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IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

Happy New Year 2021! It's the year of the Ox and it's charging ahead. Strong and spirited: that's how I see ***Crux*** entering Volume 47.

The new feature we are adding this year is a column titled "Explorations in Indigenous Mathematics" that will reside within the MathemAttic portion of the journal. In the past four years, I have been very lucky to live and work in a community with close ties to its local First Nations people: my home is Fraser Valley, the land of the Sto:lo people. The various resources and opportunities available have allowed me to explore the meaning behind reconciliation and indigenization. So naturally, I'm very excited about this new ***Crux*** column that will allow readers to explore cultural mathematics, experience the discipline through a broader humanizing approach, and engage in mathematics with societal context and history.

Welcome to the new volume!

Kseniya Garaschuk

MathemAttic

No. 21

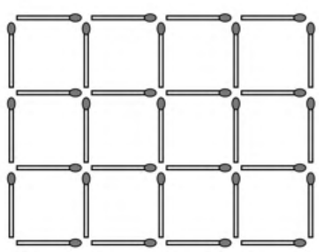
The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 30, 2021**.

MA101. Standard six-sided dice have their dots arranged so that the opposite faces add up to 7. If 27 standard dice are arranged in a $3 \times 3 \times 3$ cube on a solid table what is the maximum number of dots that can be seen from one position?

MA102. As shown in the diagram, you can create a grid of squares 3 units high and 4 units wide using 31 matches. I would like to make a grid of squares a units high and b units wide, where $a < b$ are positive integers. Determine the sum of the areas of all such rectangles that can be made, each using exactly 337 matches.



MA103. What is the largest three-digit number with the property that the number is equal to the sum of its hundreds digit, the square of its tens digit and the cube of its units digit?

MA104. The sequence

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$$

is defined by $a_1 = 2$ and $a_{n+1} = 2^{a_n}$ for all $n \geq 1$. What is the first term in the sequence greater than 1000^{1000}?

MA105. Eighteen points are equally spaced on a circle, from which you will choose a certain number at random. How many do you need to choose to guarantee that you will have the four corners of at least one rectangle?

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

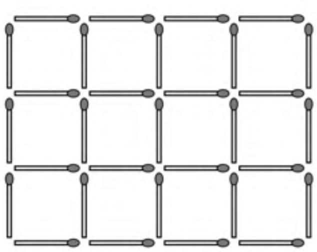
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA101. Des dés ordinaires à six côtés sont tels que les nombres de points sur les faces opposées ont une somme de 7. Si 27 tels dés sont disposés dans un cube $3 \times 3 \times 3$, quel est le nombre maximum de points visibles d'un endroit quelconque?

MA102. Comme indiqué ci-bas, il est possible de créer un grillage de taille 3 unités de haut par 4 unités de large à l'aide de 31 allumettes. On va alors créer des grillages de a unités de haut par b unités de large, où a et b sont des entiers positifs quelconques tels que $a < b$. Si exactement 337 allumettes sont utilisées, déterminer la somme des surfaces de tous les grillages possibles.



MA103. Quel est le plus gros entier à trois chiffres tels que ce nombre égale la somme de son chiffre en position de centaines, du carré de son chiffre en position de dizaines et du cube de son chiffre en position d'unités?

MA104. La suite

$$2, 2^2, 2^{2^2}, 2^{2^{2^2}}, \ldots$$

est définie par $a_1 = 2$ et $a_{n+1} = 2^{a_n}$ pour tout $n \geq 1$. Déterminer le premier terme dans la suite, plus grand que 1000^{1000}.

MA105. Parmi dix-huit points équidistants sur un cercle, on en choisit un certain nombre n. Déterminer le plus petit n qui assure qu'on y retrouvera les quatre coins d'au moins un rectangle, quels que soient les points choisis.

MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(6), p. 246–248.

MA76. The sum of two real numbers is n and the sum of their squares is $n+19$, for some positive integer n. What is the maximum possible value of n?

Originally 2018 Mathcon Finals, Grade 11, Part D, Question 43.

We received 21 submissions, of which 20 were correct. We present 3 solutions, one with a generalization.

Solution 1, by Lorenzo Benedetti.

Let x, y be the two real numbers. The relations that we are given are the following:

$$\begin{cases} x+y=n \\ x^2+y^2=n+19 \end{cases}$$

Notice that we may represent graphically the solutions to the system of equations as between the line $x+y-n=0$ and the circle $x^2+y^2=n+19$ centered at the origin and of radius $\sqrt{n+19}$. Clearly, the line and the circle have intersection points if and only if the radius of the circle is greater than or equal to the distance from its center to the line, given by $\frac{|0+0-n|}{\sqrt{1+1}}=\frac{n}{\sqrt{2}}$.

Therefore, we have solutions if and only if

$$\frac{n}{\sqrt{2}} \leq \sqrt{n+19} \Leftrightarrow n^2-2n-38 \leq 0 \Leftrightarrow 1 \leq n \leq 7$$

(because n is a positive integer). Hence the maximum possible value of n is 7.

Solution 2, by Manescu-Avram Corneliu.

We have that $x+y=n$ and $x^2+y^2=n+19$ for two real number x and y. Since $(x+y)^2 \leq 2(x^2+y^2)$, it follows that $n^2 \leq 2(n+19)$, whence $(n-1)^2 \leq 39$. We deduce that

$$n \leq 1+\lfloor\sqrt{39}\rfloor=7.$$

the maximum value $n=7$ is then attained for $x=\frac{7+\sqrt{3}}{2}$ and $y=\frac{7-\sqrt{3}}{2}$ since $x+y=7$ and $x^2+y^2=7+19=26$.

Solution 3, by Vincent Blevins.

Consider a generalization of the problem. Suppose the sum of m real numbers is n and the sum of their squares is $n+k$ where n, m, and k are positive integers; we find the maximum possible value of n by first finding the largest real value t replacing n and satisfying these conditions.

Let $a_1, a_2, \cdots, a_m$ and t be real numbers satisfying

$$\begin{aligned} \sum_{j=1}^{m} a_j &= t \\ \sum_{j=1}^{m} a_j^2 &= t + k. \end{aligned} \tag{1}$$

Then, after subtracting and completing the square,

$$\begin{aligned} (\sum_{j=1}^{m} a_j^2) - k &= \sum_{j=1}^{m} a_j \\ \sum_{j=1}^{m}(a_j - \frac{1}{2})^2 &= k + \frac{m}{4} = \frac{4k+m}{4}. \end{aligned}$$

For a given t, consider the hyperplane P_t in $\mathbb{R}^m$ with equation $\sum_{j=1}^{m} x_j = t$. Then the components of any point on the intersection of P_t and the sphere with equation $\sum_{j=1}^{m}(x_j - \frac{1}{2})^2 = \frac{4k+m}{4}$ will satisfy (1). Note that P_t is normal to the line passing through the origin and the center of the sphere as the parallel vector of the latter is parallel to the normal vector of the former. Hence, P_t is parallel to the hyperplanes tangent to the points of the sphere lying on the line, and so the largest t satisfying the given conditions is the coordinate intercepts of one of the tangent planes.

To compute the points of the sphere lying on the line, we substitute in s for each x_j in the equation of the sphere and solve for s. We have,

$$\sum_{j=1}^{m}(s - \frac{1}{2})^2 = \sum_{j=1}^{m} \frac{1}{4}(2s-1)^2 = \frac{m}{4}(2s-1)^2 = \frac{4k+m}{4}$$

Thus, $s = \frac{1 \pm \sqrt{\frac{4k+m}{m}}}{2}$. So the equations of the two tangent hyperplanes are

$$\sum_{j=1}^{m} x_j = \frac{m + m\sqrt{\frac{4k+m}{m}}}{2} \quad \text{and} \quad \sum_{j=1}^{m} x_j = \frac{m - m\sqrt{\frac{4k+m}{m}}}{2}.$$

As the right-hand side of the first equation is larger than the right-hand side of the second, it follows that the largest t satisfying the conditions of the problem is $\frac{m+m\sqrt{\frac{4k+m}{m}}}{2}$. Thus, the largest integer n satisfying the conditions is $\left\lfloor \frac{m+m\sqrt{\frac{4k+m}{m}}}{2} \right\rfloor$.

In particular, for $m = 2$ and $k = 19$:

$$n = \left\lfloor \frac{2 + 2\sqrt{\frac{76+2}{2}}}{2} \right\rfloor = 7.$$

MA77. In a regular decagon, all diagonals are drawn. If a diagonal is chosen at random, what is the probability that it is neither one of the shortest nor one of the longest?

Originally 2018 Mathcon Finals, Grade 9, Part C, Question 32.

We received 13 submissions, of which 11 solutions were correct. We present the solution with generalization provided by Lorenzo Benedetti.

We will compute the probability of choosing at random one of the shortest diagonal or one of the longest, then we will take the complement of it.

First, notice that the numbers of diagonals in a regular n-gon is given by $\frac{n(n-3)}{2}$, in our case ($n = 10$) is $\frac{10\cdot 7}{2} = 35$.

Now we will compute the number of shortest diagonals + the number of longest diagonals. The longest diagonals are those that connect two diametrally oppposite vertices, so they are just diameters of the circle in which the decagon is inscribed. It is easy to see that there are 5 such diagonals. The shortest diagonals are those that connect two vertices with just a vertex in between. It is easy to see that there are 10 such diagonals.

Hence, the required probability is given by

$$1 - \frac{5+10}{35} = \frac{20}{35} = \frac{4}{7}.$$

MA78. Let $T(n)$ be the digit sum of a positive integer n; for example, $T(5081) = 5+0+8+1 = 14$. Find the number of three-digit numbers that satisfy $T(n) + 3n = 2020$.

Originally 2018 Mathcon Finals, Grade 12, Part D, Question 43.

We received 15 submissions, all correct. We present the solution by Jaimin Patel.

We have $T(n) + 3n = 2020 \implies T(n) = 2020 - 3n$. Since n is a three-digit number,

$$\begin{aligned} 1 \le T(n) &\le 27, \\ 1 \le 2020 - 3n &\le 27, \\ -2019 \le -3n &\le -1993, \\ 1993 \le 3n &\le 2019, \\ 665 \le n &\le 673. \end{aligned}$$

Now, we know that $n \equiv T(n) \pmod 3$, so

$$\begin{aligned} & T(n) + 3n = 2020 \\ \implies & T(n) \equiv 2020 \pmod 3 \\ \implies & T(n) \equiv 1 \pmod 3 \\ \implies & n \equiv 1 \pmod 3 \\ \implies & (n-1) \equiv 0 \pmod 3 \\ \implies & 3 \mid (n-1) \end{aligned}$$

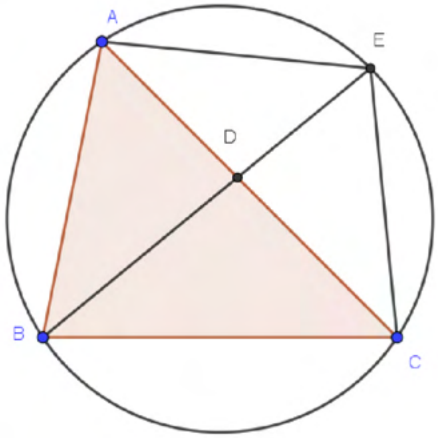
From the above two observations, we know that if such n is possible then it must be either 667, 670 or 673. By checking each possibility we can see that only 667 satisfies our condition. Hence there exists only one such three-digit number.

MA79. Suppose BD bisects $\angle ABC$, $BD = 3\sqrt{5}$, $AB = 8$ and $DC = \frac{3}{2}$. Find $AD + BC$.

Originally 2019 Mathcon Finals, Grade 12, Part C, Question 10.

We received 8 submissions of which 6 were correct and complete. We present the solution by Thinh Nguyen, modified by the editor.

Let the line extension of BD intersect the circumcircle of $\triangle ABC$ at E.



By the circle theorem,

$$\angle BCA = \angle BEA \quad \Rightarrow \quad \angle BCD = \angle BEA. \tag{1}$$

Given (1), and since $\angle DBC = \angle ABE$, we have that $\triangle BCD$ and $\triangle BEA$ are similar. Thus

$$\frac{BD}{BA} = \frac{BC}{BE} \quad \Rightarrow \quad BD \times BE = BC \times BA. \tag{2}$$

As $\angle DBC = \angle DAE$ and, by (1), $\angle BCD = \angle BEA$, $\triangle BCD$ and $\triangle ADE$ are similar. Thus

$$\frac{DA}{DB} = \frac{DE}{DC} \qquad \Rightarrow \qquad DE \times DB = DA \times DC. \tag{3}$$

Subtracting (3) from (2), we have that

$$BD\,(BE - DE) = BC \times BA - DA \times DC.$$

Given $BD = BE - DE$, the above becomes

$$\begin{aligned} BD^2 = BC \times BA - DA \times DC \qquad &\Rightarrow \qquad \left(3\sqrt{5}\right)^2 = 8BC - \frac{3}{2}DA \\ &\Rightarrow \qquad 16BC - 3DA = 90. \end{aligned} \tag{4}$$

By the angle bisector theorem,

$$\frac{AB}{BC} = \frac{AD}{DC} \qquad \Rightarrow \qquad \frac{8}{BC}\left(\frac{3}{2}\right) = AD \qquad \Rightarrow \qquad \frac{12}{BC} = AD.$$

When substituted into 4,

$$\begin{aligned} 16BC - 3\left(\frac{12}{BC}\right) = 90 \qquad &\Rightarrow \qquad 16BC^2 - 90BC - 36 = 0 \\ &\Rightarrow \qquad 2\,(BC - 6)\,(8BC + 3) = 0. \end{aligned}$$

As $BC > 0$, $BC = 6 \Rightarrow AD = 2$. Thus $AD + BC = 8$.

Editor's comment. Many of the other submissions were able to prove similar results with the law of cosines and Stewart's Theorem. This proof was selected because of its use of elementary tools.

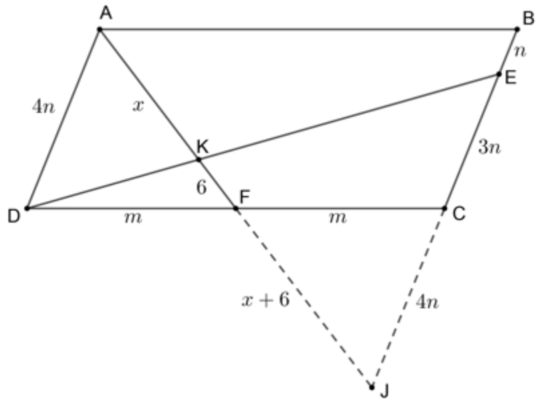
MA80. Suppose $ABCD$ is a parallelogram. Let E and F be two points on BC and CD, respectively. If $CE = 3BE$, $CF = DF$, DE intersects AF at K and $KF = 6$, find AK.

Originally 2019 Mathcon Finals, Grade 11, Part C, Question 8.

We received 6 submissions, of which 5 were correct and complete. We present the solution by Ronald Martins, modified by the editor.

Let n, m, and x denote the lengths BE, CF, and AK, respectively. Additionally, let J be the point of intersection of the line extensions of AF and BC, as shown in the figure. As $\triangle FDA$ and $\triangle FCJ$ are congruent,

$$FD = FC = m, \qquad FA = FJ = x + 6, \qquad DA = CJ = 4n.$$



As $\triangle KAD$ and $\triangle KEJ$ are similar triangles,

$$\frac{KA}{KJ} = \frac{AD}{EJ} \quad \Rightarrow \quad \frac{x}{x+12} = \frac{4n}{7n} \quad \Rightarrow \quad 7x = 4x + 48 \quad \Rightarrow \quad x = 16.$$

Thus, $AK = 16$.

PROBLEM SOLVING VIGNETTES

No. 14

Shawn Godin

Playing with Probability

Probability problems can be easily stated, but their solutions may seem elusive or counterintuitive. Careful attention has to be placed on counting the number of outcomes for each event and recognizing if the outcomes we are counting are equally likely or not. As a simple example, when we roll two dice the possible sums range from 2 to 12, inclusive, but they are not all equally likely. We also have to understand when events are independent or not. For example, if I flip a coin then roll a die, the outcome of the die does not depend on the outcome of the flip because the outcomes are independent of each other. On the other hand, if I draw a card from a deck of cards and then draw a second card, these actions are dependent. That is, if I am interested in the second card being a heart, the probability depends on whether the first card was a heart or not. It seems our intuition about probability is sometimes flawed, probably because we overlook these subtle properties of probability.

We will begin with a problem from the first *Canadian Mathematical Gray Jay Competition* (CMGC). The CMGC is a new, multiple choice competition from the CMS for elementary school students . The first competition was written on Thursday October 8, 2020 by just under 2000 students world-wide.

> **6.** Alice and Bill play a game. They go to separate rooms, flip a coin and try to predict what the other person flipped. They win if at least one of them predicts correctly. They decide that Alice will always guess the same thing that she flips and Bill will always predict the opposite of what he flips. What percentage of the time should they win?
> **(A)** 0% **(B)** 25% **(C)** 50% **(D)** 75% **(E)** 100%

First, we should determine all possible outcomes to the process of Alice and Bill flipping their coins. We will use H and T for heads and tails, and we will list the results from Alice first them Bill, so TH means Alice flipped tails and Bill flipped heads. Thus all possible outcomes are $\{HH, HT, TH, TT\}$. We will assume that we have fair coins, so heads and tails are equally likely for each person. Also note that the result of Alice's flip will in no way influence Bill's flip, so we can see that our outcomes are all equally likely, with probability of $\frac{1}{4}$ each.

The key to solving this problem is deciding which of the events constitute a "win". To do that, let us look at the information in a bit more detail:

A's Flip	A's Guess	Correct?	B's Flip	B's Guess	Correct?	Win?
H	H	✓	H	T	✗	✓
H	H	✗	T	H	✓	✓
T	T	✗	H	T	✓	✓
T	T	✓	T	H	✗	✓

We see that Alice and Bill's strategy yields a win every time! It is interesting that we are dealing with a totally random process but we set up a situation where we can guarantee at least one person guesses correctly. Can you find a strategy that guarantees at least one of three people guesses correctly if A guesses B's who guesses C's who guesses A's?

Next, we will examine problem B2 from another CMS competition, the 2020 *Canadian Open Mathematics Challenge*.

