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IN THIS ISSUE / DANS CE NUMÉRO

Crux Mathematicorum

Crux Mathematicorum with Mathematical Mayhem

EDITORIAL

New Year, New Volume! As my palindromic-loving friends would like to say, Happy $2 \times 505 \times 2$!

My New Year's resolution is to read more. I tend to flip through many books (mostly math or math education related), but I find that I don't fully read many. This year, I'm hoping to change that and, in the absence of Book Reviews, I will update you in my readings.

I am starting with something old: "On Being the Right Size And Other Essays" by J. B. S. Haldane (1892–1964). An evolutionary biologist, Haldane was a passionate science popularizer and he writes with ease and charisma. The collection was recommended to me by a colleague specifically for the first (and title) essay after I lamented that my biology students use the surface area/volume "law" as if it is something that holds independent of the shape you're considering or proportionality constants involved. "On Being the Right Size" addresses exactly that question of proportion through a variety of examples: scaling of bones, danger of falling or getting wet, how limits to gas diffusion limit insect size, why big animals don't have giant eyes and so on. "Comparative anatomy is largely the story of the struggle to increase surface in proportion to volume", hence large animals' fractal lungs and twisted guts. It is an interesting read, persuasive in its arguments (albeit not always infallible) and sprinkled with quantitative reasoning. I'm looking forward to reading the rest of the book.

Kseniya Garaschuk

The book can be found, for example, here
`https://archive.org/details/OnBeingTheRightSize-J.B.S.Haldane/`

MathemAttic

No. 11

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2020**.

MA51. *Proposed by Nguyen Viet Hung.*

Find all non-negative integers x, y, z satisfying the equation

$$2^x + 3^y = 4^z.$$

MA52. The diagram shows part of a tessellation of the plane by a quadrilateral. Khelen wants to colour each quadrilateral in the pattern.

1. What is the smallest number of colours he needs if no two quadrilaterals that meet (even at a point) can have the same colour?
2. Suppose that quadrilaterals that meet along an edge must be coloured differently, but quadrilaterals that meet just at a point may have the same colour. What is the smallest number of colours that Khelen would need in this case?
3. What is the smallest number of colours needed to colour the edges so that edges that meet at a vertex are coloured differently?

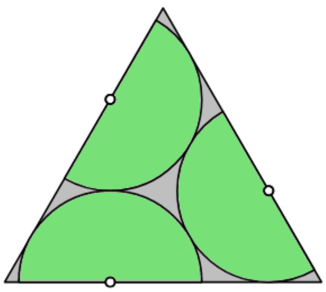
MA53.

Find all positive integers m and n which satisfy the equation

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{m^3-1}{m^3+1}=\frac{n^3-1}{n^3+2}.$$

MA54. How many six-digit numbers are there, with leading 0s allowed, such that the sum of the first three digits is equal to the sum of the last three digits, and the sum of the digits in even positions is equal to the sum of the digits in odd positions?

MA55. The diagram shows three touching semicircles with radius 1 inside an equilateral triangle, which each semicircle also touches. The diameter of each semicircle lies along a side of the triangle. What is the length of each side of the equilateral triangle?



. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA51. *Proposé par Nguyen Viet Hung.*

Déterminer tous les entiers non négatifs x, y, z satisfaisant à l'équation

$$2^x + 3^y = 4^z.$$

MA52. Le diagramme montre une partie d'un pavage du plan par un quadrilatère. Katherine désire en effectuer un colorage.

1. Déterminer le plus petit nombre de couleurs possible si Katherine exige que deux quadrilatères se touchant, même en un seul point, aient besoin d'être colorés différemment.

2. Supposons maintenant que deux quadrilatères partageant un côté doivent être colorés différemment, mais pas nécessairement ceux se touchant en un point seulement. Déterminer le plus petit nombre de couleurs requises pour colorer les quadrilatères dans ce contexte.

3. Enfin, Katherine désire colorer les côtés seulement, mais de faon à ce que les côtés se rencontrant en un point soient colorés différemment. Déterminer le plus petit nombre de couleurs requises dans ce contexte.

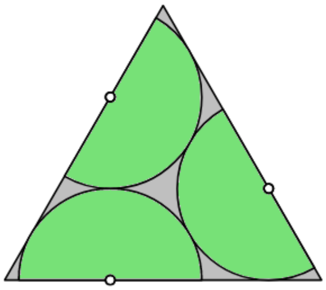
MA53.

Déterminer tous les entiers positifs m et n satisfaisant à l'équation

$$\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{m^3-1}{m^3+1}=\frac{n^3-1}{n^3+2}.$$

MA54. Combien de nombres à six chiffres y a-t-il, tels que la somme des trois premiers chiffres est égale à la somme des trois derniers chiffres et puis que la somme des chiffres en positions paires égale la somme des chiffres en positions impaires? La présence de 0s en première(s) position(s) est permise.

MA55. Le diagramme montre trois demi cercles de rayon 1 à l'intérieur d'un triangle équilatéral, les diamètres étant situés sur les côtés du triangle. Chaque demi cercle touche les deux autres et le triangle. Déterminer la longueur du côté du triangle.



MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(6), p. 303–305.

MA26. Nine (not necessarily distinct) 9-digit numbers are formed using each digit 1 through 9 exactly once. What is the maximum possible number of zeros that the sum of these nine numbers can end with?

Originally Problem M2430 of Kvant.

We received 3 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

The answer is eight. Since

$$8 \times 987654321 + 198765432 = 8100000000,$$

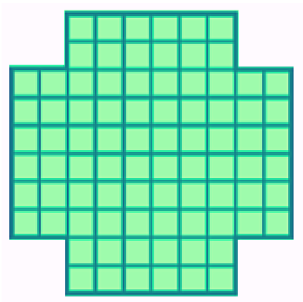
the answer is at least 8. But the maximum value the sum can be is

$$9 \times 987654321 = 8888888889,$$

so the only other possibility is to have nine zeros. Now each number whose digits are a permutation of $1, \ldots, 9$ is a multiple of 9, since the sum of their digits is. Therefore any sum of these numbers must also be a multiple of 9. But the only 10-digit number ending in nine zeros that is a multiple of 9 is 9000000000 and this is larger than our upper bound.

We note that analogous methods extend this result to base b: if $b-1$ numbers consisting of permutations of $1, \ldots, b-1$ are added, the maximum possible number of zeros that their sum can end in is $b-2$.

MA27. You want to play Battleship on a 10×10 grid with 2×2 squares removed from each of its corners:

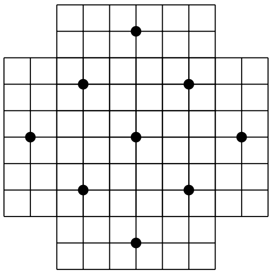


What is the maximum number of submarines (ships that occupy 3 consecutive squares arranged either horizontally or vertically) that you can position on your board if no two submarines are allowed to share any common side or corner?

Originally Problem 24 of 2018 Savin contest.

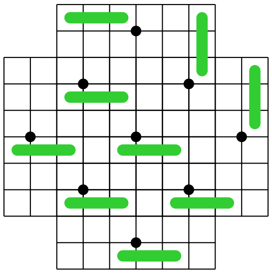
We received 1 submission, which was correct but incomplete. We present the solution by Richard Hess and Taus Brock-Nannestad, and completed by the editor.

Consider the following diagram:



There is no way to place a submarine on the grid without its touching one of the nine marked grid points. No two submarines can touch the same marked grid point so nine submarines is the most that can be placed on the grid without touching.

It is possible to place nine submarines on the grid. There are many ways to do this; here is one:



This is an example of a problem where a construction is a necessary part of the proof. Without actually demonstrating that it is possible to place nine submarines, we know only that we cannot place more than this many.

MA28. Prove that for all positive integers n, the number

$$\frac{1}{3}\left(4^{4n+1}+4^{4n+3}+1\right)$$

is not prime.

