides CA and CB (and the extension of AB) as in the accompanying figure. When P (on ℓ by definition) is inside the triangle, so is C_2 (where ℓ again meets γ), while A_2 and B_2 are necessarily outside. But A_1, B_1, C_1 (where the lines joining P to the vertices again meet the circles) are all outside the triangle. Thus C' is the only primed point on the interior of a side, as claimed. Now fix ℓ and slide P along it. Note that as P crosses a side such as AC, the points A_1 and C_1 switch their status (with A_1, A_2 on opposite arcs BC of α, and C_1, C_2 on the same arc AB of γ); consequently, A' and C' likewise exchange their status. Similarly, whenever P crosses the extension of a side, one primed point jumps out across a vertex while another jumps in, so that there is always exactly one of the sides of ΔABC that contains a primed point. A similar analysis applies to the situation when the line ℓ contains no points inside the triangle. This argument concludes the proof that the cevians AA', BB', CC' are concurrent.

We use barycentric coordinates to show that the locus of these points of concurrence is an ellipse. Take $A = (1, 0, 0), B = (0, 1, 0)$ and $C = (0, 0, 1)$, and let

$$\lambda = \frac{A'B}{A'C}, \quad \mu = \frac{B'C}{B'A}, \quad \text{and} \quad \nu = \frac{C'A}{C'B}.$$

Then in the given diagram we get $A' = (0 : 1 : -\lambda)$ and $B' = (-\mu : 0 : 1)$. Write $X = (x, y, z)$. Since X lies on AA', then

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -\lambda \\ x & y & 0 \end{vmatrix} = 0$$

giving $z = -\lambda y$. Similarly, since X lies on BB' we get $x = -\mu z$. Thus $X = (\lambda\mu : 1 : -\lambda)$. Then xy, yz, zx are in a ratio of $\lambda\mu : -\lambda : -\lambda^2\mu$. Equivalently they are in a ratio of $(-\mu) : 1 : 1/\nu$.

Let $\vartheta_A = \angle BPC, \vartheta_B = \angle CPA$ and $\vartheta_C = \angle APB$. These angles depend only on P and not on ℓ. Let also $\vartheta = \angle APB_2$. So

$$\mu = \frac{\sin(\vartheta_B - \vartheta)}{\sin\vartheta} \cdot \frac{\sin\vartheta_A}{\sin\vartheta_C} = (\sin\vartheta_B \cot\vartheta - \cos\vartheta_B)\,\frac{\sin\vartheta_A}{\sin\vartheta_C}$$

and

$$\frac{1}{\nu} = \frac{\sin(\vartheta_C + \vartheta)}{\sin\vartheta} \cdot \frac{\sin\vartheta_A}{\sin\vartheta_B} = (\sin\vartheta_C \cot\vartheta + \cos\vartheta_C)\,\frac{\sin\vartheta_A}{\sin\vartheta_B}\,.$$

Now

$$\sin^2\vartheta_A + \frac{\sin^2\vartheta_B}{\nu} + \sin^2\vartheta_C \cdot (-\mu) = \sin\vartheta_A\,(\sin\vartheta_A + \sin\vartheta_B\cos\vartheta_C + \sin\vartheta_C\cos\vartheta_B) = 0$$

because

$$\sin\vartheta_B\cos\vartheta_C + \sin\vartheta_C\cos\vartheta_B = \sin(\vartheta_B + \vartheta_C) = \sin(2\pi - \vartheta_A) = -\sin\vartheta_A\,.$$

This gives that $(\sin^2 \vartheta_A)yz + (\sin^2 \vartheta_B)zx + (\sin^2 \vartheta_C)xy = 0$ which represents a conic passing through the vertices A, B, C.

To check that the conic is an ellipse we work as follows: We vary the line ℓ continuously, rotating it about P. We observe that A_2 varies continously around the circle α. The line A_1A_2 therefore also varies continuously. The point A' with one exception varies continuously — the only discontinuity being where it jumps from one 'end' of the line BC to the other. However the line AA' does vary continuously. (The discontinuity of A' does not affect it because at the point of discontinuity AA' is parallel to BC, and close to the discontinuity the line is close to being parallel to BC). So the lines AA', BB', CC' vary continuously with the line ℓ and, therefore, so does the point X. So the image set of X is a closed and bounded subset of the plane. Furthermore, all maps are injective (distinct lines ℓ, give distinct points A_2 on α etc.). This can happen only if the conic is an ellipse and, moreover, only when the locus is the whole ellipse rather than just a part. (Of course we need to fill in some 'gaps'. For example when A_1A_2 is parallel to BC then A' and therefore X are undefined. The right way to define A' is the point at infinity on BC, which makes AA' parallel to BC.)