> **B2.** Alice places a coin, heads up, on a table then turns off the light and leaves the room. Bill enters the room with 2 coins and flips them onto the table and leaves. Carl enters the room, in the dark, and removes a coin at random. Alice reenters the room, turns on the light and notices that both coins are heads. What is the probability that the coin Carl removed was also heads?

We know that before Carl went in there were three coins and after he left there were two coins, both heads. Since Carl could have removed either a head or a tail, before he went in there was either three heads (and he removed a head), or two heads and a tail (and he removed a tail). At this point, since there were two possible "starting" states, we may be fooled into thinking that our desired probability is $\frac{1}{2}$. Unfortunately, this is not the case since, counter-intuitively, the starting points are not equally likely in a couple of ways!

Looking closer, since Alice placed a coin heads up in the room, we already have a head (H_A, for Alice's head). When Bill goes in, he flips two coins yielding

$$H_{B_1}H_{B_2},\ H_{B_1}T_{B_2},\ T_{B_1}H_{B_2} \text{ or } T_{B_1}T_{B_2}$$

(where the subscripts B_1 and B_2 indicate Bill's first and second flip). At this point we see that it was impossible for Bill to have flipped two tails, because Carl could never have left behind two heads. Thus, when Carl enters the room there are three possible, equally likely, configurations:

$$H_AH_{B_1}H_{B_2},\ H_AH_{B_1}T_{B_2},\ \text{and } H_AT_{B_1}H_{B_2}.$$

If Carl went in and there were three heads, he could remove any of them and satisfy the conditions of the problem. However, if there were two heads and a tail, he could *only* remove the tail. Hence, as shown in the table below, the probability that Carl removed a head was $\frac{3}{5}$.

Start	Remove	Left	Conditions?
$H_A H_{B_1} H_{B_2}$	H_A	$H_{B_1} H_{B_2}$	✓
$H_A H_{B_1} H_{B_2}$	H_{B_1}	$H_A H_{B_2}$	✓
$H_A H_{B_1} H_{B_2}$	H_{B_2}	$H_A H_{B_1}$	✓
$H_A H_{B_1} T_{B_2}$	H_A	$H_{B_1} T_{B_2}$	✗
$H_A H_{B_1} T_{B_2}$	H_{B_1}	$H_A T_{B_2}$	✗
$H_A H_{B_1} T_{B_2}$	T_{B_2}	$H_A H_{B_1}$	✓
$H_A T_{B_1} H_{B_2}$	H_A	$T_{B_1} H_{B_2}$	✗
$H_A T_{B_1} H_{B_2}$	T_{B_1}	$H_A H_{B_2}$	✓
$H_A T_{B_1} H_{B_2}$	H_{B_2}	$H_A T_{B_1}$	✗

The next few problems will come from the problem solving course I took with Professor Honsberger. Problems from the three assignments and problems 1 through 25 have been featured in earlier columns. Below is the next set of problems from the course.

#26. A normal die bearing the numbers $1, 2, 3, 4, 5, 6$ on its faces is thrown repeatedly until the running total first exceeds 12. What is the most likely total that will be obtained?

#27. Find all natural numbers, not ending in zero, which have the property that if the final digit is deleted, the integer obtained divides into the original.

#28. Let n denote an odd natural number greater than one. Let A denote an $n \times n$ symmetric matrix such that each row and each column consists of some permutation of the numbers $1, 2, 3, \ldots, n$. Show that each of the numbers $1, 2, 3, \ldots, n$ must occur in the main diagonal of A.

#29. A,B, and C are to fight a 3-cornered duel. All of them know that A's chance of hitting his target is 0.3, that C's chance is 0.5, and that B never misses. They are to fire at their choice of target in succession $A, B, C, A, B, \ldots$ etc. until only one man is left unhit (once a man is hit, he drops out of the duel). What is the best strategy for A?

#30. What are the final two digits of $\left(\cdots\left(7^7\right)^7\cdots\right)^7$, containing 1001 7's?

We will consider problem 26 next. To get a feel for what is happening, you may want to do an experiment. I rolled a die a number of times to perform the process four times and came up with the following:

Rolls	Total
$3, 4, 2, 1, 4$	14
$2, 3, 4, 1, 1, 3$	14
$5, 5, 4$	14
$4, 5, 6$	15

The first thing that we should notice is even though a single roll of the die does not have any effect on future rolls of the die, since our outcomes must sum to a number greater than 12, they will have different probabilities. The first entry took 5 rolls, the second 6 and the last two took 3. The probability of rolling the last two results are the same, but different from the first two which are also different from each other.

Other strange things happen as well. For the last roll $4, 5, 6$, we could have also gotten $4, 6, 5$, or any other permutation of those three numbers and we would have gotten the same result. This is useful for counting our results. On the other hand, looking at the first roll $3, 4, 2, 1, 4$, we could have also gotten $4, 1, 2, 4, 3$ for the same total of 14, but $4, 2, 4, 3, 1$ would not occur! Since $4 + 2 + 4 + 3 = 13$, we would have stopped before we reached 14. So, some of the strategies that we might have considered have to be discarded. Instead, let us look at how things could end. We will use the notation $(11, 3)$ to represent any roll whose last sum, before being greater than 12, was 11 and the last roll is 3. Hence, the second roll $2, 3, 4, 1, 1, 3$ would fall under this category. Listing all possible outcomes and their final result we get:

Final Sum

13	14	15	16	17	18
$(7, 6)$					
$(8, 5)$	$(8, 6)$				
$(9, 4)$	$(9, 5)$	$(9, 6)$			
$(10, 3)$	$(10, 4)$	$(10, 5)$	$(10, 6)$		
$(11, 2)$	$(11, 3)$	$(11, 4)$	$(11, 5)$	$(11, 6)$	
$(12, 1)$	$(12, 2)$	$(12, 3)$	$(12, 4)$	$(12, 5)$	$(12, 6)$

Looking at a total of 15 as an example we see that all the events that give us our desired sum; $(9, 6)$, $(10, 5)$, $(11, 4)$, and $(12, 3)$; all have different probabilities. On the bright side, all events that start with the same number; such as $(9, 4)$, $(9, 5)$, and $(9, 6)$; have the same probability. Hence we can easily see that

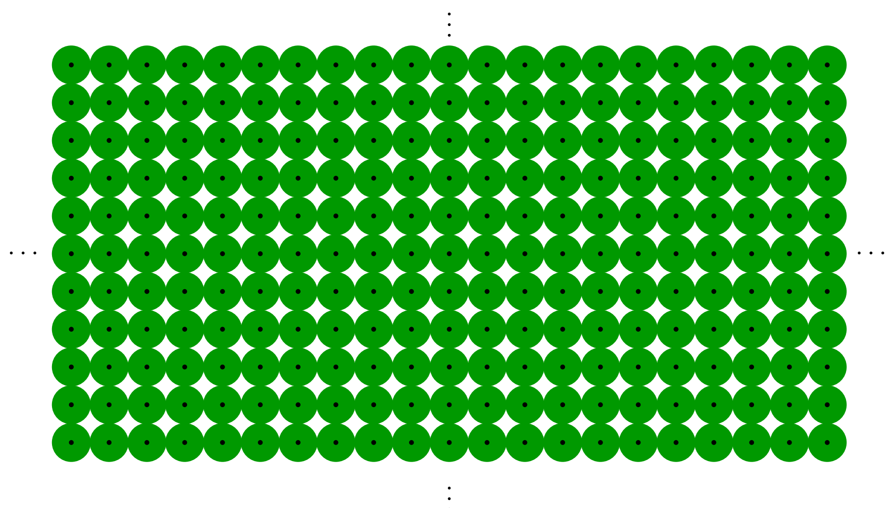
$$P(18) < P(17) < P(16) < P(15) < P(14) < P(13)$$

so 13 is the most likely sum. In this case we were able to answer the question, without actually having to calculate the probabilities involved.

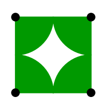
Next on the list is question 4 from the third assignment from professor Honsberger's class, featured in an earlier column [2017 : 43(10), p. 441-443].

> **4.** A circle of radius $\frac{1}{2}$ is tossed at random onto a coordinate plane. What is the probability that it covers a lattice point?

This is a geometric probability problem, where we are looking at areas rather than counting cases, since there are infinitely many! If we focus on a particular lattice point, we see that it will be covered by the circle if and only if the centre of the circle is a distance of at most $\frac{1}{2}$ away. Thus, the centre of the circle must land within a circle of radius $\frac{1}{2}$ centred on a lattice point to contain the lattice point. If we colour the possible locations of the centre of the circle green, the coordinate plane will look something like the diagram below.



Because of the symmetry, we can focus on a unit square formed by four lattice points, shown in the figure below.



Thus, for whichever unit square the centre lands in, if it lands in the green area the circle will contain a lattice point, otherwise it won't. Hence

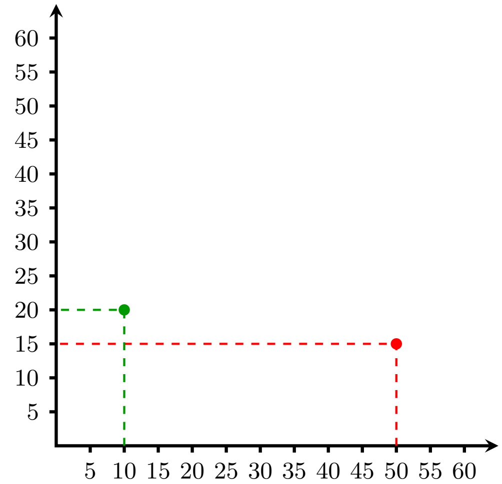
$$P = \frac{\text{Green area}}{\text{Total area}} = \frac{\pi\left(\frac{1}{2}\right)^2}{1} = \frac{\pi}{4}$$

or about 78.5%.

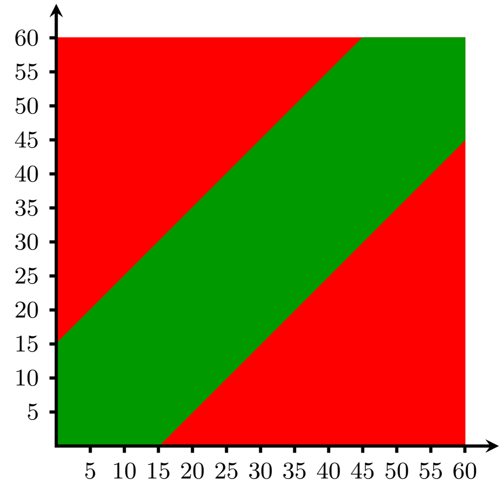
In many cases where you are dealing with continuous data, a geometric argument will work. Consider the following problem from the second assignment [2017: 43(8) 344-346].

> **3.** Two people agree to meet for lunch at their favourite restaurant, Each agrees to wait 15 minutes for the other, after which time he will leave. If each chooses his time of arrival at random between noon and l o'clock, what is the probability of a meeting taking place ?

Again, there are infinitely many possibilities to consider. Let's look at a graph where we indicate the number of minutes after noon that each person arrives on the axes. We can then colour the points in the plane either red (no meeting) or green (they meet). For example, if the first person arrives at 12 : 10 and the second at $12:20$, they would meet, indicated by the green point at $(10, 20)$. On the other hand, if the first person arrives at $12:50$ and the second at $12:15$, they would not meet, indicated by a red point at $(50, 15)$.



Colouring the rest of the grid accordingly, we can find the desired probability as the ratio of the green area to the total area which is $\frac{7}{16}$.



You may want to consider a similar problem with three friends and calculate the probability that they all meet.

Probability problems can be interesting, because their solutions can require tools from different areas of mathematics. Enjoy the rest of the problems from Professor Honsberger's class. We will revisit probability in a future column.

Explorations in Indigenous Mathematics

No. 1

Edward Doolittle

The Starblanket Design

The starblanket design is popular among the Indigenous peoples of the Plains region, particularly in quilted blanket designs, but also in other crafts. In the Plains Cree language, the word for star is atāhk and the word for blanket is akohp, so starblanket is atāhkakohp. Chief Ahtahkakoop, so named because "the stars blanketed the sky, more numerous and brighter than usual" the night he was born, was one of the first signatories of Treaty Six. Ahtahkakoop Cree Nation is named after the chief.

In Figure 1, you can see an example of a starblanket quilt. The quilt was given to me by First Nations University Elder Kohkom Bea Lavalley when I first started to work at the university. Figure 2 shows another starblanket quilt, from the collection of First Nations University, with an additional design superimposed on the centre, made for the 2014 North American Indigenous Games.

Figure 1: A Traditional Starblanket Quilt

The regularity and precision of the starblanket design makes it a potential source of mathematics for us to explore. One question I ask my mathematics students at First Nations University is, "How many tiles are there in the starblanket design in our atrium?" The ceramic tile starblanket, shown in Figure 3, was designed by Douglas Cardinal, the architect of the building. Figure 4 contains a drawing of the starblanket tiling (in which the colours are not accurately represented).

Figure 2: The NAIG Starblanket Quilt

Figure 3: The First Nations University Atrium

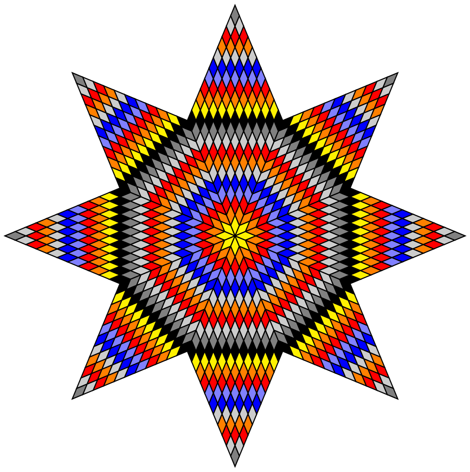


Figure 4: Drawing of the First Nations University Atrium Starblanket

There are numerous ways to solve that problem. One way that I didn't anticipate, but that some of my students have used, is to simply count all the tiles. We are going to explore another interesting way of solving the problem.

First let us consider a simpler problem, or rather a sequence of simpler problems. The Douglas Cardinal starblanket is large, but let us try to quantify its size, and then make adjustments. We define the *radius* of the design to be the number of edges from the centre of the design to its outside. A careful count of the Douglas Cardinal design gives a radius of 11. Let us look at designs with smaller radii to see if we can more easily determine the number of tiles in those designs.

One other thing to consider before we proceed: What is the simplest starblanket design? A starblanket of radius 1 might be the simplest; a mathematician would be more likely to think that a starblanket of radius 0 is really the simplest (Figure 5). How many tiles does a starblanket of radius 0 have? Clearly it has 0 tiles. Next, we see that a starblanket with radius 1 has 8 tiles (Figure 6).

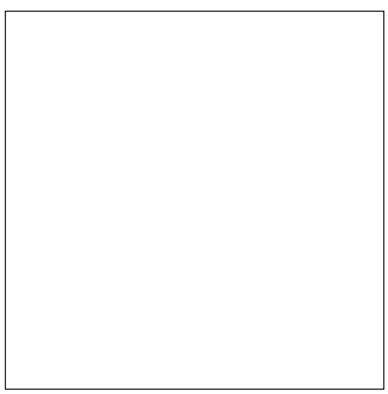


Figure 5: Starblanket of Radius 0

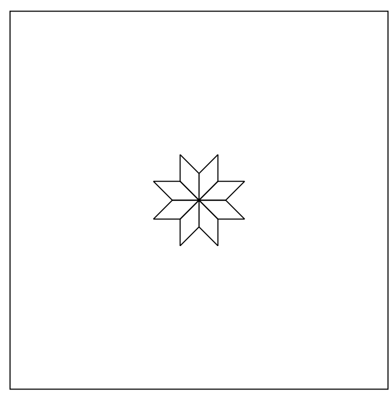


Figure 6: Starblanket of Radius 1

We can assemble the information in a "difference scheme" as in Figure 7, where the starblanket area numbers are on the first (top) row and their difference is on the second (bottom).

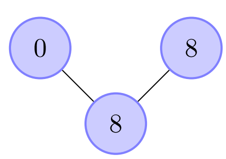


Figure 7: First Differences, Two Steps

If the differences remain constant, we continue the second row with another 8 (Figure 8), which means that next starblanket, of radius 2, should have 16 tiles (Figure 9), but it doesn't. It has 32 tiles (Figure 10).

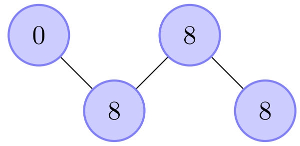


Figure 8: Continuing with Constant First Differences

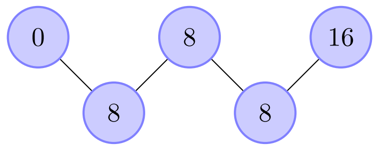


Figure 9: Constant First Differences, Next Value Predicted

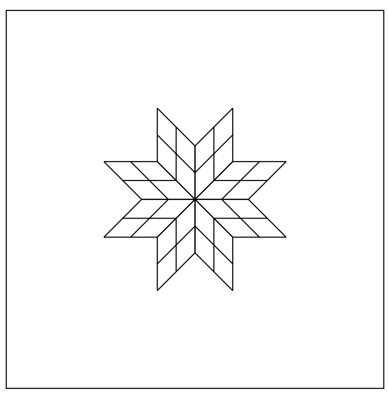


Figure 10: Starblanket of Radius 2

Our simple theory didn't work. Let's enter the information that we have into a new table and try again (Figure 11).

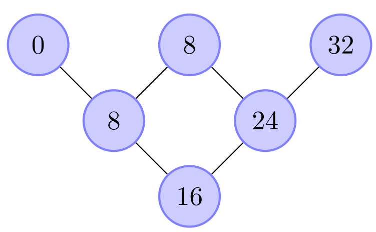


Figure 11: Three Data Points, Second Differences

Here we have taken differences of the first differences to make a third row of "second differences". Assuming that the second differences are constant, we obtain the following two predictions for the next two numbers in the series:

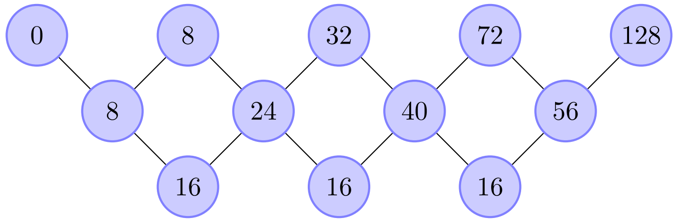


Figure 12: Predicting Fourth and Fifth Starblanket Area Numbers

Comparing with the number of tiles in a starblanket of radius 3 (72, as in Figure 13) and a starblanket of radius 4 (128, as in Figure 14), we see that our method has given correct predictions for those two numbers.