Originally Problem 27 of 2017 Savin contest.

We received 4 submissions which were correct and complete. We present the solution by the Missouri State University Problem Solving Group.

The statement is false. If $n = 6$, we have

$$\frac{1}{3}(4^{4n+1} + 4^{4n+3} + 1) = 6\,380\,099\,472\,108\,203,$$

which is prime. (*Mathematica* claims that $n = 861$ and $n = 5304$ also yield prime values).

However, it is true that if $n \not\equiv 0 \bmod 3$, then $(4^{4n+1} + 4^{4n+3} + 1)/3$ is never prime.

If $n \equiv 1 \bmod 3$, then $n = 3k + 1$, $k \in \mathbb{Z}$ and

$$\begin{aligned} 4^{4n+1} + 4^{4n+3} + 1 &= 4^{12k+5} + 4^{12k+7} + 1 \\ &= 16 \cdot 64^{4k+1} + 4 \cdot 64^{4k+2} + 1 \\ &\equiv 2 \cdot 1 + 4 \cdot 1 + 1 \bmod 7 \\ &\equiv 0 \bmod 7 \end{aligned}$$

and

$$4^{4n+1} + 4^{4n+3} + 1 \geq 4 + 4^3 + 1 = 69 > 7,$$

so 7 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)/3$.

If $n \equiv 2 \bmod 3$, then $n = 3k + 2$, $k \in \mathbb{Z}$ and

$$\begin{aligned} 4^{4n+1} + 4^{4n+3} + 1 &= 4^{12k+9} + 4^{12k+11} + 1 \\ &= 64^{4k+3} + 16 \cdot 64^{4k+3} + 1 \\ &\equiv 1 + 7 \cdot 1 + 1 \bmod 9 \\ &\equiv 0 \bmod 9 \end{aligned}$$

and

$$4^{4n+1} + 4^{4n+3} + 1 \geq 69 > 9,$$

so 9 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)$ and hence 3 is a non-trivial factor of $(4^{4n+1} + 4^{4n+3} + 1)/3$.

MA29. Find all positive integers n satisfying the following condition: numbers $1, 2, 3, \ldots, 2n$ can be split into pairs so that if numbers in each pair are added and all the sums are multiplied together, the result is a perfect square.

Originally Problem 2 of Fall Junior A-level of XL Tournament of Towns 2017.

We received 3 submissions, all of which were correct and complete. We present the solution by the Missouri State University Problem Solving Group, modified by the editor.

We claim that n satisfies the condition if $n > 1$.

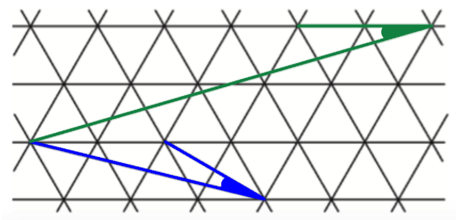
We first observe that $n = 1$ fails the condition. For $n = 1$ the only pairing is $\{1, 2\}$, the sum of which is the non-perfect square 3.

There are two cases:

1. $n = 2k$ where $k \geq 1$. By pairing i with $2n + 1 - i$ for $i = 1, 2, \ldots, n$ gives a product of $\left((2n + 1)^k\right)^2$.
2. $n = 2k + 1$ where $k \geq 1$. When $k \geq 1$, we pair 1 and 5, 2 and 4, 3 and 6, and $6 + i$ with $2n + 1 - i$ for $i = 1, 2, \ldots, n - 3 = 2k - 2$. The product is then

$$(1 + 5)(2 + 4)(3 + 6)(2n + 7)^{2k-2} = \left(18(2n + 7)^{k-1}\right)^2.$$

MA30. Consider the two marked angles on a grid of equilateral triangles.



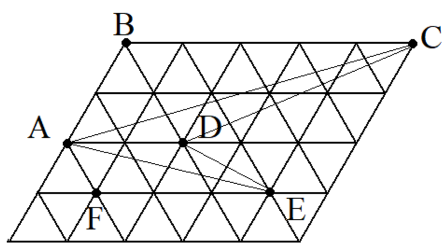
Prove that these angles are equal.

Originally Problem 18 of 2017 Savin contest.

We received 6 solutions, all of which were correct. We present the solution of Missouri State University Problem Solving Group, modified by the editor.

Let the side lengths of the equilateral triangles be 1.

Method I. Consider the figure below.



Let $\alpha = m(\angle ACB)$ and $\beta = m(\angle AED)$. Since $AB = 2$ and $BC = 5$, the Law of Cosines gives

$$AC = \sqrt{2^2 + 5^2 + 2 \cdot 5} = \sqrt{39}.$$

Applying the Law of Cosines again

$$\cos\alpha = \frac{\sqrt{39}^2 + 5^2 - 2^2}{2 \cdot 5\sqrt{39}} = \sqrt{\frac{12}{13}}.$$

Similarly, $DE = \sqrt{3}$ and applying the Law of Cosines to $\triangle AFE$ we have

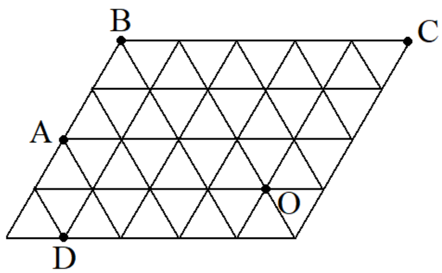
$$AE = \sqrt{1^2 + 3^2 + 1 \cdot 3} = \sqrt{13}.$$

One more use of the Law of Cosines gives

$$\cos\beta = \frac{\sqrt{13}^2 + \sqrt{3}^2 - 2^2}{2\sqrt{3}\sqrt{39}} = \sqrt{\frac{12}{13}},$$

so the angles in question are congruent.

Method II. Consider the figure below.



Triangle ABD in this figure is congruent to triangle DAE in the figure in Method I. Thus, we wish to show that $\angle ACB \cong \angle ADB$. The point marked O is equidistant from each of A, B, C, D (it lies on the intersection of the perpendicular bisectors of $\overline{AD}$, $\overline{AB}$, and $\overline{BC}$). Therefore, these points lie on a circle centered at O. Since $\angle ACB$ and $\angle ADB$ are subtended by the same arc, they must be congruent.

TEACHING PROBLEMS

No. 8

Richard Hoshino

The Calendar Problem

In the Calendar Problem, your goal is to figure out the day of the week on which you were born.

There are various YouTube videos of mathematicians (or "mathemagicians") performing this trick in their heads. For example, an audience member will call out her birthday (e.g. May 25, 2004), and the mathematician will instantly reply, "Tuesday".

In this article, we will unpack this problem and determine an algorithm to solve this problem.

First, let's investigate the day of the week that our birthday falls on in the year 2020. To do this, all we need is the knowledge that January 1, 2020 is Wednesday. Whenever I have presented this problem in a class, either with high school students or undergraduates, one student always knows the number of days in each month:

January	31
February	29 (since 2020 is a leap year)
March	31
April	30
May	31
June	30
July	31
August	31
September	30
October	31
November	30
December	31

Notice that January 1 must be the same day of the week as January 8, January 15, January 22, and January 29. This is because each of these numbers in {1, 8, 15, 22, 29} gives the same remainder when divided by 7.

Thus, for some birthdays, the Calendar Problem can be easily solved. Let's consider someone born on January 17. For the year 2020, since January 1 is a Wednesday we know that January 15 is a Wednesday, which implies that January 16 is a Thursday, from which it follows that January 17 is a Friday.

For birthdays in the month of January, notice that the answer can be found by simply taking the date, dividing by 7, and calculating the remainder. Then we can use this table to read off the answer:

0	Tuesday
1	Wednesday
2	Thursday
3	Friday
4	Saturday
5	Sunday
6	Monday

Here are two common approaches for solving the Calendar Problem.