To show that the ellipse is the circumcircle of ABC when P is the orthocenter H, observe that in that case $\vartheta_A = 180^\circ - \angle A$ and therefore $\sin \vartheta_A = \sin \angle A$. So writing a, b, c for the side lengths of the triangle, by the Sine Law the equation of the conic becomes $a^2yz + b^2zx + c^2xy = 0$, which is known to be the equation of the circumcircle.

We proceed to give a second proof that the locus of X is the circumcircle of ΔABC when P is the orthocenter:

Let D be the foot of the perpendicular from A onto BC. Then

$$\angle BA_1D = \angle BA_1H = \angle BCH = 90^\circ - \angle B = \angle BAD\,.$$

So BAA_1 is isosceles, and since $BD \perp AA_1$ then $AD = DA_1$. Letting D' be the point of intersection of HA_2 with BC we get

$$\begin{aligned}\angle CA'A = \angle CA'A_2 &= 180^\circ - \angle A'A_2H - \angle BD'A_2 \\ &= \angle A_1A_2H - \angle BD'A_2 \\ &= \angle A_1BH - \angle HD'C.\end{aligned}$$

But $\angle A_1BH = \angle A_1BD + \angle DBH = \angle B + 90^\circ - \angle C$. Therefore

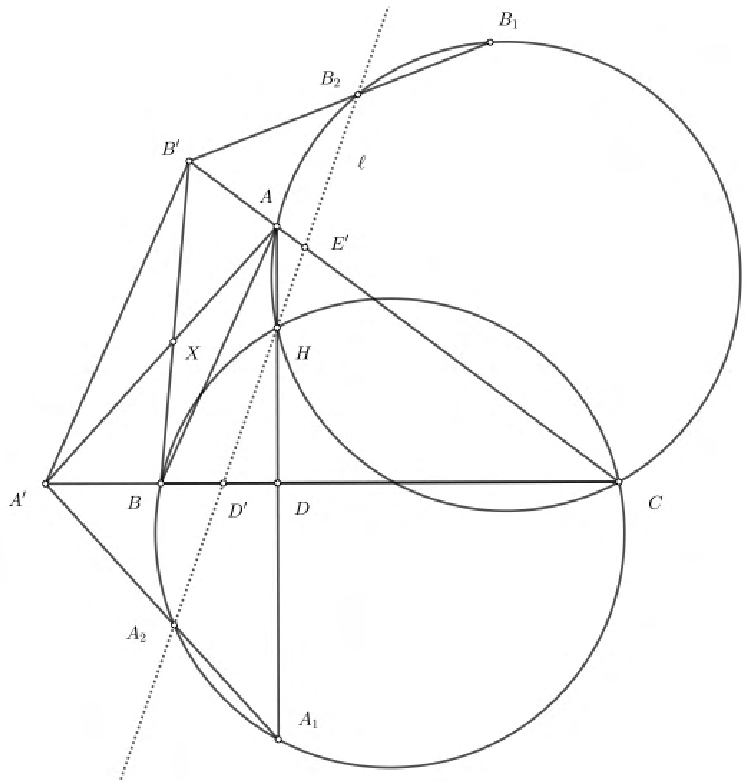
$$\angle CA'A = 90^\circ + \angle B - \angle C - \angle HD'C\,.$$

Similarly, if E' is the point of intersection of HB_2 with AC, then

$$\angle CB'B = 90^\circ + \angle A - \angle C - \angle HE'C\,.$$

Therefore

$$\begin{aligned}\angle CA'A + \angle CB'B &= 180^\circ + \angle A + \angle B - 2\angle C - \angle HD'C - \angle HE'C \\ &= \angle A + \angle B - \angle C \\ &= 180^\circ - 2\angle C.\end{aligned}$$



But

$$\angle CA'B' + \angle CB'A' = 180^\circ - \angle C$$

and therefore $\angle XA'B' + \angle XB'A' = \angle C$. Thus $\angle AXB = 180^\circ - \angle C$ and so X belongs on the circumcircle of triangle ABC.