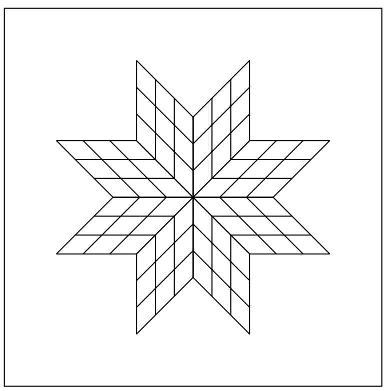


Figure 13: Starblanket of Radius 3

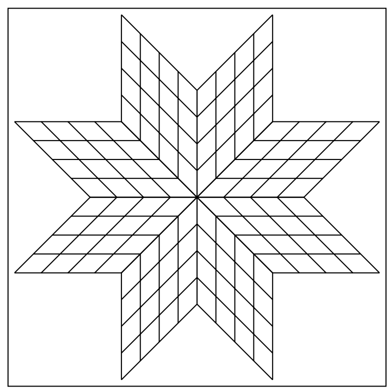


Figure 14: Starblanket of Radius 4

Of course, those predictions are not proven, but the method we have been using could form the foundation of a proof by induction. I will leave details up to you. I will also leave it up to you to continue the starblanket series up to a starblanket of radius 11; the answer you should get is 968 tiles.

The technique we have been using is known as the "calculus of finite differences". It is a powerful way to analyze sequences of numbers, and (as you may have guessed by the name) is a precursor to and a discrete analog of calculus, the kind of mathematics which is studied in many university programs such as science, engineering, and even business.

You could use the calculus of finite differences to analyze other "figurate numbers" such as the square numbers (0, 1, 4, 9, 16, ...) and triangular numbers (0, 1, 3, 6, 10, ...). An online search for "figurate numbers" will turn up many more examples, including pentagonal numbers, hexagonal numbers, and so on. There are also three-dimensional figurate numbers, the most familiar of which are the cubes (0, 1, 8, 27, 64, ...), among many other examples.

The calculus of finite differences can be used not only to predict the next value of a series, but also to find explicit formulas for series. Here are some problems to get you started.

1. Continue the difference scheme in Figure 15, and find a general formula for the nth term in the top row.
2. Continue the difference scheme in Figure 16, and find a general formula for the nth term in the top row.
3. Continue the difference scheme in Figure 17, and find a general formula for the nth term in the top row. (Hint: Pascal's Triangle might be helpful.)
4. Combine the previous three results to find a general formula for a difference scheme with any given first diagonal.
5. Use the previous result to find a formula for the nth starblanket number as a function of n. Simplify the formula as much as possible.
6. Try a similar investigation with the cube numbers. Pascal's triangle may continue to be useful.

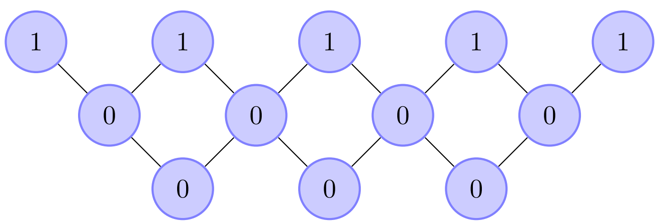


Figure 15: Difference Scheme with First Diagonal 1, 0, 0

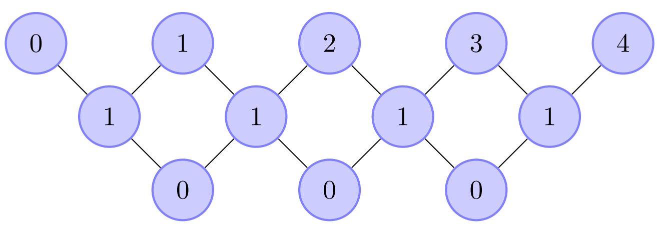


Figure 16: Difference Scheme with First Diagonal 0, 1, 0

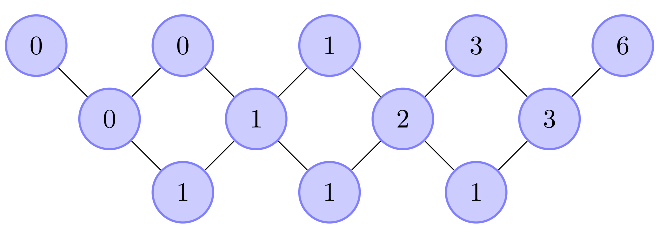


Figure 17: Difference Scheme with First Diagonal 0, 0, 1

In the end, you should find that the formula for the starblanket numbers is actually quite simple. We will explore a method for obtaining that simple formula, and other geometric aspects of the starblanket design, in a future article.

The meaning of the starblanket design

The choice of colours, and the shape of the starblanket design, are meaningful to some Indigenous communities, but the details vary from one tradition to another. If you want to know more about the cultural meanings behind the design, I suggest you seek out a local elder and invite them to speak by offering them a traditional gift such as tobacco and cloth.

One thing I can say is that, in my experience, starblankets are given as gifts to commemorate transitions in a person's life. For example, when babies are born they are often given small starblankets; Indigenous graduates from high school or degree programs are given starblankets, sometimes as part of their graduation ceremonies; and when I began work at First Nations University, I was given a starblanket, the one shown in the first illustration of this article. In that spirit, I hope that my contribution of this article, as small and as inexpertly crafted as it is, may commemorate the start of a new and better relationship between Indigenous people and the mathematics community.

References

[1] Deanna Christensen. *Ahtahkakoop: the epic account of a Plains Cree head chief, his people, and their struggle for survival 1816–1896.* Atahkakoop Publishing, Shell Lake, Saskatchewan, 2000.

[2] Kenneth S. Miller. *An introduction to the calculus of finite differences and difference equations.* Dover Publications, New York, 1966.

OLYMPIAD CORNER

No. 389

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 30, 2021**.

OC511. All the proper divisors of some composite natural number n, increased by 1, are written out on a blackboard. Find all composite natural numbers n for which the numbers on the blackboard are all the proper divisors of some natural number m. (Note: here 1 is not considered a proper divisor.)

OC512. A convex quadrilateral $ABCD$ is given. We denote by I_A, I_B, I_C and I_D the centers of the inscribed circles ω_A, ω_B, ω_C and ω_D of the triangles DAB, ABC, BCD and CDA, respectively. It is known that $\angle BI_AA + \angle I_CI_AI_D = 180^\circ$. Prove that $\angle BI_BA + \angle I_CI_BI_D = 180^\circ$.

OC513. In an acute triangle ABC the angle bisector of $\angle BAC$ intersects BC at point D. Points P and Q are orthogonal projections of D on lines AB and AC. Prove that $Area(APQ) = Area(BCQP)$ if and only if the circumcenter of ABC lies on line PQ.

OC514. Consider the set $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \mid ab = cd \right\}$.

(a) Give an example of a matrix $A \in M$ such that $A^{2017} \in M$ and $A^{2019} \in M$, but $A^{2018} \notin M$.

(b) Prove that if $A \in M$ and there exists an integer $k \geq 1$ such that $A^k \in M$, $A^{k+1} \in M$ and $A^{k+2} \in M$, then $A^n \in M$ for all integers $n \geq 1$.

OC515. Let a, b, c, d be natural numbers such that $a+b+c+d = 2018$. Find the minimum value of the expression:

$$E = (a-b)^2 + 2(a-c)^2 + 3(a-d)^2 + 4(b-c)^2 + 5(b-d)^2 + 6(c-d)^2.$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC511. Tous les diviseurs propres d'un nombre naturel composé n sont inscrits à un babillard, mais auparavant augmentés par 1, où les diviseurs propres n'incluent pas 1. Déterminer tous les nombres naturels n tels que les nombres au babillard sont tous les diviseurs propres d'un nombre naturel m.

OC512. Soit un quadrilatère convexe $ABCD$. Soient alors I_A, I_B, I_C et I_D les centres des cercles inscrits ω_A, ω_B, ω_C et ω_D des triangles DAB, ABC, BCD et CDA, respectivement. De plus, $\angle BI_AA + \angle I_CI_AI_D = 180°$. Démontrer que $\angle BI_BA + \angle I_CI_BI_D = 180°$.

OC513. Dans un triangle acutangle ABC, la bissectrice de l'angle $\angle BAC$ intersecte BC en D. Dénotons par P et Q les projections orthogonales de D vers les lignes AB et AC. Démontrer que $Aire(APQ) = Aire(BCQP)$ si et seulement si le centre du cercle circonscrit de ABC se trouve sur la ligne PQ.

OC514. Soit l'ensemble $M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}) \mid ab = cd \right\}$.

(a) Fournir un exemple de matrice $A \in M$ telle que $A^{2017} \in M$ et $A^{2019} \in M$, mais $A^{2018} \notin M$.

(b) Démontrer que si $A \in M$ et s'il existe un entier $k \geq 1$ tel que $A^k \in M$, $A^{k+1} \in M$ et $A^{k+2} \in M$, alors $A^n \in M$ pour tout entier $n \geq 1$.

OC515. Soient a, b, c, d des nombres naturels tels que $a + b + c + d = 2018$. Déterminer la valeur minimale de l'expression suivante:

$$E = (a-b)^2 + 2(a-c)^2 + 3(a-d)^2 + 4(b-c)^2 + 5(b-d)^2 + 6(c-d)^2.$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(6), p. 256–257.

OC486. There are 2017 points in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least 1009 of the given points.

Originally 2017 Czech-Slovakia Math Olympiad, 2nd Problem, Category B, Regional Round.

We received 11 correct submissions. We present two solutions.

Solution 1, by Dmitry Fleischman.

Let A_1 be one of the points. By the pigeonhole principle one of the next two situations occurs.

First, there exists 1008 points, A_2, A_3, ... , A_{1009}, such that

$$A_1A_2 \leq 1, A_1A_3 \leq 1, \ldots, A_1A_{1009} < 1.$$

Then the circle with centre A_1 and radius 1 contains 1009 points: A_1, A_2, ... , A_{1009}.

Second, there exists 1009 points, B_1, B_2, ... , B_{1009}, such that

$$A_1B_1 \geq 1, A_1B_2 \geq 1, \ldots, A_1B_{1009} \geq 1.$$

Since among the points A_1, B_1, B_2 we can select two so that their distance is less than 1, $A_1B_1 \geq 1$, and $A_1B_2 \geq 1$ we must have $B_1B_2 < 1$. Similarly,

$$B_1B_3 < 1, \ldots, B_1B_{1009} < 1.$$

Then B_1, B_2, ... , B_{1009} are 1009 points included in the circle with centre B_1 and radius 1. The statement is proved.

Solution 2 by Oliver Geupel.

We can show that the statement holds in the following generalized form. "There are N points, P_1, P_2, ..., P_N, in the plane such that among any three of them two can be selected so that their distance is less than 1. Prove that there is a circle of radius 1 containing at least $\lceil N/2 \rceil$ of the given points."

Let d denote the maximum of distances, P_iP_j, between any two points. If $d < 1$, then $P_1P_2 < 1$, $P_1P_3 < 1$, ..., $P_1P_N < 1$. Hence, the circle with centre P_1 and radius 1 contains all given points. It remains to consider the case where $d \geq 1$. There is no loss of generality in assuming that $P_1P_2 \geq 1$. By hypothesis, for

$3 \leq i \leq N$, at least one of the distances P_1P_i and P_2P_i is less than 1; hence P_i belongs to at least one of the unit circles, Γ_1 with centre P_1 or Γ_2 with centre at P_2. By the pigeonhole principle, one of the circles Γ_1 and Γ_2 must contain at least $\lceil N/2 \rceil$ of the N points. Hence, the statement holds.

OC487. Let a, b, c be real numbers such that $1 < b \leq c^2 \leq a^{10}$ and

$$\log_a b + 2\log_b c + 5\log_c a = 12.$$

Show that

$$2\log_a c + 5\log_c b + 10\log_b a \geq 21.$$

Originally 2018 Romania Math Olympiad, 3rd Problem, Grade 10, District Round.

We received 11 correct submissions. We present the solution by Roy Barbara.

Set $x = 5\log_c a$, $y = 2\log_b c$, and $z = \log_a b$. Then clearly, $xyz = 10$, and by hypothesis $x + y + z = 12$. Now $1 < b \leq c^2 \leq a^{10}$ yields $x \geq 1$, $y \geq 1$, and $z = 12 - x - y \leq 10$.

The conclusion becomes $\frac{10}{x} + \frac{10}{y} + \frac{10}{z} \geq 21$, or $\frac{10(xy+yz+zx)}{xyz} \geq 21$, which is equivalent to $xy + yz + zx \geq 21$, since $xyz = 10$.

We write

$$xy + yz + zx - 21 = xy + z(x+y) - 21 = \frac{10}{z} + z(12 - z) - 21$$

$$= \frac{-z^3 + 12z^2 - 21z + 10}{z} = \frac{-(z-1)^2(z-10)}{z} \geq 0.$$

The last inequality is true because $z \leq 10$. The conclusion follows.

OC488. Prove that the equation

$$(x^2 + 2y^2)^2 - 2(z^2 + 2t^2)^2 = 1$$

has infinitely many integer solutions.

Originally 2017 Poland Math Olympiad 9th Problem, First Round.

We received 11 submissions of which 10 were correct and complete. We present two solutions.

Solution 1, by Sergey Sadov.

Let $\mathcal{S}$ be the set of all integer numbers of the form $p^2 + 2q^2$ for some integers p and q. Let

$$u_1 = 3, \qquad v_1 = 2$$

be two numbers from $\mathcal{S}$, as $u_1 = 1^2 + 2 \cdot 1^2$ and $v_1 = 0^2 + 2 \cdot 1^2$. Moreover, $u_1^2 - 2v_1^2 = 9 - 8 = 1$.

We define recurrently two strictly increasing sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_n = u_{n-1}^2 + 2v_{n-1}^2, \qquad v_n = 2u_{n-1}v_{n-1}.$$

We prove that the numbers u_n, v_n: (1) satisfy the equation $u_n^2 - 2v_n^2 = 1$, and (2) belong to $\mathcal{S}$ for any $n \geq 1$. These imply that the statement equation has infinitely many integer solutions.

(1) A simple computation shows that

$$u_n^2 - 2v_n^2 = (u_{n-1}^2 + 2v_{n-1}^2)^2 - 2(2u_{n-1}v_{n-1})^2 = (u_{n-1}^2 - 2v_{n-1}^2)^2.$$

Since $u_1^2 - 2v_1^2 = 1$, an induction proof shows that $u_n^2 - 2v_n^2 = 1$ for any $n \geq 1$.

(2) For $n \geq 2$ we have $u_n \in \mathcal{S}$ by definition. To prove that $v_n \in \mathcal{S}$, it suffices to show that if $a, b \in \mathcal{S}$, then $2a \in \mathcal{S}$ and $ab \in \mathcal{S}$.

First, if $a = p^2 + 2q^2$, then we write $2a = (2q)^2 + 2p^2 \in \mathcal{S}$.

Second, given $a = p^2 + 2q^2$ and $b = \tilde{p}^2 + 2\tilde{q}^2$, then a computation shows that

$$ab = (p\tilde{p} - 2q\tilde{q})^2 + 2(p\tilde{q} + \tilde{p}q)^2 = (p\tilde{p} + 2q\tilde{q})^2 + 2(p\tilde{q} - \tilde{p}q)^2,$$

and so $ab \in \mathcal{S}$.

We can explain the computation above by the following fact involving complex numbers. Note $a = |p \pm i\sqrt{2}q|^2$ and $b = |\tilde{p} \pm i\sqrt{2}\tilde{q}|^2$. Then

$$ab = |(p \pm i\sqrt{2}q)(\tilde{p} \pm i\sqrt{2}\tilde{q})|^2.$$

The product of two complex numbers of the form $p + i\sqrt{2}q$ is also a number of that form. There are four possible combinations of signs in the above formula and they produce two, in general different, decompositions:

$$\begin{aligned} ab &= |(p + i\sqrt{2}q)(\tilde{p} + i\sqrt{2}\tilde{q})|^2 = (p\tilde{p} - 2q\tilde{q})^2 + 2(p\tilde{q} + \tilde{p}q)^2 \\ &= |(p + i\sqrt{2}q)(\tilde{p} - i\sqrt{2}\tilde{q})|^2 = (p\tilde{p} + 2q\tilde{q})^2 + 2(p\tilde{q} - \tilde{p}q)^2. \end{aligned}$$

We evaluated the first few terms of our sequences:

$u_1 = 3 = 1^2 + 2 \cdot 1^2$ $\qquad$ $v_1 = 2 = 0^2 + 2 \cdot 1^2$
$u_2 = 17 = 3^2 + 2 \cdot 2^2$ $\qquad$ $v_2 = 12 = 2^2 + 2 \cdot 2^2$
$u_3 = 577 = 17^2 + 2 \cdot 12^2$ $\qquad$ $v_3 = 408 = 20^2 + 2 \cdot 2^2 = 4^2 + 2 \cdot 14^2$
$u_4 = 665857 = 577^2 + 2 \cdot 408^2$ $\qquad$ $v_4 = 470832 = 572^2 + 2 \cdot 268^2$
$u_5 = 886731088897$ $\qquad$ $v_5 = 627013566048$

v_4 has other representations: $v_4 = 548^2 + 2 \cdot 292^2 = 412^2 + 2 \cdot 388^2 = 380^2 + 2 \cdot 404^2$. v_5 has 8 representations given by the following pairs (p, q):

$$(784136, 77924), (783944, 78884), (776296, 110404), (776024, 111356),$$
$$(157480, 548732), (156136, 548924), (111560, 554332), (110200, 554468).$$

Solution 2, by UCLan Cyprus Problem Solving Group and Roy Barbara, done independently.

The equation $a^2 - 2b^2 = 1$ has fundamental solution $a = 3, b = 2$, so by the theory of Pell's equation the solutions of $a^2 - 2b^2 = 1$ in non-negative integers are given by the pairs (a_n, b_n) where

$$a_n = \frac{(3+2\sqrt{2})^n + (3-2\sqrt{2})^n}{2}, \quad \text{and} \quad b_n = \frac{(3+2\sqrt{2})^n - (3-2\sqrt{2})^n}{2\sqrt{2}}.$$

Let $\mathcal{S}$ be the set of all numbers which can be written in the form $c^2 + 2d^2$ with $c, d \in \mathbb{Z}$. It is enough to show that $a_n \in \mathcal{S}$ for every $n \in \mathbb{N}$ and $b_n \in \mathcal{S}$ for every $n \in \mathbb{N}$ of the form 2^m, where $m \in \mathbb{N}$.