Approach One: Count the number of days that have elapsed from the start of the year (January 0) until the target date. For example, March 23 consists of $31 + 29 + 23$ days, since we need to add up all the days in January and February and then the twenty-three days in March. This adds up to 83. We divide by 7. Since $83 = 7 \times 11 + 6$, the remainder is 6. From the above table, we see that a remainder of 6 corresponds to Monday.

Approach Two: Determine the day of the week for the first date of each month, showing that if January 1 falls on Wednesday, then February 1 must be a Saturday, March 1 must be a Sunday, and so on. From this, students can solve their problem for any given date by adding or subtracting increments of seven. For example, March 23 has to be the same date as March 16, March 9, and March 2. Thus, March 23 has to be a Monday, since March 1 is a Sunday.

A clever approach combines these two paradigms, using the first date of each month to determine the appropriate "shift". For example, March 1 is $31 + 29 = 60 = 8\times 7+4$ days after January 1, and so March 1 is "shifted" by 4 days compared to January 1. Thus, if we know that the shift number of March is 4, then we can determine the day of week of March 23 by adding the date to the shift number ($4 + 23 = 27 = 3 \times 7 + 6$), dividing the number by 7 and taking the remainder (which is 6), and then reading the above table to conclude that the answer is Monday.

For leap years such as the year 2020, the shift dates of each month are as follows:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	4	0	2	5	0	3	6	1	4	6

Notice this table forms four sets of three digits that can be remembered this way:

$$034 = 5^2 + 3^2, \quad 025 = 5^2, \quad 036 = 6^2, \quad 146 = 12^2 + 2^2.$$

For example, the shift number for June is 5, since the number of days until the start of June is $31 + 29 + 31 + 30 + 31 = 152 = 21 \times 7 + 5$, which has a remainder of 5 upon division by 7. In other words, June 1 is exactly 21 weeks and 5 days after January 1 which implies that the shift for June is 5.

Let D be the date and S be the shift number. For example, June 15 would have $D = 15$ and $S = 5$. To perform this algorithm in our head, we just need to add $D+S$, divide by 7, and the remainder gives us our answer to the Calendar Problem. Since this remainder is 6, we can conclude that June 15, 2020 will be a Monday.

Now let's extend this by replacing the year 2020 with our birth year. In solving this harder problem, we realize that each 365-day year contributes one extra day (52 weeks plus 1 day) and each 366-day leap year contributes two extra days (52 weeks plus 2 days). Thus, if January 1, 2020 is a Wednesday, then January 1, 2019 was a Tuesday, since we have shifted back one day. And similarly, January 1, 2021 will be a Friday since we will need to shift forward two days.

In one of my school visits (in 2019), one student made the powerful insight that her birthday in 2002 must be the same day of week as her birthday in 2019, since there are 17 "extra days" in addition to the four Feb 29 "leap days" that occurred in 2004, 2008, 2012, and 2016. Since $17+4=21$, the calendar shifted 21 days between her birthday in 2002 and her birthday in 2019. And since 21 is a multiple of 7, if her birthday fell on a Tuesday in 2019, then it must have fallen on a Tuesday in 2002. This student provided a clear method for how to handle the tricky concept of leap years.

A different student from the same class observed that the calendar repeats itself every 28 years, since each year contributes one extra day (52 weeks plus 1 day), and there are 7 occurrences of February 29 during any 28-year period. Thus, the calendar shifts by $28+7=35$ days, which is a multiple of 7. This observation enabled the student to determine the day of the week on which her parents were born.

Through this process of solving the Calendar Problem and determining an algorithm that works for any birthday, students demonstrate the four principles of the *Computational Thinking* process.

(i) Decomposition: break down the problem into smaller tasks

(ii) Pattern recognition: identify similarities, differences, and patterns within the problem

(iii) Abstraction: identify general principles and filter out unnecessary information

(iv) Algorithmic design: identify and organize the steps needed to solve the problem

As mathematicians we use these four principles in our research endeavours, and the Calendar Problem offers a challenge for enabling our students to have similar experiences.

During the 2018-2019 sabbatical year, I worked with the Callysto Project, a federally-funded initiative to bring computational thinking and mathematical problem solving into Grade 5-12 Canadian classrooms (`www.callysto.ca`). Through my work with Callysto, I visited over a dozen schools and worked with 700+ students, sharing rich math problems that incorporated the Callysto technology (a web-based platform known as a Jupyter Notebook, freely accessible to anyone with an Internet connection). I created a Notebook for the Calendar Problem, to be used by teachers and students. This free resource, which also includes a lesson plan for teachers, can be found at `www.bit.ly/CallystoCalendar`.

We end with three questions for consideration.

Communications, including solutions, concerning these questions are welcomed via email at `richard.hoshino@questu.ca`.

Question #1

Here is an algorithm that determines the correct day of week for any date in the 20th century (Jan 1, 1901 to Dec 31, 2000).

Let Y be the last two digits of the year, D be the day, and S be the "shift" value according to the following table that is correct for *non-leap years*:

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
0	3	3	6	1	4	6	2	5	0	3	5

For example, the author's birthday (June 15, 1978) has $Y = 78$, $D = 15$, and $S = 4$.

Now calculate the sum $T = Y + \lfloor Y/4 \rfloor + D + S$.

If the year corresponds to a leap year (i.e., Y is a multiple of 4) and the month is January or February, subtract 1 from T. (Why do we need to do this?)

Divide T by 7 and determine its remainder. The remainder tells us our answer:

0	Sunday
1	Monday
2	Tuesday
3	Wednesday
4	Thursday
5	Friday
6	Saturday

For example, October 29, 1929 has $T = 29 + 7 + 29 + 0 = 65$, which gives a remainder of 2 when divided by 7. Therefore, this date in history (known as Black Tuesday) was indeed a Tuesday.

Here is the question: why does this algorithm work?

Question #2

What day of the week would it be on your 100th birthday?

Question #3

Create your own algorithm for other famous dates before the 20th century, and apply it to the famous dates such as the following:

(i) July 1, 1867 (Confederation Day in Canada)

(ii) July 4, 1776 (Independence Day in the USA)

(iii) April 23, 1616 (Death of William Shakespeare)

(iv) September 30, 1207 (Birthday of Rumi)

Note that you will need to be careful about ensuring the correct calculation of leap years, due to the quirky rules that occur when the year is a multiple of 100 but not a multiple of 400. Specifically, the years 1600 and 2000 are leap years, while the years 1700, 1800, 1900 are not leap years.

Richard Hoshino teaches at Quest University Canada in Squamish, BC. He can be reached via email at `richard.hoshino@questu.ca`.

Note: Submissions for consideration in Teaching Problems are welcomed. Please feel free to send along a contribution concerning a valuable teaching example from your experience. It is also appreciated if you can include some related problems for consideration as has been done here. Our readers welcome opportunities to solve problems.

OLYMPIAD CORNER

No. 379

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **March 15, 2020**.

OC461. Let A and B be two finite sets. Determine the number of functions $f : A \to A$ with the property that there exist two functions $g : A \to B$ and $h : B \to A$ such that $g(h(x)) = x \; \forall x \in B$ and $h(g(x)) = f(x) \; \forall x \in A$.

OC462. The integers $a_1, a_2, \ldots, a_n$ satisfy

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

If m is the number of distinct prime factors of $a_1 a_2 \cdot \ldots \cdot a_n$, then prove that

$$(a_1 a_2 \cdot \ldots \cdot a_n)^{m-1} \geq (n!)^m.$$

OC463. A 6×6 table is filled with the integers from 1 to 36.

(a) Give an example of such a fill of the table so that the sum of every two numbers in the same row or column is greater than 11.

(b) Prove that in some row or column, no matter how you fill the table, you will always find two numbers whose sum does not exceed 12.

OC464. Given an acute triangle ABC with orthocenter H. The angle bisector of $\angle BHC$ intersects side BC at D. Let E and F be the symmetric points of D with respect to lines AB and AC, respectively. Prove that the circumcircle of triangle AEF passes through the midpoint G of arc BAC.