Let

$$x_n = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}, \quad \text{and} \quad y_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}.$$

Expanding using the Binomial Theorem and cancelling terms, we see that $x_n, y_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Furthermore, it is a matter of simple algebra to check that $x_n^2 + 2y_n^2 = a_n^2$. So $a_n \in \mathcal{S}$.

It remains to show that $b_n \in \mathcal{S}$ whenever $n = 2^m$ with $m \in \mathbb{N}$. We proceed by induction on m. For $m = 0$, we have $n = 1$, so $b_n = 2$ and we can write $2 = 0^2 + 2 \cdot 1^2$. So the case $m = 0$ is true.

Assume the result is true for $m = k$ and note that $b_{2^{k+1}} = 2a_{2^k}b_{2^k}$. By the induction hypothesis (and earlier results) $b_{2^{k+1}}$ is a product of three elements of $\mathcal{S}$. So to complete the inductive step it is enough to prove that the product of any two elements of $\mathcal{S}$ is also an element of $\mathcal{S}$. This is a consequence of the identity

$$(a^2 + 2b^2)(c^2 + 2d^2) = (ac + 2bd)^2 + 2(ad - bc)^2.$$

This completes the induction and the proof that the original equation has infinitely many solutions in integers.

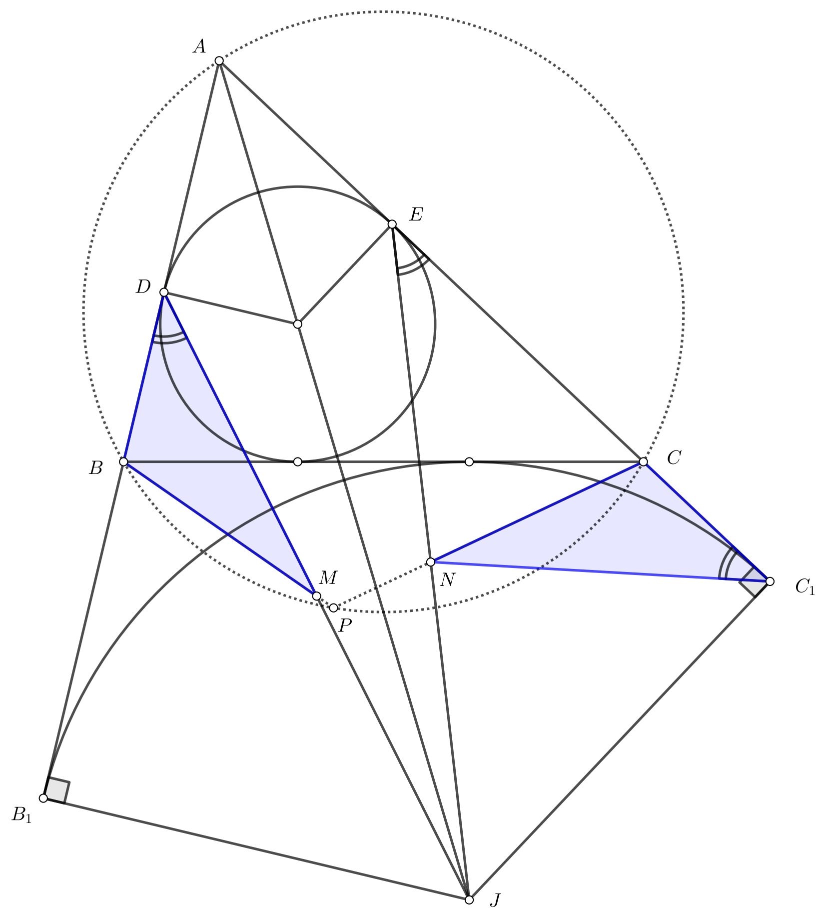
OC489. The incircle of a triangle ABC touches AB and AC at points D and E, respectively. Point J is the center of the excircle of triangle ABC tangent to side BC. Points M and N are midpoints of segments JD and JE, respectively. Lines BM and CN intersect at point P. Prove that P lies on the circumcircle of triangle ABC.

Originally 2017 Poland Math Olympiad, 4rd Problem, Second Round.

We received 8 *submissions. We present* 2 *solutions.*

Solution 1, by UCLan Cyprus Problem Solving Group.

Let B_1 and C_1 be the points where the A-excircle meets (the extensions of) AB and AC respectively.



It is well-known that

$$BD = CC_1 = (a + c - b)/2 \quad \text{and} \quad CE = BB_1 = (a + b - c)/2.$$

Therefore $DB_1 = EC_1$.

Since $JB_1 = JC_1, DB_1 = EC_1$, and $\angle JB_1D = \angle JC_1E = 90^\circ$ then the triangles $\triangle DB_1J$ and $\triangle EC_1J$ are equal. In particular $DJ = EJ$ and so $DM = EN$.

Since N is the midpoint of EJ, which is the hypotenuse of $\triangle EC_1J$, we have that $C_1N = EN = DM$ and $\angle CC_1N = \angle CEN = \angle MDB$.

The above show that the triangles $\triangle CC_1N$ and $\triangle BDM$ are equal. Therefore

$$\angle PBA = \angle MBD = \angle NCC_1 = 180^\circ - \angle PCA\,.$$

Therefore P lies on the circumcircle of ABC, as required.

Solution 2, by Sergey Sadov.

We will prove the following generalization, where M and N are relabeled as F and F' and become (dependently) movable.

Let Q be the midpoint of the segment IJ, where I the incenter of $\triangle ABC$ and J is the center of the excircle of triangle ABC tangent to side BC. Denote by q the line through Q perpendicular to AB. Let F be any point on that line and F' be the point symmetrical to F about the bisector $k = AJ$ of $\angle BAC$. Then the point $P = BF \cap CF'$ lies on the circumcircle of $\triangle ABC$. Moreover, P also lies on the circle QFF'.

Note two interesting cases in addition to the one proposed in the Problem.

I. The roles of the incircle and the excircle in the Problem can be interchanged. Thus, the lines connecting B to the midpoint of IT and C to the midpoint of IS meet at the circumcircle. (See Figure for notation.)

II. If $F \in q$ is such that $BF \perp k$, then $F' = P$ (since BF is invariant under symmetry about k), so F' lies on the circle ABC.

Denote by σ_q the reflection about the line q and by σ_k the reflection about the line k.

The composite transformation $\rho = \sigma_k \circ \sigma_q$ (performed from right to left) is a rotation about the intersection point of the reflection axes, which is Q. Consider the action of ρ on some objects in our figure.

(1) $\sigma_q(F) = F$ and $\sigma_k(F) = F'$, hence $\rho(F) = F'$.

(2) $\rho(q) = q'$ (the line symmetric to q under σ_k). Therefore ρ is a rotation by $\pi - \alpha$ clockwise, where $\alpha = \angle BAC$.

(3) The line AB as a whole is mapped by σ_q to itself and by σ_k to the line AC. Hence $\rho(AB) = AC$. (This is not to say that $\rho(\{A, B\}) = \{A, C\}$.)

(4) Yet, $\rho(B) = C$. This fact is not immediately obvious. First, we show that the angle between the rays QB and QC equals $\pi - \alpha$.

Q lies on the circumcircle ABC. Indeed, the angles IBJ and ICJ, being formed by the inner and outer bisectors of the angles B and C of the triangle, are right. Therefore the points I, B, J, C lie on the circle with diameter IJ whose centre is Q. In this circle, the central angle BQC subtends the same arc as the inscribed angle BJC. In $\triangle BJC$ the angles B and C are, respectively, $(\pi - \beta)/2$ and $(\pi - \gamma)/2$. Hence $\angle BJC = (\beta + \gamma)/2 = (\pi - \alpha)/2$ and $\angle BQC = \pi - \alpha$.

It follows that $\rho(QB) = QC$. Therefore, taking (3) into account, $\rho(B) = \rho(AB \cap QB) = AC \cap QC = C$.

The preparatory work is complete. By (1) and (4) we see that $\rho(BP) = CP$. By (2), $\rho(F) = \rho(q \cap BP) = q' \cap CP = F'$. Using (4) again, we get $\rho(BF) = CF'$. Therefore the angles formed by the lines BF and CF' are $\pi - \alpha$ and α. This leaves two logical possibilities: $\angle BPC = \pi - \alpha$ (which, in the arrangement where P and

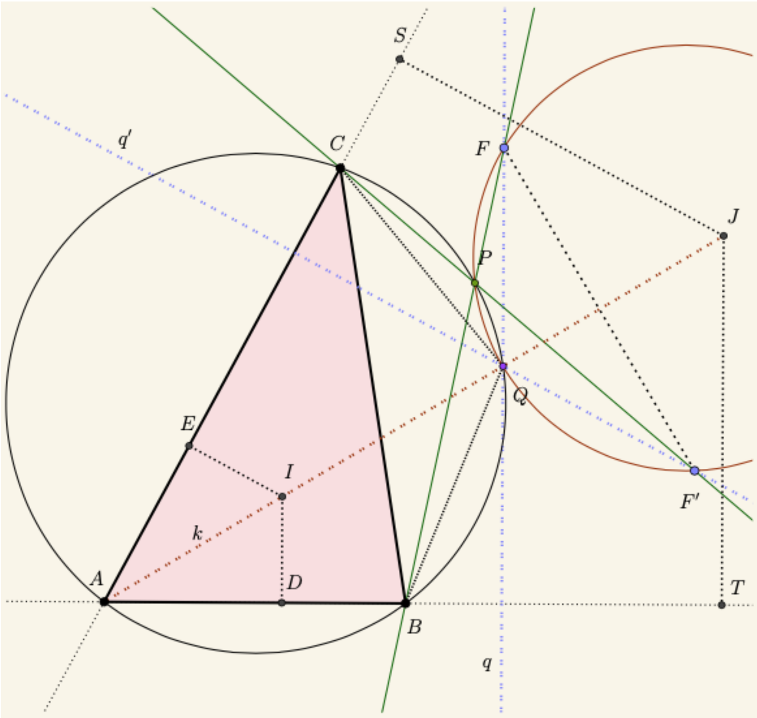
A are separated by the line BC as in our Figure, implies that P lies on the circle ABC) or $\angle BPC = \alpha$ (which would imply the same if A and P were on the same side from BC).

The statement that P lies on the circle QFF' follows similarly from the fact that $\angle FPF' = \pi - \alpha = \angle FQF'$ (in the arrangement as in our Figure).

Let us show that for F near Q the case $\angle BPA = \pi - \alpha$ is realized. As F starts moving continuously away from Q along the line q, the angle BPA evolves continuously. Therefore it is constant, the same as $\angle BQP = \pi - \alpha$, at least as long as neither F nor F' goes to infinity.

Thus, for F sufficiently close to Q our claim is proved. To complete the proof for the general case, one can either consider different positions of P relative to the sides of $\triangle ABC$ with help of a picture, or use some suitable formal approach.

We propose the following argument. Suppose the coordinates of the points A, B, C are fixed and the coordinates of the variable point $F \in q$ depend linearly on the parameter t. By construction, the coordinates of P are rational functions $x = \xi(t)$, $y = \eta(t)$. If $f(x, y) = 0$ is the equation of the circle ABC, then $g(t) = f(\xi(t), \eta(t))$ is a rational function of t. We know that $g(t) = 0$ when t runs through some interval of values (corresponding to F near Q). Therefore $g(t) = 0$ identically, so P remains on the circle ABC for any position of F on the line q.



Remark. The last argument can be expanded into a full alternative proof in the style of the 19th century analytical (algebraic) geometry. It is compact, but assumes familiarity with projective coordinates.

Let the triangle ABC be fixed and F be a variable point on the line q. Suppose its projective coordinates depend linearly on the parameter t. Then the (projective) coordinates of the point F' depend linearly on t; the same is true for the coordinates (coefficients) of the lines BF and CF'. The coordinates of the point P are quadratic in t. (The projective coordinates of the intersection of lines can be found as cross-product of the coefficient vectors of the lines.)

A curve parametrized (projectively) by quadratic functions is a quadric. (There exists a linear combination of two coordinates of degree 1 in parameter; express the parameter linearly via those two coordinates and substitute into the parametric expression of the third coordinate.) A quadric is uniquely determined by five points. Let us indicate the five points to show that in our case the quadric coincides with the circle ABC.

1) $F = q \cap AB$ yields $P = A$. (Because $F' \in AC$.)

2) $F = q \cap BC$ yields $P = C$.

3) $F = q \cap \sigma_k(BC)$ yields $P = B$.

4) $F = q \cap \infty$ (point at infinity on the line q) corresponds to the intersection of perpendiculars to AB at B and to AC and C. Since they meet at the angle $\pi - \alpha$, the point of intersection lies on the circle ABC.

5) $F = Q$ yields $P = Q$. (One needs to know in advance that Q lies on the circumcircle, i.e. the theorem: the bisector of angle A meets the circumcircle at the midpoint of the arc BC (not containing A).)

OC490. Find the smallest prime number that cannot be written in the form $\left|2^a - 3^b\right|$ with nonnegative integers a, b.

Originally 2017 Germany Math Olympiad, 3rd Problem, Grade 11-12, Day 1, State Round.

We received 9 *submissions. We present the solution by Oliver Geupel.*

We show that the desired prime is 41.

For all the smaller primes we have the following equalities:

$$\begin{array}{llll} 2 = 3^1 - 2^0, & 7 = 2^3 - 3^0, & 17 = 3^4 - 2^6, & 29 = 2^5 - 3^1, \\ 3 = 2^2 - 3^0, & 11 = 3^3 - 2^4, & 19 = 3^3 - 2^3, & 31 = 2^5 - 3^0, \\ 5 = 2^3 - 3^1, & 13 = 2^4 - 3^1, & 23 = 2^5 - 3^2, & 37 = 2^6 - 3^3. \end{array}$$

It remains to show that 41 cannot be written in the required form.

Assume the contrary. Then there are nonnegative integers a and b such that $2^a - 3^b = 41$ or $3^b - 2^a = 41$.

First suppose that $2^a - 3^b = 41$. By inspection, it holds $a > 2$. Hence

$$3^b \equiv 2^a - 41 \equiv 7 \pmod 8.$$

But 3^b is congruent to either 1 or 3 modulo 8, a contradiction.

Therefore, we have $3^b - 2^a = 41$. By inspection, it holds $a > 1$ and $b > 0$. We obtain

$$2^a \equiv 3^b - 41 \equiv 1 \pmod 3 \qquad \text{and} \qquad 3^b \equiv 2^a + 41 \equiv 1 \pmod 4.$$

Thus, a and b are even numbers, say $a = 2m$ and $b = 2n$. It follows that $41 = 3^b - 2^a = (3^n - 2^m)(3^n + 2^m)$; whence $3^n - 2^m = 1$ and $3^n + 2^m = 41$. Then, $3^n = 21$, which is impossible.

This completes the proof that 41 cannot be written in the desired form.

FOCUS ON...

No. 44

Michel Bataille

Quadratics (I)

Introduction

The last problem of the 1988 IMO has been considered as one of the most difficult problems ever posed in an IMO. Its statement is short and attractive:

> Let a, b, be positive integers such that $ab+1$ divides a^2+b^2. Show that $\dfrac{a^2+b^2}{ab+1}$ is a perfect square.

Not surprisingly, the solution calls for a lot of ingenuity, but surprisingly, it rests upon very simple results about the quadratic equation. [For the details, we refer the reader to **[1]**.]

Prompted by this famous example, the intent of this number is to show through various, simpler examples how elementary properties of the quadratic polynomial (those everybody learns as a 15-year-old student!) can intervene, sometimes unexpectedly, in the solutions to problems. In a second part, our next number will more specifically address applications to the polynomials of degree three or four.

Ending up at a quadratic equation

We start with a selection of examples that directly lead to solving a quadratic equation. The first one is from **[2]**:

> Find all real numbers x such that $x > 1$ and $\{x\} + \left\{\frac{1}{x}\right\} = 1$.

Here the notation $\{x\}$ refers to the fractional part of x, that is, $\{x\} = x - \lfloor x \rfloor$.

Suppose that the conditions on x hold. Since $x > 1$, we have $\left\lfloor \frac{1}{x} \right\rfloor = 0$, hence $\left\{\frac{1}{x}\right\} = \frac{1}{x}$. It follows that $x + \frac{1}{x} = \lfloor x \rfloor + 1$. If m denotes the integer $\lfloor x \rfloor + 1$, we have $x^2 - mx + 1 = 0$ and $m \geq 2$; even $m > 2$ since $x = 1$ if $x^2 - 2x + 1 = 0$.

Conversely, consider the equation $x^2 - mx + 1 = 0$ where m is an integer satisfying $m \geq 3$. This equation has two positive solutions

$$x_1 = \frac{m - \sqrt{m^2-4}}{2} \quad \text{and} \quad x_2 = \frac{m + \sqrt{m^2-4}}{2}$$

and $x_1 < 1 < x_2$ (note that $x_1 x_2 = 1$). Then, we have $x_2 = m - x_1 = m - \frac{1}{x_2}$, hence $m - 1 < x_2 < m$. We deduce that $\lfloor x_2 \rfloor = m - 1$ and so

$$\{x_2\} + \left\{\frac{1}{x_2}\right\} = x_2 - \lfloor x_2 \rfloor + \frac{1}{x_2} = 1.$$

Thus, the desired x's are the numbers $\dfrac{m+\sqrt{m^2-4}}{2}$ where m is an integer with $m \geq 3$.

Our second example offers a variant of solution to problem **3732** [2012 : 149, 151 ; 2013 : 190]:

> A circle of radius 1 is rolling on the x-axis in the first quadrant towards the parabola with equation $y = x^2$. Find the coordinates of the point of contact when the circle hits the parabola.

Let Γ be the circle and $U(a,1)$ its centre (where $a > 1$). Let T be the point of contact when the circle hits the parabola. We have $T(w, w^2)$ for some positive real number w. Since the equation of Γ is $x^2 + y^2 - 2ax - 2y + a^2 = 0$, the following relation holds:

$$w^2 + w^4 - 2aw - 2w^2 + a^2 = 0. \tag{1}$$

Since UT is perpendicular to the tangent to the parabola at T, we also have

$$(w-a) + 2w(w^2-1) = 0. \tag{2}$$

Note that $w > 1$ because $2w^3 - w = a > 1$ and $2w^3 - w - 1 \leq 0$ for $w \leq 1$, as showed by a quick study of $x \mapsto 2x^3 - x - 1$.

The elimination of a between (1) and (2) gives $4w^4 - 7w^2 + 2 = 0$ and hence $w^2 = \frac{7+\sqrt{17}}{8}$. We conclude that

$$T\left(\sqrt{\frac{7+\sqrt{17}}{8}}\ ,\ \frac{7+\sqrt{17}}{8}\right).$$

Another example of problem leading to a biquadratic equation is problem **3298** [2007 : 486, 489 ; 2008 : 500]:

> Let ABC be a triangle of area $\frac{1}{2}$ in which a is the side opposite vertex A. Prove that $a^2 + \csc A \geq \sqrt{5}$.

Let $F = \frac{1}{2}$ be the area of ΔABC and let $b = CA, c = AB$. Since $\dfrac{a}{\sin A} = \dfrac{abc}{2F}$, we have $bc = \csc A \geq 1$ and the required inequality rewrites as

$$a^2 + bc \geq \sqrt{5}.$$

The well-known relation $16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$ yields

$$a^4 - 2(b^2+c^2)a^2 + 4 + (b^2-c^2)^2 = 0.$$

Solving for a^2, we obtain

$$a^2 = b^2 + c^2 + 2\sqrt{b^2c^2-1} \quad \text{or} \quad a^2 = b^2 + c^2 - 2\sqrt{b^2c^2-1}.$$

In any event, we have

$$a^2 + bc \geq b^2 + c^2 + bc - 2\sqrt{b^2c^2 - 1} \geq 3bc - 2\sqrt{b^2c^2 - 1}.$$

Now, the inequality $3bc - 2\sqrt{(bc)^2 - 1} \geq \sqrt{5}$ holds since it is equivalent to inequality $(bc\sqrt{5} - 3)^2 \geq 0$ (easily checked) and we deduce that $a^2 + bc \geq \sqrt{5}$, as desired.

We conclude this section with problem 841 of *The College Mathematics Journal* (Vol. 38 No 1):

> Assume that the quadratic polynomial $f(x) = ax^2 + bx + c,\ a \neq 0$, has two fixed points x_1 and x_2, $x_1 \neq x_2$. If 1 and -1 are two fixed points of the function $f(f(x))$, but not of $f(x)$, then find the exact values of x_1 and x_2.

Since 1 is a fixed point of $f(f(x))$, the quadratic polynomial

$$f(x) - 1 = ax^2 + bx + c - 1$$

vanishes for $x = f(1)$. Its other root must be $-\frac{b}{a} - f(1)$ and so

$$f(x) - 1 = (x - f(1))(ax + b + af(1)).$$

With $x = 1$, the latter yields

$$a + b + af(1) = -1 \tag{1}$$

(note that $f(1) - 1 \neq 0$ because 1 is not a fixed point of $f(x)$). Reasoning in the same way with the quadratic polynomial $f(x) + 1$ one of whose root is $f(-1)$, we obtain

$$-a + b + af(-1) = -1. \tag{2}$$

By difference and addition, (1) and (2) lead to

$$a(f(1) - f(-1)) = -2a \quad \text{and} \quad a(f(1) + f(-1)) = -2(b + 1).$$

Now, recalling that $f(1) = a + b + c$ and $f(-1) = a - b + c$, we easily deduce $b = -1$ and $a + c = 0$, which shows that $f(x)$ must be the polynomial $ax^2 - x - a$.

Conversely, taking $f(x) = ax^2 - x - a$, we have $f(1) \neq 1,\ f(-1) \neq -1$,

$$f(f(x)) = a(ax^2 - x - a)^2 - ax^2 + x$$

and it is readily checked that $f(f(1)) = 1,\ f(f(-1)) = -1$.

As a result, x_1, x_2 are the roots of $f(x) - x = ax^2 - 2x - a$ that is,

$$\frac{1 + \sqrt{1 + a^2}}{a} \quad \text{and} \quad \frac{1 - \sqrt{1 + a^2}}{a}.$$

Two numbers from their sum and product

In the examples that follow, we illustrate the fact that the knowledge of the sum and the product of two numbers allows one to find the numbers by solving a quadratic equation.

First, an easy exercise:

> Evaluate $w + w^2 + w^4$ where $w = \exp(2\pi i/7)$.

With $S = w + w^2 + w^4$ we associate $T = w^3 + w^5 + w^6$ (the key idea!). Since $w \neq 1$ and

$$0 = w^7 - 1 = (w-1)(w^6 + w^5 + w^4 + w^3 + w^2 + w + 1),$$

we have $S + T = -1$. Using $w^7 = 1$, we find the product

$$ST = 3 + (w^6 + w^5 + w^4 + w^3 + w^2 + w) = 3 - 1 = 2.$$

Therefore S, T are the solutions of the quadratic equation $z^2 + z + 2 = 0$, namely $\dfrac{-1+i\sqrt{7}}{2}$ and $\dfrac{-1-i\sqrt{7}}{2}$.

Next, we observe that

$$\mathrm{Im}(S) = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = (\sin\frac{2\pi}{7} - \sin\frac{\pi}{7}) + \sin\frac{4\pi}{7}$$

is a positive real number and we conclude that $S = \frac{-1+i\sqrt{7}}{2}$.

We continue with problem 5501 proposed by *School Science and Mathematics Association* in May 2018:

> Determine all real numbers a, b, x, y that simultaneously satisfy the following relations:
>
> $$\begin{cases} (1) & ax + by & = & 5 \\ (2) & ax^2 + by^2 & = & 9 \\ (3) & ax^3 + by^3 & = & 17 \\ (4) & ax^4 + by^4 & = & 33 \end{cases}$$

It is readily checked that $(2,1,2,1)$ and $(1,2,1,2)$ are solutions for (a,b,x,y). We show that there are no other solutions. To this aim, let a, b, x, y be real numbers satisfying the four equations. The reader will easily check that we must have $xy(y-x) \neq 0$.

From equations (1) and (2), we obtain

$$a = \frac{1}{xy(y-x)}\begin{vmatrix} 5 & y \\ 9 & y^2 \end{vmatrix} = \frac{5y-9}{x(y-x)}.$$

In the same way, (2) and (3) give $a = \dfrac{9y-17}{x^2(y-x)}$ and it follows that

$$9y - 17 = x(5y-9)(= ax^2(y-x)).$$

In consequence,

$$5xy = 9(x+y) - 17. \tag{*}$$

With equations (3) and (4), we get $a = \dfrac{17y-33}{x^3(y-x)}$ and so

$$17y - 33 = x(9y-17)(= ax^3(y-x)),$$

which gives

$$9xy = 17(x+y) - 33. \tag{**}$$

From $(*)$ and $(**)$, we obtain $x + y = 3$ and $xy = 2$. The quadratic equation $X^2 - 3X + 2 = 0$ then shows that $(x, y) = (2, 1)$ or $(1, 2)$. In the former case, we easily find $a = 2, b = 1$ and in the latter case $a = 1, b = 2$ and so

$$(a, b, x, y) = (2, 1, 2, 1) \quad \text{or} \quad (1, 2, 1, 2).$$

The sign of $mx^2 + nx + p$

In this section we suppose that the coefficients of the quadratic polynomial $p(x)$ are real. The discussion about the sign of $p(x)$ when x describes $\mathbb{R}$ is well-known and of frequent use. Take for example the classical inequality $a^2+b^2+c^2 \geq ab+bc+ca$ for real a, b, c.

Young students can find a bit far-fetched the usual proof using $a^2 + b^2 \geq 2ab$, etc. (this was once the case of my students who responded by: "We would never have thought of that..."). The following might sound more direct and familiar: Consider the difference between the left and the right sides:

$$a^2 - a(b+c) + b^2 + c^2 - bc = p(a),$$

where

$$p(x) = x^2 - x(b+c) + b^2 + c^2 - bc.$$

The discriminant of $p(x)$ is

$$\Delta = (b+c)^2 - 4(b^2+c^2-bc) = 6bc - 3b^2 - 3c^2 = -3(b-c)^2$$

and is not positive, hence $p(x)$ is nonnegative for all x and therefore $p(a) \geq 0$, the desired inequality. [Alternatively, one can "complete the square" to obtain: $p(a) = \left(a - \frac{b+c}{2}\right)^2 + \frac{3(b-c)^2}{4}$.]

A similar, more elaborate example is the following geometric inequality from [**2**]:

Let ABC be a triangle with sides $BC = a, CA = b, AB = c$ and semiperimeter s. If r and R are the inradius and circumradius, respectively, prove that
$$\frac{(b+c)^2}{4bc} \le \frac{s^2}{3r(4R+r)}.$$

Let $X = \dfrac{s^2}{r(4R+r)}$. Using $a^2+b^2+c^2 = 2s^2-2r^2-8rR$, it is easily checked that

$$X = \frac{(a+b+c)^2}{(a+b+c)^2-2(a^2+b^2+c^2)} = \frac{a^2+2a(b+c)+(b+c)^2}{-a^2+2a(b+c)-(b-c)^2}.$$

Note that $-a^2+2a(b+c)-(b-c)^2 = 4r^2+16rR > 0$, hence the inequality is equivalent to

$$4bc(a^2+2a(b+c)+(b+c)^2) \ge 3(b+c)^2(-a^2+2a(b+c)-(b-c)^2),$$

which rewrites as $p(a) \ge 0$ where

$$p(x) = x^2(3b^2+3c^2+10bc) - 2x(b+c)(3b^2+3c^2+2bc) + (b+c)^2(3b^2+3c^2-2bc).$$

Now, the discriminant is 4δ where

$$\delta = (b+c)^2(3b^2+3c^2+2bc)^2 - (b+c)^2(3b^2+3c^2-2bc)(3b^2+3c^2+10bc).$$

A simple calculation shows that $\delta = -12bc(b-c)^2(b+c)^2$, hence $\delta \le 0$. Since in addition $3b^2+3c^2+10bc > 0$, we have $p(x) \ge 0$ for all real x and in particular $p(a) \ge 0$, as desired.

In our final illustration, slightly adapted from a problem of the 23rd Russian Olympiad for Secondary Schools [2000 : 388 ; 2002 : 493], the sign of the quadratic polynomial plays an incidental but important role.

Solve in positive integers the equation $(x^2-y^2)^2 = 1+16y$.

Let (x,y) be a solution. Clearly, we have $x \neq y$. We distinguish two mutually exclusive cases:

• if $1 \le x < y$, then $y \ge 2$ and with $k = y-x$, the equation becomes $k^2(2y-k)^2 = 1+16y$. Since $k \le y-1$, the assumption $k \ge 2$ leads to $1+16y \ge 4(y+1)^2$, that is, $4y^2-8y+3 \le 0$. The roots of the polynomial $4y^2-8y+3$ being $\frac{1}{2}$ and $\frac{3}{2}$, the latter inequality does not hold for $y \ge 2$. We deduce that we must have $k=1$ and so $(2y-1)^2 = 1+16y$. Therefore $y=5$ and $x = y-k = 5-1 = 4$.

• if $1 \le y < x$, similarly we set $\ell = x-y$ so that $1 \le \ell \le x$ and $\ell^2(2y+\ell)^2 = 1+16y$. Again, $\ell \ge 2$ would imply that $1+16y \ge 4(2y+2)^2$, that is $16y^2+16y+15 \le 0$, which does not hold. Hence $\ell = 1$ and so $y=3$ and $x=4$.

Conversely, we check that $(4,3)$ and $(4,5)$ are indeed solutions and so are the solutions.

Exercises

1. Let u be a complex number with $|u| = 1$. Show that the solutions to the equation

$$z^2 - 2z(1-u) - u = 0$$

are unimodular if and only if $|1-u| \leq 1$.

2. (Problem E.386 proposed in the French journal *Quadrature* No 101, 2016). Let x, y, z, a, b be positive real numbers satisfying

$$\begin{cases} x^2 + xy + y^2 & = & a^2 \\ y^2 + yz + z^2 & = & b^2 \\ z^2 + zx + x^2 & = & a^2 + b^2. \end{cases}$$

Express $s = x + y + z$ as a function of a and b (hint: obtain a biquadratic equation of which s is a solution).

References

[1] Yimin Ge, The Method of Vieta Jumping, *Mathematical Reflections*, 2007, No 5.

[2] G. Apostolopoulos, personal communication.

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 30, 2021**.

4601. *Proposed by Bill Sands.*

One or more pieces of clothing are hanging on a clothesline. Each piece of clothing is held up by either 1, 2 or 3 clothespins. Clothes do not overlap and each clothespin holds up one piece of clothing. You want to remove all the clothing from the line, obeying the following rules:

(i) you must remove the clothing in the order that they are hanging on the line;

(ii) all the pins holding up a piece of clothing must be removed at the same time;

(iii) the number of clothespins you remove each time must belong to the set $\{n+1, n+2, \ldots, n+c\}$, where n and c are given positive integers.

Find the smallest positive integer c so that, for any positive integer n, all sufficiently long lines of clothing can be removed.

4602. *Proposed by Nguyen Viet Hung.*

Let ABC be an acute triangle. Let h_a be the length of the altitude from vertex A to side BC and let w_a be the length of the internal bisector of $\angle A$ to side BC. Define h_b, h_c, w_b and w_c similarly. Also let r be the inradius and R the circumradius of ABC. Prove that

$$\frac{h_b h_c}{a^2} + \frac{h_c h_a}{b^2} + \frac{h_a h_b}{c^2} = \frac{r}{2R} + \frac{2h_a h_b h_c}{w_a w_b w_c}.$$

4603. *Proposed by Michel Bataille.*

Let ABC be a triangle. The perpendiculars to AB through A and to AC through C intersect at D. The perpendiculars to AC through A and to AB through B intersect at E. Prove that the altitude from A in ΔDAE is a symmedian of ΔABC.

4604. *Proposed by Nguyen Viet Hung.*

Prove that the triangle ABC is equilateral if and only if

$$a\sin(A - \frac{\pi}{3}) + b\sin(B - \frac{\pi}{3}) + c\sin(C - \frac{\pi}{3}) = 0.$$

4605. *Proposed by George Stoica.*

Let $\{x_i\}_{i=1}^m$ be any set of non-zero vectors in $\mathbb{R}^n$. Prove the following:

(1) If $\langle x_i, x_j \rangle < 0$ for all $i \neq j$, then $m \leq n+1$.

(2) If $\langle x_i, x_j \rangle \leq 0$ for all $i \neq j$, then $m \leq 2n$.

4606. *Proposed by Garcia Antonio.*

For $a, b, c, n > 0$, show that

$$(a+b)\sqrt{\frac{na+b}{a+nb}} + (b+c)\sqrt{\frac{nb+c}{b+nc}} + (c+a)\sqrt{\frac{nc+a}{c+na}} \geq 2(a+b+c).$$

4607. *Proposed by Ted Barbeau.*

a) Determine all polynomials $q(x)$ that satisfy the functional equation

$$q(x)q(x+1) = q(x^2+x).$$

b) Determine all polynomials $p(x)$ that satisfy the functional equation

$$p(x)p(x+1) = p(x+p(x)).$$

c) $\star$ Prove or disprove the conjecture: Let $p(x)$ be a polynomial solution of the functional equation in (b). Then, if $q(x)$ satisfies the functional equation

$$q(x)q(x+1) = q(x+p(x)),$$

then $q(x) = p(x)^n$ for some nonnegative integer n.

4608. *Proposed by Florin Stanescu.*

Calculate

$$\lim_{n\to\infty} \frac{H_{n+1} + H_{n+2} + \cdots + H_{2n}}{nH_n},$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 1$.

4609. *Proposed by George Apostolopoulos.*

Triangle ABC has internal angle bisectors AD, BE and CF, where points D, E and F lie on the sides BC, AC and AB, respectively. Prove that

$$\frac{AB^4 + BC^4 + CA^4}{DE^4 + EF^4 + FD^4} \geq 16.$$

4610. *Proposed by Albert Natian.*

Find the smallest positive number x so that the following three quantities a, b and c are all integers:

$$a = \sqrt[4]{72 + \sqrt{3x} + \sqrt{16 + 275x} + \sqrt{19 + 288x}}\,,$$
$$b = 5\sqrt[3]{\frac{9x}{20}} + \sqrt{16 + 275x}\,,$$
$$c = 7\sqrt[3]{\frac{2x}{15}} + 2\sqrt{3x}\,.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 mars 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4601. *Proposée par Bill Sands.*

Au moins une pièce de linge est suspendue sur une corde à linge, chaque pièce de linge y étant retenue par 1, 2 ou 3 épingles à linge. Les pièces de linge ne se chevauchent pas et une épingle à linge retient une seule pièce de linge. Maintenant, on voudrait enlever le linge de la corde, en respectant les règles suivantes :

(i) on doit enlever les pièces de linge dans le même ordre auquel ils se trouvent sur la corde;

(ii) toutes les épingles retenant une pièce de linge doivent être enlevées au même moment;

(iii) le nombre d'épingles à linge enlevées en un même moment doit se trouver dans l'ensemble $\{n+1, n+2, \dots, n+c\}$, où n et c sont des entiers.

Déterminer la valeur du plus petit entier positif c tel que, pour tout entier positif n, tout linge sur une corde à linge suffisamment longue puisse être enlevé.

4602. *Proposée par Nguyen Viet Hung.*

Soit ABC un triangle acutangle. Soit h_a la longueur de l'altitude du sommet A au côté BC et w_a la longueur de la bissectrice de l'angle A jusqu'au côté BC. Les longueurs h_b, h_c, w_b et w_c sont définies de la même façon. Si r est le rayon du cercle inscrit et R est le rayon du cercle circonscrit au triangle ABC, démontrer que

$$\frac{h_b h_c}{a^2} + \frac{h_c h_a}{b^2} + \frac{h_a h_b}{c^2} = \frac{r}{2R} + \frac{2h_a h_b h_c}{w_a w_b w_c}.$$

4603. *Proposée par Michel Bataille.*

Soit ABC un triangle. Les perpendiculaires vers AB passant par A, puis vers AC passant par C, intersectent en D. De façon similaire, les perpendiculaires vers AC passant par A, puis vers AB passant par B, intersectent en E. Démontrer que l'altitude émanant de A dans ΔDAE est sym-médiane dans ΔABC.