OC465. The sequence (a_n) is defined by

$$a_1 = 1, \qquad a_n = \lfloor \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1} \rfloor \qquad \text{if } n > 1.$$

Find a_{2017}.

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC461. Soient A et B deux ensembles finis. Déterminer le nombre de fonctions $f : A \to A$ telles qu'il existe deux fonctions $g : A \to B$ et $h : B \to A$ pour lesquelles $g(h(x)) = x \ \forall x \in B$ et $h(g(x)) = f(x) \ \forall x \in A$.

OC462. Les entiers $a_1, a_2, \ldots, a_n$ an satisfont à

$$1 < a_1 < a_2 < \ldots < a_n < 2a_1.$$

Si m est le nombre de facteurs premiers distincts de $a_1 a_2 \cdot \ldots \cdot a_n$, démontrer que

$$(a_1 a_2 \cdot \ldots \cdot a_n)^{m-1} \geq (n!)^m.$$

OC463. Les cases d'une matrice de taille 6×6 sont remplies par les entiers de 1 à 36.

(a) Déterminer une manière de remplir les cases de la matrice de façon à ce que la somme de deux quelconques nombres d'une même rangée ou colonne donne toujours supérieur à 11.

(b) Quelle que soit la façon de remplir les cases de la matrice, démontrer qu'au moins une somme de deux éléments de même rangée ou colonne sera inférieure ou égale à 12.

OC464. Soit ABC un triangle acutangle avec orthocentre H. La bissectrice de $\angle BCH$ intersecte le côté BC en D. Soient E et F less points symétriques à D par rapport aux lignes AB et AC, respectivement. Démontrer que le cercle circonscrit du triangle AEF passe par le mi point de l'arc BAC.

OC465. La suite (a_n) est définie par

$$a_1 = 1, \qquad a_n = \lfloor \sqrt{2a_{n-1} + a_{n-2} + \cdots + a_1} \rfloor \qquad \text{if } n > 1.$$

Déterminer a_{2017}.

OLYMPIAD CORNER SOLUTIONS

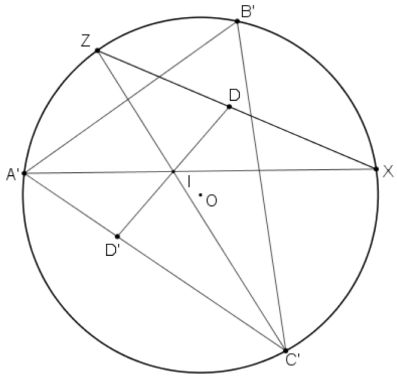
Statements of the problems in this section originally appear in 2019: 45(6), p. 320–321.

OC436. In a non-isosceles triangle ABC, let O and I be its circumcenter and incenter, respectively. Point B', which is symmetric to point B with respect to line OI, lies inside $\angle ABI$. Prove that the tangents to the circumcircle of the triangle $BB'I$ at points B' and I intersect on the line AC.

Originally Russia MO, 8th Problem, Grade 10, Final Round 2017 (Geometry).

We received 3 correct submissions. We present all three solutions.

Solution 1, by Lee Jang Yong



Let ω be the circumcircle of $\triangle ABC$. Let A', B', and C' be the points that are symmetric with respect to line OI to A, B, and C, respectively. Because the symmetry line, OI, passes through the centre of ω we have that the symmetric images A', B', and C' belong to ω, as well.

Let D be the intersection of AC with the tangent at I to the circumcircle of $\triangle BB'I$. We show that $DI = DB'$.

Let D' be the point symmetric to D with respect to line OI. The centre of the circumcircle of triangle $BB'I$ lies on OI, given that $\triangle BB'I$ is isosceles. Therefore the tangent ID is perpendicular to OI, and I, D, and D' are co-linear with I being the middle point of DD'. Moreover, if we extend DD' to intersect ω at M and M' we find that I is the middle of the new segment, MM'.

We are in the setting of the butterfly theorem. Let X be the intersection of $A'I$ with ω, and Z be the intersection of $C'I$ with ω. D' is a point on $A'C'$, D, I, and D' are co-linear, and I is the middle point of DD'. Due to the butterfly theorem we conclude that D belongs to XZ.

However, XZ is the perpendicular bisector of $B'I$, so $DI = DB'$. This is because I is the centre of the incircle and the intersection of the bisectors of $\triangle A'B'C'$. These properties imply the equality of the following $\angle B'IX = \angle IB'X = (\angle C'A'B' + \angle C'B'A')/2$ and that $\triangle B'IX$ is isosceles. Similarly, $\triangle B'IZ$ is isosceles.

Since $DI = DB'$ it follows that DB' is tangent to the circumcircle of $\triangle BB'I$ at B', and that the tangents to the circumcircle of $\triangle BB'I$ at points B' and I intersect on the line AC at the point D.

Solution 2, by Ivko Dimitrić.

Consider, without loss of generality, triangle ABC in the plane of complex numbers, whose circumcircle is the unit circle centered at the origin O. For any point included in our proof we associate a unique capital letter and a complex number. The capital letter is used to refer to the complex number, as well. Let $A = e^{i\alpha}$, $B = e^{i\beta}$, and $C = e^{i\gamma}$ be the complex numbers that identify the triangle vertices, with $0 < \alpha < \beta < \gamma < 2\pi$. Moreover, denote by $a = e^{i\alpha/2}$, $b = -e^{i\beta/2}$, $c = e^{i\gamma/2}$ so that $\bar{a} = 1/a$, $\bar{b} = 1/b$, $\bar{c} = 1/c$, and $A = a^2$, $B = b^2$, $C = c^2$. Then,

$$I = -(ab + bc + ca) \qquad \text{and} \qquad \overline{I} = -\frac{a+b+c}{abc} \tag{1}$$

(see p. 262 of M. Bataille's article in *Crux Mathematicorum*, Vol 45:5 (May 2019)).

In general, the orthogonal projection S of a point X to a line PQ is given by

$$S = \frac{1}{2}\left(\frac{P\overline{Q} - \overline{P}Q}{\overline{Q} - \overline{P}} + X + \frac{Q-P}{\overline{Q} - \overline{P}}\,\overline{X}\right) \tag{2}$$

and the point Y symmetric to X with respect to $\overleftrightarrow{PQ}$ is

$$Y = 2S - X = \frac{P\overline{Q} - \overline{P}Q}{\overline{Q} - \overline{P}} + \frac{Q-P}{\overline{Q} - \overline{P}}\,\overline{X}. \tag{3}$$

Thus, when $P = O$, $Q = I$ and $X = B$ we get $B' = (I/\overline{I})\,\overline{B}$.

Let $D = \frac{1}{2}(I + B') = \frac{I}{2}\left(1 + \frac{\overline{B}}{\overline{I}}\right)$ be the midpoint of segment $B'I$ and let K be the center of the circumcircle of $BB'I$. The perpendicular bisector of $B'I$ consists of points Z for which

$$\overline{\left(\frac{Z-D}{I-B'}\right)} = -\frac{Z-D}{I-B'},$$

yielding

$$\left[Z - \frac{I}{2}\left(1 + \frac{\overline{B}}{\overline{I}}\right)\right]\overline{I}\left(1 - \frac{B}{I}\right) + \left[\overline{Z} - \frac{\overline{I}}{2}\left(1 + \frac{B}{I}\right)\right] I \left(1 - \frac{\overline{B}}{\overline{I}}\right) = 0,$$

After multiplying out, simplifying, and dividing by $I\overline{I}$ the equation of the bisector, KD, reduces to

$$\left(1 - \frac{B}{I}\right)\frac{Z}{I} + \left(1 - \frac{\overline{B}}{\overline{I}}\right)\frac{\overline{Z}}{\overline{I}} = 1 - \frac{1}{I\overline{I}}. \tag{4}$$

Since $KB' = KI$ and KD is the angle bisector of $\angle B'KI$, the two tangents to the circumcircle of $BB'I$ at points B' and I intersect on the bisector KD. Therefore, to prove that the two tangents intersect on the line AC it suffices to show that the lines DK, AC and the perpendicular to IO at I intersect at one point.