4604. *Proposée par Nguyen Viet Hung.*

Démontrer que le triangle ABC est équilatéral si et seulement si

$$a\sin(A - \frac{\pi}{3}) + b\sin(B - \frac{\pi}{3}) + c\sin(C - \frac{\pi}{3}) = 0.$$

4605. *Proposée par George Stoica.*

Soit $\{x_i\}_{i=1}^m$ un ensemble de vecteurs non nuls dans $\mathbb{R}^n$. Démontrer les suivantes.

(1) Si $\langle x_i, x_j\rangle < 0$ pour tout $i \neq j$, alors $m \leq n + 1$.

(2) Si $\langle x_i, x_j\rangle \leq 0$ pour tout $i \neq j$, alors $m \leq 2n$.

4606. *Proposée par Garcia Antonio.*

Pour $a, b, c, n > 0$, démontrer que

$$(a + b)\sqrt{\frac{na + b}{a + nb}} + (b + c)\sqrt{\frac{nb + c}{b + nc}} + (c + a)\sqrt{\frac{nc + a}{c + na}} \geq 2(a + b + c).$$

4607. *Proposée par Ted Barbeau.*

a) Déterminer tout polynôme $q(x)$ satisfaisant à l'équation fonctionnelle

$$q(x)q(x + 1) = q(x^2 + x).$$

b) Déterminer tout polynôme $p(x)$ satisfaisant à l'équation fonctionnelle

$$p(x)p(x + 1) = p(x + p(x)).$$

c) $\star$ Démontrer la conjecture suivante ou présenter un contre exemple: Soit $p(x)$ une solution polynomiale à l'équation fonctionnelle en (b) ; si $q(x)$ satisfait à l'équation fonctionnelle

$$q(x)q(x+1) = q(x+p(x)),$$

alors $q(x) = p(x)^n$ pour un certain entier non négatif n.

4608. *Proposée par Florin Stanescu.*

Calculer

$$\lim_{n\to\infty} \frac{H_{n+1} + H_{n+2} + \cdots + H_{2n}}{nH_n},$$

où $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $n \geq 1$.

4609. *Proposée par George Apostolopoulos.*

Les bissectrices internes du triangle ABC sont AD, BE et CF, où les points D, E et F se trouvent sur les côtés BC, AC et AB, respectivement. Démontrer que

$$\frac{AB^4 + BC^4 + CA^4}{DE^4 + EF^4 + FD^4} \geq 16.$$

4610. *Proposée par Albert Natian.*

Déterminer le plus petit nombre positif x tel que les trois valeurs a, b et c suivantes sont entières:

$$\begin{aligned} a &= \sqrt[4]{72 + \sqrt{3x} + \sqrt{16 + 275x} + \sqrt{19 + 288x}}\ , \\ b &= 5\sqrt[3]{\frac{9x}{20}} + \sqrt{16 + 275x}\,, \\ c &= 7\sqrt[3]{\frac{2x}{15}} + 2\sqrt{3x}\,. \end{aligned}$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(6), p. 264–269.

4551. *Proposed by Michel Bataille.*

Let ABC be a triangle with sides $BC = a, CA = b$ and $AB = c$. Suppose $b > c$ and let A_1, A_2 be the two points such that ΔA_1BC and ΔA_2BC are equilateral. Express the circumradius of ΔAA_1A_2 as a function of a, b, c.

There were 12 correct solutions. Two other submissions provided a process for getting the solution, but did not work out the final answer. We present 4 solutions.

The circumradius of triangle AA_1A_2 is equal to

$$\begin{aligned}&\frac{a}{b^2-c^2}\left[a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2\right]^{\frac{1}{2}}\\&=\frac{a\sqrt{2}}{2(b^2-c^2)}\left[(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)\right]^{\frac{1}{2}}.\end{aligned}$$

Solution 1, by UCLan Cyprus Problem Solving Group.

Note that B and A lie on the same side of A_1A_2. Let O be the circumcentre of triangle AA_1A_2 and let R, the length of OA and OA_1, be the circumradius. Since BC is the right bisector of the chord A_1A_2 of the circumcircle, then O lies on BC produced.

First, suppose that O lies between B and C, x and y are the respective lengths of OC and BO, and a, b, c have their conventional meaning. By Stewart's theorem applied to triangles ABC and A_1BC,

$$c^2x + b^2y = a(R^2 + xy) = a^2x + a^2y,$$

whence $(c^2 - a^2)x = (a^2 - b^2)y = (a^2 - b^2)(a - x)$ and

$$x = \frac{a(b^2-a^2)}{b^2-c^2}.$$

Therefore

$$\begin{aligned}R^2 &= ax + ay - xy = ax + (a-x)^2 = a^2 - ax + x^2\\&= \frac{a^2}{(b^2-c^2)^2}\left[(b^2-c^2)^2-(b^2-c^2)(b^2-a^2)+(b^2-a^2)^2\right]\\&= \frac{a^2}{2(b^2-c^2)^2}\left[(b^2-c^2)^2+((b^2-c^2)-(b^2-a^2))^2+(b^2-a^2)^2\right].\end{aligned}$$

Secondly, suppose that B lies between O and C and z is the length of BO. Then

$$R^2a + b^2z = (a+z)(c^2+za) \quad \text{and} \quad R^2a + a^2z = (a+z)(a^2+za),$$

whence $(b^2-a^2)z = (a+z)(c^2-a^2)$ and

$$z = \frac{a(c^2-a^2)}{b^2-c^2}.$$

Then

$$R^2 = a^2 + az + z^2 = \frac{a^2}{b^2-c^2}\left[(b^2-c^2)^2 + (c^2-a^2)(b^2-c^2) + (c^2-a^2)^2\right].$$

Finally, suppose that C lies between B and O and w is the length of CO. Then

$$R^2a + c^2w = (a+w)(b^2+aw) \quad \text{and} \quad R^2a + a^2w = (a+w)(a^2+aw),$$

whence $(c^2-a^2)w = (a+w)(b^2-a^2)$, and

$$w = \frac{a(a^2-b^2)}{b^2-c^2}.$$

Then

$$R^2 = a^2 + aw + w^2 = \frac{a^2}{(b^2-c^2)^2}\left[(b^2-c^2)^2 + (a^2-b^2)(b^2-c^2) + (a^2-b^2)^2\right].$$

These all give the desired expression.

Solution 2, by Eugen Ionascu, Walther Janous, C.R. Pranesachar, and Joel Schlosberg (independently).

Let a_1 and a_2 be the respective lengths of AA_1 and AA_2, with $a_1 < a_2$. From the Law of Cosines, we have that $2ab\cos C = a^2+b^2-c^2$, $2ac\cos B = a^2+c^2-b^2$ and

$$\begin{aligned} a_1^2 &= a^2+b^2-2ab\cos|C-60^\circ| \\ &= a^2+b^2-ab\cos C - \sqrt{3}ab\sin C \\ &= \frac{a^2+b^2+c^2}{2} - \sqrt{3}ab\sin C, \end{aligned}$$

and

$$\begin{aligned} a_2^2 &= a^2+b^2-2ab\cos(C+60^\circ) \\ &= a^2+b^2-ab\cos C + \sqrt{3}ab\sin C \\ &= \frac{a^2+b^2+c^2}{2} + \sqrt{3}ab\sin C, \end{aligned}$$

whence $a_1^2 + a_2^2 = a^2 + b^2 + c^2$ and

$$\begin{aligned} a_1^2 \cdot a_2^2 &= \left(\frac{a^2+b^2+c^2}{2}\right)^2 - 3a^2b^2(1-\cos^2 C) \\ &= \left(\frac{a^2+b^2+c^2}{2}\right)^2 - 3a^2b^2 + \frac{3}{4}(a^2+b^2-c^2)^2 \\ &= a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2. \end{aligned}$$

(Alternatively, $a_1^2 = (b\cos C - a/2)^2 + (b\sin C \;\; - \sqrt{3}a/2)^2$ and $a_2^2 = (b\cos C - a/2)^2 + (b\sin C + \sqrt{3}a/2)^2$.)

Using the formula for the area of the triangle in terms of the sides of the triangle and noting that the length of A_1A_2 is $a\sqrt{3}$, we find that

$$\begin{aligned} [AA_1A_2] &= \frac{1}{4}\sqrt{4a_1^2a_2^2 - (a_1^2+a_2^2-3a^2)^2} \\ &= \frac{1}{4}\sqrt{4(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2) - (b^2+c^2-2a^2)^2} \\ &= \frac{1}{4}\sqrt{3b^4+3c^4-6b^2c^2} = \frac{\sqrt{3}}{4}(b^2-c^2). \end{aligned}$$

Alternatively, with the $\pm$ sign allowing for different configurations,

$$\begin{aligned} [AA_1A_2] &= [BA_1A_2] + [BAA_1] \pm [BAA_2] \\ &= \frac{1}{2}a^2\sin 120^\circ + \frac{1}{2}ac\sin(B-60^\circ) - \frac{1}{2}ac\sin(B+60^\circ) \\ &= \frac{\sqrt{3}}{4}[a^2 - 2ac\cos B] = \frac{\sqrt{3}}{4}[b^2-c^2]. \end{aligned}$$

Using the determination of the circumradius as $(\sqrt{3}aa_1a_2)/(4[AA_1A_2])$, we obtain the desired result.

Solution 3, by Oliver Geupel.

Let D be the midpoint of BC, m be the length of the median AD, and let a, b, c, a_1, a_2 have the foregoing meanings. Applying the Law of Sines in triangle ACD, we find that

$$\cos\angle A_1DA = \sin\angle CDA = \frac{b}{m}\sin C,$$

whence

$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}am\cos\angle A_1DA.$$

By the Law of Cosines,

$$\begin{aligned}
a_1^2 = |AA_1|^2 &= |A_1D|^2 + |AD|^2 - 2|A_1D| \cdot |AD| \cdot \cos\angle A_1DA \\
&= \frac{3a^2}{4} + m^2 - 2\left(\frac{a\sqrt{3}}{2}\right)\left(\frac{2[ABC]}{a}\right) \\
&= \frac{3a^2}{4} + \frac{2b^2+2c^2-a^2}{4} - 2\sqrt{3}[ABC] \\
&= \frac{a^2+b^2+c^2}{2} - 2\sqrt{3}[ABC].
\end{aligned}$$

Similarly,

$$a_2^2 = \frac{a^2+b^2+c^2}{2} + 2\sqrt{3}[ABC].$$

$$\begin{aligned}
a_1^2a_2^2 &= \frac{1}{4}((a^2+b^2+c^2)^2) - 12[ABC]^2 \\
&= \frac{1}{4}[(a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2) \\
&\quad - 3(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)] \\
&= a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2.
\end{aligned}$$

Let h and d be the respective distances of A from A_1A_2 and BC. Then

$$[AA_1A_2] = \frac{ah\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\left[\left(\frac{a}{2}+h\right)^2 + d^2\right) - \left[\left(\frac{a}{2}-h\right)^2 + d^2\right) = \frac{(b^2-c^2)\sqrt{3}}{4}.$$

The formula for the circumradius of triangle AA_1A_2 yields the desired result.

Solution 4, by Marie-Nicole Gras.

Assign coordinates

$$A \sim (c\cos\theta, c\sin\theta),\ B \sim (0,0),\ C \sim (a,0), \quad \text{and} \quad A_1 \sim (a/2, a\sqrt{3}/2).$$

The right bisector of AA_1 contains the circumcentre of triangle AA_1A_2 and has equation

$$\left(c\cos\theta - \frac{a}{2}\right)x + \left(c\sin\theta - \frac{a\sqrt{3}}{2}\right)y = \frac{c^2-a^2}{2}.$$

The circumcentre is located at the intersection of this line and BC produced:

$$\left(\frac{a(a^2-c^2)}{b^2-c^2}, 0\right).$$

The distance from this point to A_1 yields the desired expression for the circumradius.

4552. *Proposed by Anupam Datta.*

Given positive integers a, b and n, prove that the following are equivalent:

1. $b \equiv ax \pmod{n}$ has a solution with $\gcd(x, n) = 1$;
2. $b \equiv ax \pmod{n}$ and $a \equiv by \pmod{n}$ have solutions $x, y \in \mathbb{Z}$;
3. $\gcd(a, n) = \gcd(b, n)$.

We received 15 submissions, out of which 7 were correct and complete. We present the solution by Eugen Ionascu, modified by the editor.

$(1) \Rightarrow (2)$

We need to show that $a \equiv by \pmod{n}$ has a solution $y \in \mathbb{Z}$. Since $\gcd(x, n) = 1$ we know that x is in the multiplicative group of integers modulo n. Using its inverse in this group gives us $y \in \mathbb{Z}$ with $xy \equiv 1 \pmod{n}$. Thus, since $b \equiv ax \pmod{n}$ by assumption,

$$by \equiv axy \equiv a \pmod{n}.$$

$(2) \Rightarrow (3)$

By assumption we can write $b = ax + rn$ for some $r \in \mathbb{Z}$. Since $\gcd(a, n)$ divides both ax and rn, we have $\gcd(a, n)|b$. Together with $\gcd(a, n)|n$ we obtain $\gcd(a, n)|\gcd(b, n)$. By symmetry we also have $\gcd(b, n)|\gcd(a, n)$ and thus $\gcd(a, n) = \gcd(b, n)$.

$(3) \Rightarrow (1)$

Let d be the greatest common divisor of a and n. Then

$$a = da' \qquad b = db' \qquad n = dn'$$

for some positive integers a', b', n' with $\gcd(a', n') = \gcd(b', n') = 1$. Then a' has an inverse in the multiplicative group of integers modulo n', say z. Hence $a'z \equiv 1 \pmod{n'}$. Let

$$x = b'z + n't$$

for any integer t. Then

$$a'x \equiv a'(b'z + n't) \equiv b' \pmod{n'}.$$

Multiplying everything by d, we obtain

$$ax \equiv b \pmod{n}.$$

It remains to show that there exists a t such that $\gcd(x, n) = 1$. Since $b'z$ and n' are coprime, by Dirichlet's Theorem the set of integers $b'z + n't$ $(t \in \mathbb{N})$ contains infinitely many primes. In particular, there exists a t such that $x = b'z + n't$ is a prime greater than n, which implies $\gcd(x, n) = 1$.

4553. *Proposed by Daniel Sitaru.*

Find

$$\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)$$

We received 33 submissions, of which 29 were complete and correct. We present the solution by Devis Alvarado, lightly edited.

The limit is equivalent to

$$\begin{aligned}\lim_{n\to\infty}\frac{\int_0^1 x^2(x+n)^n\,dx}{(n+1)^n} &= \lim_{n\to\infty}\int_0^1 \frac{x^2(x+n)^n}{(n+1)^n}\,dx\\ &= \lim_{n\to\infty}\int_0^1 x^2\,\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}\,dx.\end{aligned}$$

For $n\ge 1$ define $f_n:[0,1]\to\mathbb{R}$ by $f_n(x)=x^2\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}$. We have

$$|f_n(x)|=f_n(x)=x^2\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}\le x^2\left(1+\frac{x}{n}\right)^n.$$

It is well known that $\left\{\left(1+\frac{x}{n}\right)^n\right\}_{n\ge1}$ converges to e^x. Since $x\in[0,1]$ it is also an increasing sequence. We conclude that $|f_n(x)|\le x^2e^x\le e$.

Apply the Bounded Convergence Theorem to get

$$\begin{aligned}\lim_{n\to\infty}\int_0^1 x^2\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}\,dx &= \int_0^1 \lim_{n\to\infty}\left(x^2\,\frac{\left(1+\frac{x}{n}\right)^n}{\left(1+\frac{1}{n}\right)^n}\right)dx\\ &= \int_0^1 x^2\,\frac{\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}\,dx\\ &= \int_0^1 x^2\cdot\frac{e^x}{e}\,dx\\ &= \frac{1}{e}\left[x^2e^x-2xe^x+2e^x\right]\Big|_0^1\\ &= 1-\frac{2}{e}.\end{aligned}$$

Therefore $\lim_{n\to\infty}\left(\frac{\int_0^1 x^2(x+n)^n dx}{(n+1)^n}\right)=1-\frac{2}{e}.$

4554. *Proposed by George Stoica.*

Let ϵ be a given constant with $0 < \epsilon < 1$, and let (a_n) be a sequence with $0 \leq a_n < \epsilon$ for all $n \geq 1$. Prove that $(1 - a_n)^n \to 1$ as $n \to \infty$ if and only if $na_n \to 0$ as $n \to \infty$.

There were 19 correct solutions. Several solvers noted that it was sufficient that each a_n be less than 1. We present 4 solutions.

Solution 1, by Vincent Blevins.

Applying the Bernoulli inequality $1 + nx \leq (1 + x)^n$ for $n \geq 1$ and $x > -1$ and using $(1 - a_n)^{2n} \leq 1$, we have that

$$1 \leq 1 + na_n \leq (1 + a_n)^n \leq \frac{1}{(1 - a_n)^n} \leq \frac{1}{1 - na_n}.$$

The desired result follows from the squeeze principle.

Solution 2, by Zoltan Retkes.

The result follows from

$$\begin{aligned} \ln(1 - a_n)^{-n} &= -n \ln(1 - a_n) = n \left(a_n + \frac{a_n^2}{2} + \frac{a_n^3}{3} + \cdots \right) \\ &\geq na_n = a_n(1 + 1 + \cdots + 1) \\ &\geq a_n[1 + (1 - a_n) + \cdots + (1 - a_n)^{n-1}] \\ &= 1 - (1 - a_n)^n \geq 0. \end{aligned}$$

Solution 3, by Dmitry Fleischman and Walther Janous (independently).

The function $f(t) = \ln(1-t)/t$ is decreasing on $(0, \epsilon)$ with $\lim_{t \to 0} f(t) = -1$. Since $f(\epsilon) < f(a_n) < -1$, and

$$na_n f(\epsilon) \leq na_n f(a_n) = na_n \left(\frac{\ln(1 - a_n)}{a_n} \right) = \ln(1 - a_n)^n \leq -na_n < 0,$$

the desired result follows.

Solution 4, by Michel Bataille and Eugen Ionascu (independently).

Recall that $\ln x \leq x - 1$ for $x > 0$. Replacing x by $1/x$ yields $\ln x \geq \dfrac{x-1}{x}$. Setting $x = 1 - a_n$ leads to

$$-\frac{na_n}{1 - \epsilon} \leq \frac{-na_n}{1 - a_n} \leq \ln(1 - a_n)^n \leq -na_n \leq 0.$$

Applying the squeeze principle yields the result.