The line perpendicular to IO at I is the locus of points Z such that $Z - I$ is a real multiple of $i(I - O)$, i. e.

$$\frac{\overline{Z} - \overline{I}}{\overline{I}} = -\frac{Z - I}{I} \quad \Longleftrightarrow \quad \overline{I}Z + I\overline{Z} = 2I\overline{I}. \tag{5}$$

The line through arbitrary two points P and Q has an equation

$$(\overline{Q} - \overline{P})Z - (Q - P)\overline{Z} = P\overline{Q} - \overline{P}Q, \tag{6}$$

so that the line AC through $A = a^2$ and $C = c^2$ is

$$Z + c^2a^2\overline{Z} = a^2 + c^2. \tag{7}$$

Using (1) and combining (5) and (7), we find the intersection of the perpendicular to IO at I and the line AC to be the point Z whose affix satisfies

$$\overline{Z} = \frac{2I\overline{I} - (a^2 + c^2)\,\overline{I}}{I - c^2a^2\overline{I}} = \frac{(a + b + c)(a + 2b + c)}{ca\,(ca - b^2)}, \tag{8}$$

by factoring out $a + c$ on the top and the bottom. Consequently,

$$Z = \frac{(ab + bc + ca)(ab + bc + 2ca)}{b^2 - ca} = \frac{I(ca - I)}{ca - b^2}. \tag{9}$$

It can be now shown that this point satisfies the equation (4) so the lines DK, AC and IZ are concurrent at Z. Namely, using (1) we compute

$$1 - \frac{1}{I\overline{I}} = \frac{(a + c)(ab + bc + ca + b^2)}{(a + b + c)(ab + bc + ca)} = -\frac{a + c}{abc\,\overline{I}} \cdot \frac{I - B}{I},$$

so that

$$\left(1 - \frac{B}{I}\right) = -\frac{abc\,\overline{I}}{a + c}\left(1 - \frac{1}{I\overline{I}}\right) \quad \text{and} \quad \left(1 - \frac{\overline{B}}{\overline{I}}\right) = \frac{-I}{b(a + c)}\left(1 - \frac{1}{I\overline{I}}\right).$$

Also, from (9) we get

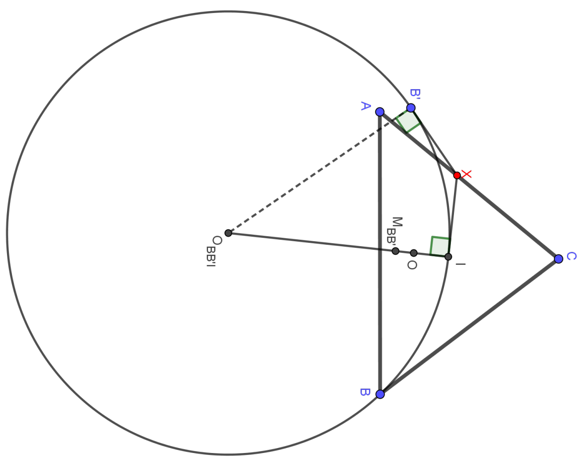
$$\frac{Z}{I} = \frac{ca - I}{ca - b^2} \qquad \text{and} \qquad \frac{\overline{Z}}{\overline{I}} = \frac{b^2(ca\,\overline{I} - 1)}{ca - b^2}.$$

Substituting these into the left-hand side of (4) we get

$$\begin{aligned}
&\left[-\frac{abc\,\overline{I}}{a+c}\,\frac{ca-I}{ca-b^2}+\frac{I}{b(a+c)}\,\frac{b^2(1-ca\,\overline{I})}{ca-b^2}\right]\left(1-\frac{1}{I\overline{I}}\right)\\
&=\frac{1}{(a+c)(ca-b^2)}\left[-c^2a^2b\,\overline{I}+abc\,I\overline{I}+bI-abc\,I\overline{I}\,\right]\left(1-\frac{1}{I\overline{I}}\right)\\
&=\frac{1}{(a+c)(ca-b^2)}\left[ca(a+b+c)-b(ab+bc+ca)\,\right]\left(1-\frac{1}{I\overline{I}}\right)\\
&=\frac{1}{(a+c)(ca-b^2)}\left[ca(a+c)-b^2(a+c)\,\right]\left(1-\frac{1}{I\overline{I}}\right)\\
&=1-\frac{1}{I\overline{I}}.
\end{aligned}$$

This verifies the equation (4) and proves the claim.

Solution 3, by Andrea Fanchini.



We use barycentric coordinates with reference to $\triangle ABC$. The line

$$IO : bc(cS_C-bS_B)x+ac(aS_A-cS_C)y+ab(bS_B-aS_A)z=0$$

has infinite perpendicular point

$$IO_{\infty\perp}(a(b-c):b(c-a):c(a-b)).$$

Therefore, the tangent to the circumcircle of $\triangle BB'I$ at point I is given by

$$IIO_{\infty\perp} : bc(b+c-2a)x+ac(a+c-2b)y+ab(a+b-2c)z=0.$$

Point B', which is the symmetric image of point B with respect to line OI is identified by

$$B'\left(2a(s-b)(c-a)(c-b):b^2(b-a)(b-c):2c(s-b)(b-a)(c-a)\right).$$

Therefore, the tangent to the circumcircle of $\triangle BB'I$ at B' is given by

$$\begin{aligned}B'X :&b^2c(b-a)(b-c)(b+c-2a)x - 2ac(s-b)(c-a)^2(a+c-2b)y\\ &+ ab^2(b-a)(b-c)(a+b-2c)z = 0.\end{aligned}$$

In conclusion, the two tangent lines intersect at the point identified by

$$X = IIO_{\infty\perp} \cap B'X = \left(a(2c-a-b) : 0 : c(b+c-2a)\right).$$

Clearly, this point lies on the line AC, since its second coordinate is 0.

OC437. The magician and his helper have a deck of cards. The cards all have the same back, but their faces are coloured in one of 2017 colours (there are 1000000 cards of each colour). The magician and the helper are going to show the following trick. The magician leaves the room; volunteers from the audience place $n > 1$ cards in a row on a table, all face up. The helper looks at these cards, then he turns all but one card face down (without changing their order). The magician returns, looks at the cards, points to one of the face-down cards and states its colour. What is the minimum number n such that the magician and his helper can have a strategy to do the magic trick successfully?

Originally Russia MO, 4th Problem, Grade 11, Final Round 2017 (Game Theory).

No solutions were received.

OC438. A teacher gives the students a task of the following kind. He informs them that he thought of a monic polynomial $P(x)$ of degree 2017 with integer coefficients. Then he tells them k integers $n_1, n_2, \ldots, n_k$ and the value of the expression $P(n_1)P(n_2) \cdot \ldots \cdot P(n_k)$. According to these data, the students should then find teacher's polynomial. Find the smallest k for which the teacher can compose such a problem so that the polynomial found by the students must necessarily coincide with the one he thought of.

Originally Russia MO, 3rd Problem, Grade 11, Regional Round 2017 (Algebra).

No solutions were received.

OC439. Let $(G, \cdot)$ be a group and let m and n be two nonzero natural numbers that are relatively prime. Prove that if the functions $f : G \to G$, $f(x) = x^{m+1}$ and $g : G \to G$, $g(x) = x^{n+1}$ are surjective endomorphisms, then the group G is abelian.

Originally Romania MO, 2nd Problem, Grade 12, District Round 2017 (Abstract Algebra).

We received 2 correct submissions. We present the solution by Oliver Geupel. Independently, Corneliu Manescu-Avram submitted a similar solution.