4555. *Proposed by Michael Rozenberg and Leonard Giugiuc.*

Prove that if a, b, c and d are positive numbers that satisfy

$$ab + bc + cd + da + ac + bd = 6,$$

then

$$a + b + c + d \geq 2\sqrt{(a^2 + b^2 + c^2 + d^2)abcd}.$$

When does the equality hold?

We received 11 submissions of which 6 were correct and complete. We present the solution by the proposer Leonard Giugiuc.

By MacLaurin's inequality,

$$\left(\frac{a+b+c+d}{4}\right)^2 \geqslant \frac{ab+bc+cd+da+ac+bd}{6} = 1.$$

So, $a + b + c + d \geqslant 4$ and there exists a unique t for which $0 < t \leqslant 1$ and $a + b + c + d = 2(t + \frac{1}{t})$. Let $s = \frac{t^2+1}{2t}$ so that $s \geqslant 1$ and $a + b + c + d = 4s$.

Next, define $q = abc + abd + acd + bcd$, and $r = abcd$. We will prove that

$$(a + b + c + d)^2 \geqslant 4(a^2 + b^2 + c^2 + d^2)abcd,$$

which is equivalent to each of

$$16s^2 \geqslant 4(16s^2 - 12)r \quad \text{and} \quad s^2 \geqslant (4s^2 - 3)r.$$

To that end, we will find an upper bound for r in terms of t.

Define

$$p(x) = (x - a)(x - b)(x - c)(x - d) = x^4 - 4sx^3 + 6x^2 - qx + r$$

and

$$f(x) = \frac{p(x)}{x} = x^3 - 4sx^2 + 6x - q + \frac{r}{x}$$

for $x > 0$. Then the zeroes of f are a, b, c, and d. Let $g(x) = 3x^4 - 8sx^3 + 6x^2 - r$ so that $f'(x) = \frac{g(x)}{x^2} = 0$ if and only if $g(x) = 0$. Since g is a fourth degree polynomial with a positive coefficient of x^4 and $g(0) = -r < 0$, then g has at least one negative zero. By Rolle's theorem applied to f, f' has at least three positive zeroes, and thus, so does g. Since

$$g'(x) = 12x^3 - 24sx^2 + 12x = 12x(x^2 - \frac{t^2+1}{t}x + 1) = 12x(x - t)(x - \frac{1}{t}),$$

then $g'(x) = 0$ when $x = 0$, $x = t$, and $x = \frac{1}{t}$. Thus $g(t) \geqslant 0$ and $g(\frac{1}{t}) \leqslant 0$. But

$$g(t) = 3t^4 - 4(t^4 + t^2) + 6t^2 - r = 2t^2 - t^4 - r$$

and so, $r \leqslant 2t^2 - t^4$. Thus,

$$(4s^2-3)(2t^2-t^4) \geqslant (4s^2-3)r$$

and it is sufficient to show that $s^2 \geqslant (4s^2-3)(2t^2-t^4)$, which is equivalent to

$$(\frac{t^2+1}{2t})^2 \geqslant ((t^2+1)^2-3t^2)(2-t^2)$$

and

$$(t^2-1)^2(2t^2-1)^2 \geqslant 0.$$

Note that $r = 2t^2 - t^4$ if and only if three of $a, b, c,$ and d are equal to t and the fourth is $\frac{2-t^2}{t}$.

Equality holds for $t = 1$ or $t = \dfrac{1}{\sqrt{2}}$. That is, when $(a,b,c,d) = (1,1,1,1)$ or $(a,b,c,d) = (\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, \dfrac{3}{\sqrt{2}})$ or a permutation thereof.

4556. *Proposed by Marian Cucoanes and Lorean Saceanu.*

Let $x \geq 1$ be a real number and consider a triangle ABC. Prove that

$$\frac{(x-\cos A)(x-\cos B)(x-\cos C)}{(x+\cos A)(x+\cos B)(x+\cos C)} \leq \left[\frac{(3x-1)R-r}{(3x+1)R+r}\right]^3.$$

When does the equality hold?

We received 9 solutions, two of which were incorrect. We present the solution by Theo Koupelis.

Setting $c_1 = \sum_{\text{cyc}} \cos A$, $c_2 = \sum_{\text{cyc}} \cos A \cos B$, and $c_3 = \prod_{\text{cyc}} \cos A$, and noting the well-known expression $c_1 = 1 + (r/R)$, we rewrite the given inequality as

$$\frac{x^3 - c_1x^2 + c_2x - c_3}{x^3 + c_1x^2 + c_2x + c_3} \quad \leq \quad \left(\frac{3x-c_1}{3x+c_1}\right)^3. \tag{1}$$

The denominators of the fractions on both sides of (1) are positive (because $x \geq 1$ and the triangle is not trivial, that is $r > 0$). Therefore, after clearing denominators, (1) is equivalent to showing that

$$x^2(8c_1^3 - 27c_1c_2 + 27c_3) \quad \geq \quad c_1^2(c_1c_2 - 9c_3). \tag{2}$$

We will first show that

$$c_1c_2 - 9c_3 \quad \geq \quad 0. \tag{3}$$

If the triangle is not obtuse (and not trivial) then the values of $\cos A, \cos B, \cos C$ are in $[0, 1)$, and by AM-GM we have

$$c_1c_2 - 9c_3 = \sum_{\text{sym}} \cos^2 A \cos B - 6 \prod_{\text{cyc}} \cos A \geq 0,$$

with equality when the triangle is equilateral.

If the triangle is obtuse, say $90^\circ < \angle C < 180^\circ$, then $\cos C = -\cos D$, where $\angle D = \pi - \angle C$; however, we still have

$$\cos A + \cos B - \cos D = 1 + (r/R) > 1,$$

and therefore $\cos D < \cos A + \cos B - 1$, and thus

$$\begin{aligned} c_1c_2 - 9c_3 \geq \cos^2 A(1 - \cos A) + \cos^2 B(1 - \cos B) + \\ + \cos^2 D(\cos A + \cos B) + 6 \cos A \cdot \cos B \cdot \cos D > 0. \end{aligned}$$

Therefore, inequality (3) holds, with equality for an equilateral triangle.

We will now show that

$$8c_1^3 - 27c_1c_2 + 27c_3 \quad \geq \quad c_1^2(c_1c_2 - 9c_3). \tag{4}$$

We can rewrite this expression as

$$2(c_1^2 - 3c_2) - \frac{c_1^2 + 3}{4c_1} \cdot (c_1c_2 - 9c_3) \geq 0.$$

But with $1 < c_1 \leq 3/2$, we have $(c_1 - 1)(c_1 - 3) < 0$, and thus $0 < \frac{c_1^2+3}{4c_1} < 1$. Therefore, inequality (4) will hold if we can show the stronger inequality

$$2(c_1^2 - 3c_2) - (c_1c_2 - 9c_3) \quad \geq \quad 0. \tag{5}$$

But this is obvious because it is equivalent to showing that

$$2\left(\sum_{\text{cyc}} \cos^2 A - \sum_{\text{cyc}} \cos A \cdot \cos B\right) + 6 \cos A \cdot \cos B \cdot \cos C - \sum_{\text{sym}} \cos^2 A \cdot \cos B \geq 0,$$

or

$$\sum_{cyc} (\cos A - \cos B)^2 (1 - \cos C) \geq 0.$$

Therefore, inequality (4) holds; for a non-trivial triangle, equality in (4) and (5) occurs for an equilateral triangle. That is because in such a case we have

$$c_1^2 = 3c_2 = \tfrac{9}{4} \quad \text{and} \quad c_1c_2 = 9c_3 = \tfrac{9}{8}.$$

From (3) and (4) we conclude that inequality (2), and therefore (1), holds for all $x \geq 1$. Equality occurs for an equilateral triangle.

4557. *Proposed by George Apostolopoulos.*

Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC with circumradius R and inradius r. Let a, b and c be the lengths of the sides of ABC. Prove

$$\frac{24r^2}{R} \le \frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le \frac{4r^2 - 2Rr}{r}.$$

We received 20 submissions all of which noticed a small typo on the right side of the given inequality, and gave valid proof for the intended correct version. We present the solution by Marie-Nicole Gras.

By Cauchy-Schwarz Inequality we have

$$(m_a + m_b + m_c)\left(\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c}\right) \ge (a+b+c)^2,$$

so

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \ge \frac{4s^2}{m_a + m_b + m_c},$$

where s denotes the semiperimeter of $\triangle ABC$.

The following inequalities are all well known [see items 5.1, 5.11, and 8.2 on pp. 48, 52, and 73 of *Geometric Inequalities* by O. Bottema et al.]: $2r \le R$, $s^2 \ge 27r^2$, and $m_a + m_b + m_c \le 4R + r \le \frac{9}{2}R$. Hence we have,

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \ge \frac{4(27r^2)}{\frac{9}{2}R} = \frac{24r^2}{R}.$$

To prove the right inequality, we let T, h_a, h_b and h_c denote the area and heights of $\triangle ABC$, respectively. Since $m_x \ge h_x$ and $h_x = \dfrac{2T}{x}$ for all $x \in \{a, b, c\}$ we have

$$\frac{a^2}{m_a} + \frac{b^2}{m_b} + \frac{c^2}{m_c} \le \frac{a^2}{h_a} + \frac{b^2}{h_b} + \frac{c^2}{h_c} = \frac{a^3 + b^3 + c^3}{2T}. \tag{1}$$

The results below are all well known: $abc = 4srR$, $ab + bc + ca = s^2 + 4rR + r^2$, and $s^2 \le 4R^2 + 4rR + 3r^2$ [Gerretsen Inequality; see item 5.8 on p. 50 of the aforementioned reference]. Using these together with $a + b + c = 2s$, we then obtain

$$\begin{aligned}
a^3 + b^3 + c^3 &= (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc \\
&= (2s)^3 - 3(2r)(s^2 + 4rR + r^2) + 3(4srR) \\
&= 2s^3 - 12srR - 6sr^2 = 2s(s^2 - 6rR - 3r^2) \\
&\le 2s(4R^2 + 4rR + 3r^2 - 6rR - 3r^2) = 2s(4R^2 - 2rR).
\end{aligned}$$

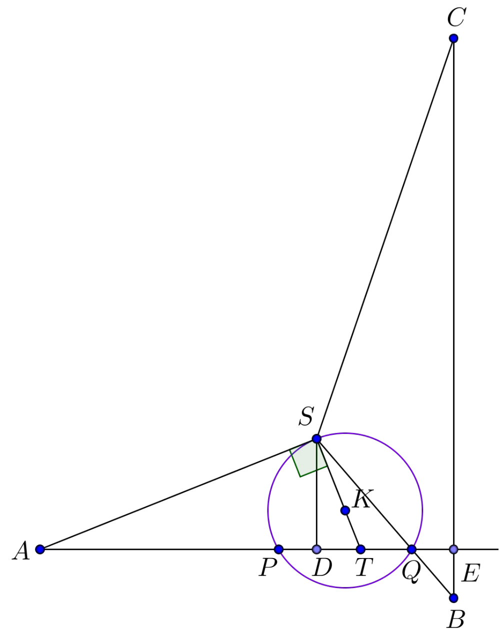
Since $T = sr$ we finally obtain

$$\frac{a^3 + b^3 + c^3}{2T} \le \frac{4R^2 - 2rR}{r}. \tag{2}$$

From (1) and (2), the given right inequality follows and the proof is complete.

4558. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

Consider a diagram below, where triangle SAT is right-angled and $\tan T > 2$. The point K lies on the segment ST such that $SK = 2KT$. The circle centered at K with radius KS intersects the line AT at P and Q. Point D is the projection of S on AT and E is a point on AT such that D lies on AE and $AD = 2DE$. Finally, suppose that SQ and SP intersect the perpendicular at E on AT at B and C respectively. Prove that S is the incenter of the triangle ABC.



We received 12 submissions, all correct. Most solutions used coordinates for the proof, and we feature one example of that approach. We then present the unique submission that exploited similar triangles and considerable persistence.

Solution 1, by Eugen J. Ionascu.

Let us employ an analytical proof. We let D be the origin of the coordinate axes, S be the point $(0, 1)$ on the y-axis, and assign A the coordinates $(-a, 0)$, with $a > 0$. Similarly, we assign the points P, T, and Q of the $x-$axis the respective coordinates $(p, 0), (t, 0)$, and $(q, 0)$. Since SD is an altitude of the right triangle AST, we have that $ta = 1$, so that T has coordinates $\left(\frac{1}{a}, 0\right)$, and K (which is $\frac{2}{3}$ of the distance from S to T) is the point $\left(\frac{2}{3a}, \frac{1}{3}\right)$. Since $ST = \frac{\sqrt{1+a^2}}{a}$, we find that the equation of the circle with center K and radius SK is given by

$$(x - \frac{2}{3a})^2 + (y - \frac{1}{3})^2 = \frac{4(1 + a^2)}{9a^2}.$$

(We see that $\tan \angle ATS = \frac{AS}{ST} = a$; although one of the stated assumptions implies that $a > 2$, we shall see that no such condition is required for the conclusion to hold. Specifically, we shall assume only that $a > 0$.) Setting $y = 0$ and solving for x gives the x-coordinates of P and Q:

$$p = \frac{2}{3a} - \sqrt{\frac{4}{9a^2} + \frac{1}{3}} \quad \text{and} \quad q = \frac{2}{3a} + \sqrt{\frac{4}{9a^2} + \frac{1}{3}}. \tag{1}$$

We observe that the product of these x-coordinates is

$$pq = -\frac{1}{3}. \tag{2}$$

By hypothesis, E is the point $(\frac{a}{2}, 0)$. Since BC is parallel to the y-axis, the distance from $S = (0, 1)$ to BC equals $DE = \frac{a}{2}$. Consequently, to prove that S is the incenter of ΔABC we must show that the distance to sides AB and AC is also $\frac{a}{2}$.

To determine the coordinates of $B = \left(\frac{a}{2}, b\right)$, we find where SQ, with equation $x + qy - q = 0$, meets the line $x = \frac{a}{2}$, namely $b = 1 - \frac{a}{2q}$. But from (2) we have $\frac{1}{q} = -3p$, so that from (1) we have finally that

$$b = 2 - \sqrt{1 + \frac{3a^2}{4}} < 1.$$

The equation of the line AB is $-bx + \frac{3a}{2}y - ab = 0$; consequently, the distance from S to AB equals

$$\frac{|\frac{3a}{2} - ab|}{\sqrt{b^2 + (\frac{3a}{2})^2}} = \frac{-\frac{a}{2} + a\sqrt{1 + \frac{3a^2}{4}}}{\sqrt{\left(2 - \sqrt{1 + \frac{3a^2}{4}}\right)^2 + (\frac{3a}{2})^2}} = \frac{a}{2} \cdot \frac{\sqrt{4 + 3a^2} - 1}{\sqrt{5 + 3a^2 - 2\sqrt{4 + 3a^2}}} = \frac{a}{2}.$$

Similarly, with C the point $\left(\frac{a}{2}, c\right)$ we have $c = 2 + \sqrt{1 + \frac{3a^2}{4}} > 1$, the line AC satisfies $-cx + \frac{3a}{2}y - ac = 0$, and the distance from S to AC equals $\frac{a}{2}$, as desired.

We have proved that S is a tritangent center. It remains to show that S is in the interior of ΔABC (and therefore not an excenter). Because BC is parallel to the y-axis, which separates the vertex A from the side BC, S must lie on an angle bisector between A and the opposite side. It remains to show that $S = (0, 1)$ lies between the points where $x = 0$ intersects the sides AB and AC. We see that

$x = 0$ meets AB where $y = \frac{2}{3}\left(2 - \sqrt{1 + \frac{3a^2}{4}}\right) < 1$,

and

$x = 0$ meets AC where $y = \frac{2}{3}\left(2 + \sqrt{1 + \frac{3a^2}{4}}\right) > 1$, which concludes the proof.

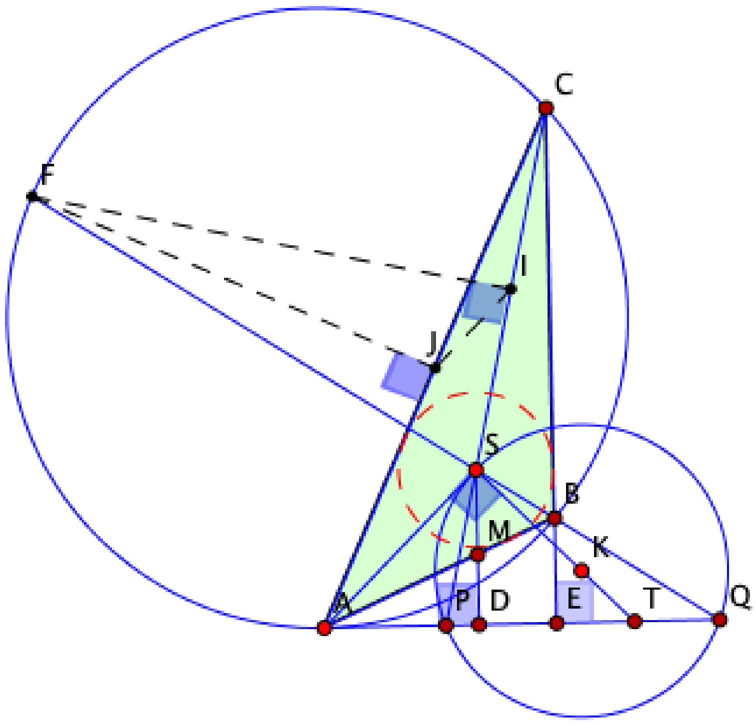
Solution 2, by Thinh Nguyen, supplemented and modified by the editor.

Editor's comment. Nguyen tacitly uses the same symbol XY to indicate the line determined by the points X and Y, and the directed distance from X to

Y. The context always make clear the intended meaning. In addition, the editor has introduced the symbol $\angle XYZ$ to indicate a directed angle, namely the angle required to rotate the line YX in the positive direction about Y to coincide with the line YZ.