Let a be an arbitrary element of G. Since f is a surjective endomorphism, we deduce that, for every $b \in G$, there is a $c \in G$ such that $b = f(c)$, and it holds

$$\begin{aligned} a^m b &= a^{-1} f(a) f(c) = a^{-1} f(ac) = a^{-1}(ac)^{m+1} = a^{-1}(ac)\dots(ac) = (ca)\dots(ca)a^{-1} \\ &= (ca)^{m+1}a^{-1} = f(ca)a^{-1} = f(c)f(a)a^{-1} = ba^m. \end{aligned}$$

Hence, a^m commutes with every element of the group. Similarly, a^n commutes with every element of the group.

It is well-known and easy to verify that the set of elements of a group, G, that commute with every element of G is a subgroup of G, called the centre $Z(G)$ of the group. Thus, for integers q and r, we have $a^{mq+nr} \in Z(G)$. Since m and n are co-prime, integers q and r can be chosen such that $mq + nr = 1$. Consequently, $Z(G) = G$, that is, G is abelian.

OC440. Let $f : [a, b] \to [a, b]$ be a differentiable function with continuous and positive first derivative. Prove that there exists $c \in (a, b)$ such that

$$f(f(b)) - f(f(a)) = (f'(c))^2 (b - a).$$

Originally Romania MO, 4th Problem, Grade 11, Final Round 2017.

We received 4 correct submissions. We present the solution by Ivko Dimitrić. Similar solutions were submitted independently by Brian Bradie and Corneliu Manescu-Avram.

Since $f([a, b]) \subset [a, b]$ and f is increasing and differentiable, the Mean Value Theorem for f applied to the interval $[f(a), f(b)]$ guarantees the existence of a number $q,\ a \le f(a) < q < f(b) \le b,$ such that

$$f(f(b)) - f(f(a)) = f'(q)\,(f(b) - f(a)).$$

Another application of the same theorem on the interval $[a, b]$ tells us that

$$f(b) - f(a) = f'(p)\,(b - a)$$

for some number $p,\ a < p < b$. Combining the two formulas we get

$$f(f(b)) - f(f(a)) = f'(p) f'(q)\,(b - a), \tag{1}$$

where $p, q \in (a, b)$.

Next, we can assume that $f'(p) \le f'(q)$. Since f' is positive we have

$$f'(p) \le \sqrt{f'(p)f'(q)} \le f'(q).$$

Then, the value $\sqrt{f'(p)f'(q)}$ is between $f'(p)$ and $f'(q)$. Since f' is continuous, by the Intermediate Value Theorem for f' on the interval $[p, q]$, there exists $c \in [p, q] \subset (a, b)$ such that $f'(c) = \sqrt{f'(p)f'(q)}$. Combining with (1)

$$f(f(b)) - f(f(a)) = (f'(c))^2 (b - a),$$

and the statement follows.

FOCUS ON...

No. 39

Michel Bataille

Introducing S_A, S_B, S_C in Barycentric Coordinates

Introduction

The use of barycentric coordinates relative to a triangle ABC is quite appropriate when solving problems involving *affine* properties such as collinearity of points, concurrency of lines or even ratio of areas, but does not seem adapted to *euclidean* properties such as lengths or perpendicularity. However, if $a = BC, b = CA, c = AB$, a few results linked to the numbers

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}$$

(Conway's triangle notation) sometimes lead to a simple coordinate solution to euclidean problems. Besides, this is less surprising once one has remarked that S_A, S_B, S_C are nothing but the dot products $\overrightarrow{AB} \cdot \overrightarrow{AC}$, $\overrightarrow{BC} \cdot \overrightarrow{BA}$, $\overrightarrow{CA} \cdot \overrightarrow{CB}$, respectively!

After a paragraph offering useful relations concerning S_A, S_B, S_C, we will present some examples of situations that can prompt a resort to these numbers.

Becoming more familiar with S_A, S_B, S_C

The obvious equalities

$$\begin{aligned} S_B + S_C &= a^2, & S_B - S_C &= c^2 - b^2, \\ S_C + S_A &= b^2, & S_C - S_A &= a^2 - c^2, \\ S_A + S_B &= c^2, & S_A - S_B &= b^2 - a^2 \end{aligned}$$

are of constant use and should be kept in mind from now on!

Other interesting, readily checked relations are

$$\begin{aligned} c^2 S_C - b^2 S_B &= (b^2 - c^2) S_A, \\ a^2 S_A - c^2 S_C &= (c^2 - a^2) S_B, \\ b^2 S_B - a^2 S_A &= (a^2 - b^2) S_C \end{aligned}$$

and, denoting by s the semiperimeter of ΔABC,

$$\begin{aligned} cS_C - bS_B &= 2s(s-a)(b-c), \\ aS_A - cS_C &= 2s(s-b)(c-a), \\ bS_B - aS_A &= 2s(s-c)(a-b). \end{aligned}$$

A connection to the area F of the triangle ABC is obtained with

$$S_BS_C + a^2S_A = S_CS_A + b^2S_B = S_AS_B + c^2S_C = 4F^2$$

and

$$2(S_BS_C + S_CS_A + S_AS_B) = a^2S_A + b^2S_B + c^2S_C = 8F^2.$$

These formulas are easily proved with the help of the known

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4).$$

See also exercise 1 for more relations.

The coordinates of O and H

The numbers S_A, S_B, S_C prove very useful when the barycentric coordinates of the circumcentre O and the orthocenter H of ΔABC are needed. In terms of sidelengths and angles of ΔABC, the coordinates are known to be

$$O = (a\cos A : b\cos B : c\cos C)$$

and

$$H = (a\cos B\cos C : b\cos C\cos A : c\cos A\cos B).$$

Observing that for example

$$a\cos A = a\cdot\frac{b^2+c^2-a^2}{2bc} = a^2\frac{S_A}{abc} \quad \text{and} \quad a\cos B\cos C = a\cdot\frac{bS_B}{abc}\cdot\frac{cS_C}{abc} = \frac{S_BS_C}{abc}$$

we obtain that

$$O = (a^2S_A : b^2S_B : c^2S_C), \qquad H = (S_BS_C : S_CS_A : S_AS_B).$$

These coordinates readily yield those of the centre N of the nine-point circle: since N is the midpoint of OH, we have

$$\begin{aligned}(8F^2)2N &= (8F^2)O + (8F^2)H\\ &= (a^2S_A + 2S_BS_C)A + (b^2S_B + 2S_CS_A)B + (c^2S_C + 2S_AS_B)C\end{aligned}$$

hence

$$\begin{aligned}N &= (2S_BS_C + a^2S_A : 2S_CS_A + b^2S_B : 2S_AS_B + c^2S_C)\\ &= (S_BS_C + 4F^2 : S_CS_A + 4F^2 : S_AS_B + 4F^2),\end{aligned}$$

a result to be used in our first example, problem **OC 311** [2017 : 12 ; 2018 : 102]:

> Let ΔABC be an acute-scalene triangle, and let N be the center of the circle which passes through the feet of the altitudes. Let D be the intersection of the tangents to the circumcircle of ΔABC at B and C. Prove that A, D and N are collinear if and only if $\angle BAC = 45°$.

Clearly, N is the point above and the point D lies on the symmedian through A of ΔABC (a well-known result), hence it is sufficient to prove that N is on this symmedian if and only if $\angle BAC = 45°$.

With the previous notations, the symmedian point K is $(a^2 : b^2 : c^2)$ and so the equation of the symmedian AK is $c^2y - b^2z = 0$.

Therefore N is on the symmedian AD if and only if

$$c^2(S_CS_A + 4F^2) = b^2(S_AS_B + 4F^2),$$

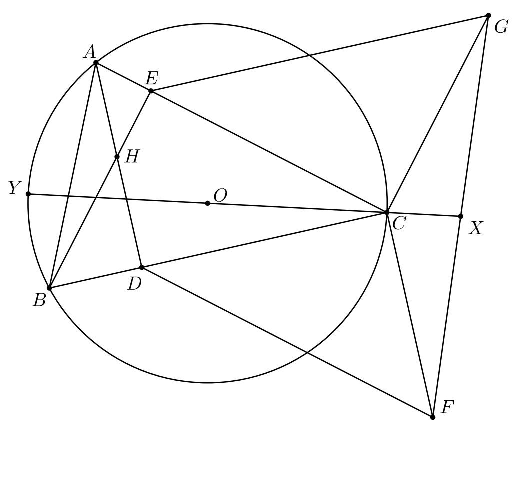
which successively rewrites as

$$\begin{aligned}
&S_A(c^2S_C - b^2S_B) = 4F^2(b^2 - c^2)\\
&S_A^2 = 4F^2 \quad \text{(since } ABC \text{ is scalene)}\\
&(b^2 + c^2 - a^2)^2 = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\\
&\left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 = \frac{1}{2}\\
&\cos^2(\angle BAC) = \frac{1}{2}
\end{aligned}$$

and since ABC is acute-angled, the latter means that $\angle BAC = 45°$.

As a second example, we next present an alternative solution to problem **4258** [2017 : 265 ; 2018 : 270]:

> Let ABC be an acute triangle with circumcentre O, orthocentre H, $D \in BC$, $AD \perp BC$, $E \in AC$, $BE \perp AC$. Define points F and G to be the fourth vertices of parallelograms $CADF$ and $CBEG$. If X is the midpoint of FG, and Y is the point where XC intersects the circumcircle again, prove that $AHBY$ is a parallelogram.



Keeping the notations of this paragraph, we have

$$a^2D = (S_C)B + (S_B)C, \qquad b^2E = (S_C)A + (S_A)C$$

and, since $G = E + C - B$ and $F = D + C - A$, we then deduce

$$b^2G = (S_C)A - b^2B + (b^2 + S_A)C$$

and

$$a^2F = -a^2A + (S_C)B + (a^2 + S_B)C.$$

A simple calculation then yields the midpoint X of FG:

$$\begin{aligned} 2a^2b^2X &= b^2(a^2F) + a^2(b^2G) \\ &= (-a^2S_A)A + (-b^2S_B)B + (2a^2b^2 + b^2S_B + a^2S_A)C. \end{aligned}$$

Now, noticing that the equation of the line CO is $(b^2S_B)x - (a^2S_A)y = 0$, we see that X is on CO. It follows that the line CY is a diameter of the circumcircle and therefore $CB \perp BY$ and $CA \perp AY$. Thus, $BY \parallel AH$ and $AY \parallel BH$ and $AHBY$ is a parallelogram.

About perpendiculars

We shall illustrate the following result: If $(f : g : h)$ is the infinite point of the line ℓ, then the infinite point $(f' : g' : h')$ of the perpendiculars to ℓ is given by

$$f' = gS_B - hS_C, \quad g' = hS_C - fS_A, \quad h' = fS_A - gS_B.$$

We quickly repeat the known proof for completeness. Expressing that the vectors $g\overrightarrow{AB} + h\overrightarrow{AC}$ and $g'\overrightarrow{AB} + h'\overrightarrow{AC}$ are orthogonal yields

$$0 = (g\overrightarrow{AB} + h\overrightarrow{AC}) \cdot (g'\overrightarrow{AB} + h'\overrightarrow{AC}) = g'(gc^2 + hS_A) + h'(gS_A + hb^2).$$

Since $f + g + h = f' + g' + h' = 0$, we obtain

$$\frac{g'}{gS_A + hb^2} = \frac{-h'}{hS_A + gc^2} = \frac{f'}{-gS_A - hb^2 + hS_A + gc^2},$$

that is,

$$\frac{f'}{gS_B - hS_C} = \frac{g'}{hS_C - fS_A} = \frac{h'}{fS_A - gS_B}.$$

To see this at work through a simple example, consider the line BC whose point at infinity is $(0 : 1 : -1)$. The point at infinity of the perpendiculars to BC then is $(S_B + S_C : -S_C : -S_B) = (-a^2 : S_C : S_B)$. It follows that the equation of the perpendicular bisector δ_A of BC is

$$\begin{vmatrix} x & -a^2 & 0 \\ y & S_C & 1 \\ z & S_B & 1 \end{vmatrix} = 0,$$

that is, $x(c^2 - b^2) - a^2y + a^2z = 0$.

With the help of this result, we can offer a variant of solution to the following problem extracted from **3910** [2014 : 30 ; 2015 : 42]:

> Two triangles ABC and $A'B'C'$ are homothetic. Show that if B' and C' are on the perpendicular bisectors of CA and AB respectively, then A' is on the perpendicular bisector of BC.

From the above equation of δ_A, we get cyclically the equations of δ_B and δ_C, the perpendicular bisectors of CA and AB:

$$b^2x + (a^2 - c^2)y - b^2z = 0 \quad \text{and} \quad c^2x - c^2y + (a^2 - b^2)z = 0.$$

Now, let $\Omega = (\alpha : \beta : \gamma)$, with $\alpha + \beta + \gamma = 1$, be the centre of the homothety h transforming A, B, C into A', B', C', respectively. If λ denotes the factor of h, then $A' - \Omega = \lambda(A - \Omega)$, hence $A' = (\lambda + (1-\lambda)\alpha : (1-\lambda)\beta : (1-\lambda)\gamma)$. Similarly, we have $B' = ((1-\lambda)\alpha : \lambda + (1-\lambda)\beta : (1-\lambda)\gamma)$ and $C' = ((1-\lambda)\alpha : (1-\lambda)\beta : \lambda + (1-\lambda)\gamma)$.

Expressing that B' and C' are on the lines δ_B and δ_C, respectively, we obtain

$$\lambda(a^2 - c^2) + (1 - \lambda)(\alpha b^2 + \beta(a^2 - c^2) - \gamma b^2) = 0 \qquad (1)$$

and

$$\lambda(a^2 - b^2) + (1 - \lambda)(\alpha c^2 - \beta c^2 + \gamma(a^2 - b^2)) = 0 \qquad (2).$$

The difference $(2) - (1)$ gives $\lambda(c^2 - b^2) + (1 - \lambda)(\alpha(c^2 - b^2) - \beta a^2 + \gamma a^2) = 0$, which implies that A' is on δ_A, as desired.

Our second example gives a solution to problem **4313** [2018 : 71 ; 2019 : 93]:

> Let I be the incenter of triangle ABC, and denote by H_a, H_b and H_c the orthocenters of triangles IBC, ICA and IAB, respectively. Prove that triangles ABC and $H_aH_bH_c$ have the same area.

Let $I = (a : b : c)$ be the incentre of ΔABC. The equation of the line IB then is $cx - az = 0$, its point at infinity is $(a : -(a + c) : c)$ and so the perpendicular to IB through C is

$$\begin{vmatrix} x & -ca^2 - aS_B & 0 \\ y & cS_C - aS_A & 0 \\ z & ac^2 + cS_B & 1 \end{vmatrix} = 0,$$

that is,

$$(cS_C - aS_A)x + (ca^2 + aS_B)y = 0.$$

Similarly, the perpendicular to IC through B is $(bS_B - aS_A)x + (ba^2 + aS_C)z = 0$. This provides their point of intersection H_a: $H_a = (a : c - a : b - a)$. Cyclically, we obtain $H_b = (c - b : b : a - b)$ and $H_c = (b - c : a - c : c)$. It follows that

$\text{Area}(H_aH_bH_c) = |\delta|\text{Area}(ABC)$ where

$$\delta = \frac{1}{(b+c-a)(c+a-b)(a+b-c)} \cdot \begin{vmatrix} a & c-b & b-c \\ c-a & b & a-c \\ b-a & a-b & c \end{vmatrix}$$
$$= \frac{1}{8(s-a)(s-b)(s-c)} \cdot \delta'.$$

It is not difficult to check that $\delta' = 8(s-a)(s-b)(s-c)$ and the conclusion follows.

For another illustration of this paragraph and the previous one, we refer the reader to my solution to problem **3878** [2013 : 371 ; 2014 : 359].