Denote by M the point where SD intersects AB. From the similar right triangles ABE and AMD we have $\frac{DM}{EB} = \frac{AD}{AE} = \frac{2}{3}$, so that $DM = \frac{2}{3}EB$, and

$$SM = SD + DM = SD + \frac{2}{3}EB. \tag{3}$$



Similarly, using the similar right triangles SDQ and BEQ we have $\frac{EB}{SD} = \frac{EQ}{QD}$, so that

$$\frac{EB}{SD} + 1 = \frac{EQ}{QD} + \frac{QD}{QD} = \frac{ED}{QD} = \frac{DA}{2QD}$$

and, therefore,

$$EB = \left(\frac{DA}{2QD} - 1\right) SD.$$

We plug this expression for EB into (3) to get

$$SM = SD + \frac{2}{3}\left(\frac{DA}{2QD} - 1\right) SD = \frac{1}{3}SD\left(\frac{DA}{QD} + 1\right) = \frac{SD}{3} \cdot \frac{QA}{QD},$$

so that, finally,

$$\frac{SA}{SM} = \frac{SA}{\frac{SD \cdot QA}{3QD}} = \frac{3SA \cdot QD}{SD \cdot QA}. \tag{4}$$

The altitude SD partitions the right triangle AST into two triangles similar to it: $\Delta AST \sim \Delta ADS \sim \Delta SDT$. Consequently,

$$AS^2 = AD \cdot AT, \quad ST^2 = AT \cdot DT, \quad \text{and} \quad \frac{SA}{SD} = \frac{ST}{DT}.$$

But AS is tangent to the circle SPQ, so that we also have $AS^2 = AP \cdot AQ$ (the power of A with respect to this circle); it follows that $\frac{AD}{AQ} = \frac{AP}{AT}$, whence (using $AD = AQ + QD$ and $AP = AT + TP$)

$$\frac{AQ + QD}{AQ} = \frac{AT + TP}{AT},$$

or

$$\frac{QD}{AQ} = \frac{TP}{AT} = \frac{TP \cdot DT}{AT \cdot DT} = \frac{TP \cdot DT}{ST^2}.$$

Consequently, equation (4) becomes

$$\frac{SA}{SM} = \frac{3SA}{SD} \cdot \frac{TP \cdot DT}{ST^2} = \frac{3ST}{DT} \cdot \frac{TP \cdot DT}{ST^2} = \frac{3TP}{ST} = \frac{3TP}{3KT} = \frac{TP}{TK}.$$

Furthermore, because $\angle ASM = \angle KTP$ (their corresponding sides are perpendicular), the triangles ASM and PTK are oppositely similar (by side-angle-side), whence

$$\begin{aligned}
\angle CBA &= \angle SMA && \text{(corresponding sides parallel)}\\
&= \angle PKT && \text{(corresponding angles of similar triangles)}\\
&= \angle PKS &&\\
&= 2\angle PQS && \text{(angle inscribed in circle with center } K)\\
&= 2\angle EQB &&\\
&= 2(90^\circ - \angle QBE) && (\Delta EQB \text{ is a right triangle})\\
&= 2\angle EBQ = 2\angle CBS.
\end{aligned}$$

Thus,

$$\angle SBA = \angle CBA - \angle CBS = 2\angle CBS - \angle CBS = \angle CBS;$$

that is, BS is the bisector of $\angle CBA$.

It remains to show that CS is the bisector of $\angle ACB$. To that end, let F be the second point where BS meets the circle ABC, and let I and J be the midpoints of CS and CA. Note that IJ is parallel to AS. The chords FA and FC have equal lengths (because they subtend equal inscribed angles), which implies that $FJ \perp AC$. Much as before we have

$$\begin{aligned}\angle JFC = \frac{1}{2}\angle AFC = \frac{1}{2}\angle ABC = \frac{1}{2}\angle ABE \\ &= 90^\circ - \angle EBQ \\ &= \angle BQE = \angle SQP = \angle ASD = \angle JIP = \angle JIC.\end{aligned}$$

This equality implies that points F, C, I, J are concyclic, so $\angle FIC = \angle FJC = 90^\circ$. Moreover, since I is the midpoint of SC, we deduce that ΔFSC is isosceles; that is, $\angle CSF = \angle FCS$. It follows that

$$\angle ACS = \angle FCS - \angle FCA = \angle CSF - \angle FBA = (\angle SCB + \angle CBS) - \angle SBA = \angle SCB.$$

We conclude that CS bisects $\angle ACB$, which completes the proof that S is the incenter of ΔABC.

Editor's comments. The UCLan Cyprus Problem Solving Group proved the result in the more general setting with $\frac{SK}{KT} = \frac{AD}{DE} = k$, where k can be any real number greater than 1. (They explain that their restriction of k to $k > 1$ was a convenient way to insure the existence of the points P and Q, and to keep the tritangent center S inside ΔABC.) Both of our featured proofs can be modified to accommodate this generalization.

As a byproduct of his solution, Bataille observed that the incenter S of ΔABC lies on the line segment DG joining D to the centroid G, $\frac{3}{4}$ of the way from D to G.

4559. *Proposed by Nho Nguyen Van.*

Let x_k be positive real numbers. Prove that for every natural number $n \geq 2$, we have

$$\left(\sum_{k=1}^{n} x_k^{10}\right)^3 \geq \left(\sum_{k=1}^{n} x_k^{15}\right)^2$$

We received 26 submissions and they were all correct. There are many different ways to prove the inequality. We present the following 4 solutions, slightly modified by the editor.

Solution 1, by Michel Bataille, Oliver Geupel, Vivek Mehra, Ángel Plaza, Digby Smith, and the proposer (independently).

Noting that the proposed inequality is homogeneous, we may assume in addition that

$$\sum_{k=1}^{n} x_k^{10} = 1.$$

So the inequality reads

$$\sum_{k=1}^{n} x_k^{15} \leq 1,$$

which follows because for each k, $x_k^{15} \leq x_k^{10}$, since $x_k \in (0,1)$.

Solution 2, by Devis Alvarado and Vivek Mehra (independently).

For each $1 \leq k \leq n$, we set $y_k = x_k^5$. By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\left(\sum_{k=1}^{n} y_k^3\right)^2 = \left(\sum_{k=1}^{n} y_k y_k^2\right)^2 &\leq \left(\sum_{k=1}^{n} y_k^2\right)\left(\sum_{k=1}^{n} y_k^4\right) \\
&= \left(\sum_{k=1}^{n} y_k^2\right)\left(\left(\sum_{k=1}^{n} y_k^2\right)^2 - 2\sum_{k<j} y_k^2 y_j^2\right) \\
&\leq \left(\sum_{k=1}^{n} y_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right)^2 \\
&= \left(\sum_{k=1}^{n} y_k^2\right)^3 .
\end{aligned}$$

Substituting the x_k's, the result is obtained.

Solution 3, by Carl Libis and UCLan Cyprus Problem Solving Group (independently).

After expanding, we see that the right-hand side of the inequality is equal to

$$\sum_{k=1}^{n} x_k^{30} + 2 \sum_{1 \leqslant k < \ell \leqslant n} x_k^{15} x_\ell^{15} .$$

Since all terms are positive, the left-hand side of the inequality is at least

$$\sum_{k=1}^{n} x_k^{30} + 3 \sum_{1 \leqslant k < \ell \leqslant n} \left(x_k^{20} x_\ell^{10} + x_k^{10} x_\ell^{20}\right) .$$

where we have omitted terms of the form $x_k^{10} x_\ell^{10} x_m^{10}$. The inequality now follows immediately by AM-GM inequality, as

$$x_k^{20} x_\ell^{10} + x_k^{10} x_\ell^{20} \geqslant 2 x_k^{15} x_\ell^{15}$$

for each k, ℓ.

We conclude by noting that the inequality holds for $n = 1$. Furthermore, since the terms are positive, the case $n = 1$ is the only case in which we can have equality.

Solution 4, by Charles Diminnie, Marie-Nicole Gras, Richard Hess, Theo Koupelis, and Shrinivas Udpikar (independently).

For all $k = 1, \ldots, n$, we put $a_k = x_k^5$; then the given inequality is equivalent to

$$\left(\sum_{k=1}^{n} a_k^2\right)^3 \geq \left(\sum_{k=1}^{n} a_k^3\right)^2. \tag{1}$$

We prove (1) by induction. Obviously, it is true for $n = 1$. Suppose that (1) holds for $n \geq 1$; then by inductive hypothesis, we have

$$\begin{aligned}
\left(a_{n+1}^2 + \sum_{k=1}^{n} a_k^2\right)^3 &= a_{n+1}^6 + 3a_{n+1}^4\left(\sum_{k=1}^{n} a_k^2\right) + 3a_{n+1}^2\left(\sum_{k=1}^{n} a_k^2\right)^2 + \left(\sum_{k=1}^{n} a_k^2\right)^3 \\
&\geq a_{n+1}^6 + a_{n+1}^4\left(\sum_{k=1}^{n} a_k^2\right) + a_{n+1}^2\left(\sum_{k=1}^{n} a_k^4\right) + \left(\sum_{k=1}^{n} a_k^3\right)^2 \\
&= a_{n+1}^6 + a_{n+1}^2\left(\sum_{k=1}^{n} a_k^2(a_{n+1}^2 + a_k^2)\right) + \left(\sum_{k=1}^{n} a_k^3\right)^2 \\
&\geq a_{n+1}^6 + a_{n+1}^2\left(\sum_{k=1}^{n} a_k^2(2a_{n+1}a_k)\right) + \left(\sum_{k=1}^{n} a_k^3\right)^2 \\
&= a_{n+1}^6 + 2a_{n+1}^3\left(\sum_{k=1}^{n} a_k^3\right) + \left(\sum_{k=1}^{n} a_k^3\right)^2 \\
&= \left(a_{n+1}^3 + \sum_{k=1}^{n} a_k^3\right)^2.
\end{aligned}$$

Thus we have proved (1) and the required result follows.

Editor's Comment. As pointed out by Walther Janous, Oliver Geupel, and Missouri State University Problem Solving Group, the proposed inequality is a special case of the following classical inequality (see for example page 4 of G. H. Hardy, J.E. Littlewood; G. Pólya, *Inequalities*, 2nd ed, Cambridge University Press, 1952): for any $0 < r < s$, and any positive numbers $x_1, x_2, \ldots, x_n$, we have

$$\left(\sum_{k=1}^{n} x_k^s\right)^{1/s} \leq \left(\sum_{k=1}^{n} x_k^r\right)^{1/r}. \tag{2}$$

The idea in the solution 1 presented above can be used to prove the above inequality.

As pointed out by Zoltan Retkes, the proposed inequality can be also treated in the p-norm setting. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, for each $p \geq 1$,

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}$$

defines the p-norm on $\mathbb{R}^n$. In general, we have the following relations between p-norms: if $1 \leq q \leq p$, we have

$$\|\mathbf{x}\|_p \leq \|\mathbf{x}\|_q \leq n^{\frac{1}{q}-\frac{1}{p}}\ \|\mathbf{x}\|_p. \tag{3}$$

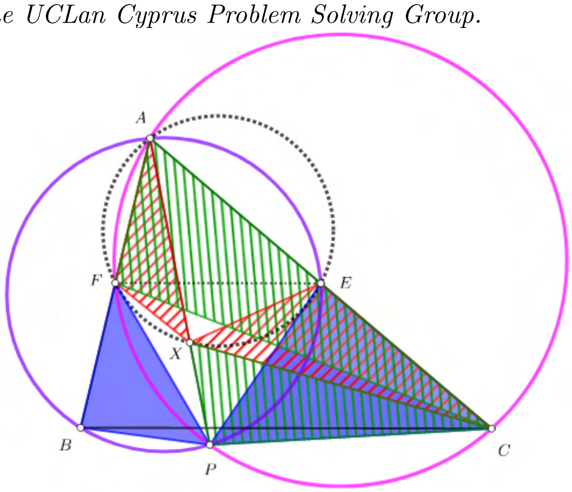
Note that the first inequality in (3) is just inequality (2), and the second inequality in (3) is a consequence of Hölder's inequality.

4560. *Proposed by Mihaela Berindeanu.*

Let E and F be midpoints on the respective sides CA and AB of triangle ABC, and let P be the second point of intersection of the circles ABE and ACF. Prove that the circle AEF intersects the line AP again in the point X for which $AX = 2XP$.

We received 15 submissions, all of which were correct, and will sample three of the various approaches.

Solution 1 by the UCLan Cyprus Problem Solving Group.



Since A, E, P, B are concyclic (in that order), then $\angle PBF = \angle PBA = \angle PEC$. Since A, C, P, F are concyclic, then $\angle ECP = \angle ACP = \angle BFP$. It follows that the triangles $\triangle PBF$ and $\triangle PEC$ are similar. Letting h_B and h_C be the altitudes from P for this pair of similar triangles, we have

$$\frac{\sin(\angle BAP)}{\sin(\angle CAP)} = \frac{h_B/AP}{h_C/AP} = \frac{h_B}{h_C} = \frac{FB}{EC} = \frac{AB}{AC}\,.$$

Since A, E, X, F are concyclic, say on a circle of radius R, then

$$\frac{FX}{EX} = \frac{2R\sin(\angle BAP)}{2R\sin(\angle CAP)} = \frac{AB}{AC}\,.$$

Again because A, E, X, F are concyclic, we have $\angle XFA = \angle XEC$. Since also $FA/EC = AB/AC = FX/EX$, it follows that the triangles $\triangle XFA$ and $\triangle XEC$ are similar. Therefore $\angle CXE = \angle AXF = \angle AEF = \angle C$ (of ΔABC). Since also $\angle AXE = \angle AFE = \angle B$, it follows that $\angle PXC = \angle A$.

Since A, C, P, F are concyclic, we then have $\angle CPX = \angle CPA = \angle CFA$. Since also $\angle PXC = \angle A$ it follows that the triangles $\triangle PXC$ and $\triangle FAC$ are similar. Using also the similarity of the triangles $\triangle XFA$ and $\triangle XEC$ we get

$$XP = \frac{(AF)(XC)}{AC} = \frac{(AF)(AX)(EC)}{(AC)(AF)} = \frac{AX}{2},$$

as required.

Solution 2 by Theo Koupelis.

Let O, $\bar{E}$, $\bar{F}$, and $\bar{O}$ be the circumcenters of the circles ABC, ABE, ACF, and AEF, respectively. The chord AE is common to the circles AEF and ABE, and thus points $\bar{O}$ and $\bar{E}$ are on the perpendicular bisector of AE. The chord AC is common to the circles ABC and ACF, and thus points O and $\bar{F}$ are on the perpendicular bisector of AC. Therefore $\bar{O}\bar{E}$ is parallel to $\bar{F}O$. Similarly, points $\bar{O}$ and $\bar{F}$ are on the perpendicular bisector of AF, and points O and $\bar{E}$ are on the perpendicular bisector of AB; therefore $\bar{F}\bar{O}$ is parallel to $O\bar{E}$. As a result, $\bar{O}\bar{E}O\bar{F}$ is a parallelogram.

Because E and F are the midpoints of the sides AC and AB, respectively, the triangles AFE and ABC are similar, with a ratio of similitude of 2. Therefore the points A, $\bar{O}$, and O are colinear with $A\bar{O} = \bar{O}O$. Thus AO is the diameter of the circle $(\bar{O}) = (AEF)$, and with X being on this circle we have $OX \perp AX$.

Let Q be the intersection point of the diagonals $\bar{O}O$ and $\bar{E}\bar{F}$ of the parallelogram $\bar{O}\bar{E}O\bar{F}$. The chord AP is the common chord of the circles $(\bar{E})$ and $(\bar{F})$, and thus $\bar{E}\bar{F} \perp AP$ and therefore $Q\bar{E} \perp AP$. Let $\bar{O}_1$ and Q_1 be the projections of the points $\bar{O}$ and Q, respectively, on AP. Then $AQ_1 = Q_1P$, $A\bar{O}_1 = \bar{O}_1X$, and $\bar{O}_1Q_1 = Q_1X$. Thus, $A\bar{O}_1 = AQ_1 - \bar{O}_1Q_1 = Q_1P - Q_1X = XP$. Therefore $AX = 2A\bar{O}_1 = 2XP$.

Solution 3 by Sorin Rubinescu.

We shall prove that $AX = \frac{2}{3}AP$ (which, because X is between A and P, is equivalent to $AX = 2XP$). Consider the inversion in the unit circle with center A. Because A is on the three given circles, these transform into lines. Specifically, the circumcircle of $\triangle AFC$ is interchanged with the line $F'C'$, while the circumcircle of $\triangle AEB$ is interchanged with the line $E'B'$; therefore, point P is sent to $P' = F'C' \cap E'B'$, while X is sent to $X' = AP' \cap E'F'$.

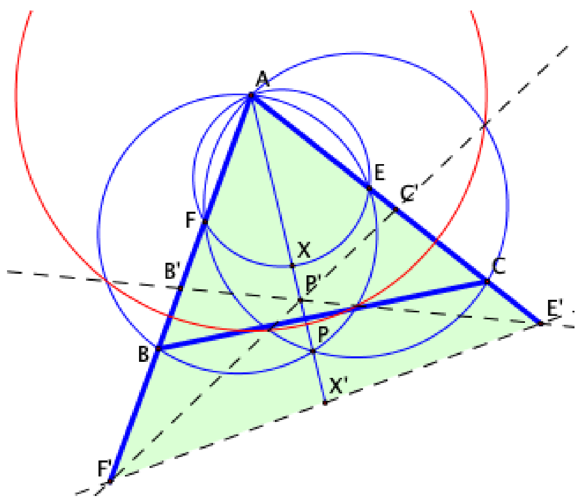
The relations $AE \cdot AE' = 1 = AC \cdot AC'$ imply that

$$\frac{AE'}{AC'} = \frac{AC}{AE} = 2.$$

Similarly,

$$\frac{AF'}{AB'} = \frac{AB}{AF} = 2.$$

As a result, P' must be the centroid of triangle $AF'E'$ (with its medians $F'C'$, $E'B'$ and, therefore, AX'). But P' is $\frac{2}{3}$ of the way along the median from the vertex A to the midpoint X' of the opposite side; because $AP \cdot AP' = AX \cdot AX'$, we see that also $\frac{AX}{AP} = \frac{2}{3}$, as desired.



Editor's comments. As part of his solution, Ivko Dimitrić observed that AP is the symmedian from A of ΔABC. This follows easily from the third of our featured solutions: The reflection of ΔABC in the line that bisects $\angle A$ (which takes that triangle to a triangle homothetic to the triangle $AE'F'$ of Solution 3) takes the median from A into the corresponding symmedian.