As usual, we end the number with a series of exercises.

Exercises

1. (Adapted from problem 11958 of *The American Mathematical Monthly*) Prove the relations

$$a^4S_A + b^4S_B + c^4S_C - 3S_AS_BS_C = 2(a^2+b^2+c^2)F^2 = S_AS_BS_C + a^2b^2c^2$$

and deduce a condition on a, b, c for the nine-point centre N to lie on the circumcircle of ΔABC.

2. Use S_A, S_B, S_C to show that O, H and the incenter I are collinear if and only if the triangle ABC is isosceles.

3. Find the point at infinity of the perpendiculars to OI, where O and I are the circumcentre and the incentre of a scalene triangle ABC.

4. If $M_1 = (x_1 : y_1 : z_1),\ M_2 = (x_2 : y_2 : z_2)$ with $x_1+y_1+z_1 = x_2+y_2+z_2 = 1$, show that

$$M_1M_2^2 = S_A(x_2-x_1)^2 + S_B(y_2-y_1)^2 + S_C(z_2-z_1)^2.$$

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **March 15, 2020**.

4501. *Proposed by Vaclav Konecny, modified by the Board.*

Given the rectangle whose vertices have Cartesian coordinates $A(0, b)$, $B(0, 0)$, $C(a, 0)$, $D(a, b)$, find the equation of the locus of points $P(x, y)$ in the third quadrant (with $x, y < 0$) for which $\angle BPA = \angle CPD$.

Comment from the proposer: this problem was inspired by problem #4301 in Crux 44(1) proposed by Bill Sands.

4502. *Proposed by George Apostolopoulos.*

Let a, b, c be the side lengths of triangle ABC with inradius r and circumradius R. Prove that

$$\frac{3}{2} \cdot \frac{r}{R} \leq \sum_{cyclic} \frac{a}{2a+b+c} \leq \frac{3}{8} \cdot \frac{R}{r}.$$

4503. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC = 90^\circ$ and let Γ be the circle with centre B and radius BC. A circle γ passing through B and A intersects Γ at X, Y with $X \neq Y$. Let E and F be the orthogonal projections of X and Y onto CY and CX, respectively. Prove that the line CA bisects EF.

4504. *Proposed by Warut Suksompong.*

Find all positive integers (a, b, c, x, y, z), $a \leq b \leq c$ and $x \leq y \leq z$, for which the following two equations hold:

$$\begin{aligned} a+b+c &= xy+yz+zx, \\ x+y+z &= abc. \end{aligned}$$

4505. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral such that $AC \perp BD$ and $AB = BC$. Let I denote the point of intersection of AC and BD. A straight line l_1 passes through I and intersects BC and AD in R and S, respectively. Similarly, straight line l_2 passes through I and intersects AB and CD in M and N, respectively. The lines MS and RN intersect AC at P and Q, respectively. Prove that $IP = IQ$.

4506. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a$, where $a \in \mathbb{R}_+^*$. Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}}.$$

4507. *Proposed by Eduardo Silva.*

Suppose that $a_0 < \cdots < a_r$ are integers. If $\{b_i\}$ are distinct integers with $a_i \leq b_i$, for each i, and σ is a permutation so that $b_{\sigma(0)} < \cdots < b_{\sigma(r)}$, prove that $a_i \leq b_{\sigma(i)}$ for each i. Further, if $a_j = b_{\sigma(j)}$ for some j, then $\sigma(j) = j$, so that $a_j = b_j$.

4508. *Proposed by Hung Nguyen Viet.*

Let x, y, z be nonzero real numbers such that $x + y + z = 0$. Find the minimum possible value of

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

4509. *Proposed by Leonard Giugiuc and Dan Stefan Marinescu.*

Let B and C be two distinct fixed points that lie in the plane α and let M be the midpoint of BC. Find the locus of points $A \in \alpha$, $A \notin BC$, for which $4R \cdot AM = AB^2 + AC^2$, where R is the circumradius of ABC.

4510. *Proposed by Leonard Giugiuc and Daniel Sitaru.*

Let ABC be a non-obtuse triangle. Prove that

$$\cos A \cos B + \cos A \cos C + \cos B \cos C > 2\sqrt{\cos A \cos B \cos C}.$$

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 mars 2019***.*

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4501. *Proposé par Vaclav Konecny, modifié par le conseul.*

Pour un rectangle dont les sommets ont les coordonnées cartésiennes $A(0,b)$, $B(0,0)$, $C(a,0)$ et $D(a,b)$, déterminer l'équation vérifiée par les points dans le troisième quadrant (où $x,y<0$) pour lesquels $\angle BPA=\angle CPD$.

Note du proposeur: ce problème tire son inspiration du problème # 4301 dans Crux 44(1), proposé par Bill Sands.

4502. *Proposé par George Apostolopoulos.*

Soient a,b et c les longueurs des côtés du triangle ABC, dont les rayons des cercles inscrit et circonscrit sont r et R respectivement. Démontrer que

$$\frac{3}{2}\cdot\frac{r}{R}\leq\sum_{cyclic}\frac{a}{2a+b+c}\leq\frac{3}{8}\cdot\frac{R}{r}.$$

4503. *Proposé par Michel Bataille.*

Soit ABC un triangle tel que $\angle BAC=90^\circ$ et soit Γ le cercle de centre B et rayon BC. Un cercle γ, passant par B et A, intersecte Γ en X et Y où $X\neq Y$. Soient E et F les projections orthogonales de X et Y vers CY et CX, respectivement. Démontrer que la ligne CA bissecte EF.

4504. *Proposé par Warut Suksompong.*

Déterminer tous les entiers positifs (a,b,c,x,y,z), $a\leq b\leq c$ et $x\leq y\leq z$, pour lesquels les deux équations suivantes tiennent:

$$\begin{aligned}a+b+c&=xy+yz+zx,\\x+y+z&=abc.\end{aligned}$$

4505. *Proposé par Miguel Ochoa Sanchez et Leonard Giugiuc.*

Soit $ABCD$ un quadrilatère convexe tel que $AC\perp BD$ et $AB=BC$. Dénotons par I le point d'intersection de AC et BD. Une ligne l_1 passe par I et intersecte

BC et AD en R et S respectivement. De façon similaire, la ligne l_2 passe par I et intersecte AB et CD en M et N, respectivement. Les lignes MS et RN intersectent AC en P et Q respectivement. Démontrer que $IP = IQ$.

4506. *Proposé par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soit (a_n) une suite de nombres réels positifs telle que $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{na_n} = a$, où $a \in \mathbb{R}_+^*$. Calculer

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=2}^{n} \frac{k}{\sqrt[k]{a_k}}.$$

4507. *Proposé par Eduardo Silva.*

Supposons que $a_0 < \cdots < a_r$ sont des entiers. Si $\{b_i\}$ sont des entiers distincts tels que $a_i \leq b_i$ pour tout i, et σ est une permutation telle que $b_{\sigma(0)} < \cdots < b_{\sigma(r)}$, démontrer que $a_i \leq b_{\sigma(i)}$ pour tout i. De plus, si $a_j = b_{\sigma(j)}$ pour un certain j, alors $\sigma(j) = j$, d'où $a_j = b_j$.

4508. *Proposé par Hung Nguyen Viet.*

Soient x, y, z des nombres réels non nuls tels que $x + y + z = 0$. Déterminer la valeur minimale de

$$(x^2 + y^2 + z^2)\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right).$$

4509. *Proposé par Leonard Giugiuc et Dan Stefan Marinescu.*

Soient B et C des points distincts dans le plan α et soit M le mi point de BC. Déterminer le lieu des points $A \in \alpha$, $A \notin BC$, pour lesquels $4R\cdot AM = AB^2 + AC^2$, où R est le rayon du cercle circonscrit de ABC.

4510. *Proposé par Leonard Giugiuc et Daniel Sitaru.*

Soit ABC un triangle non obtus. Démontrer que

$$\cos A \cos B + \cos A \cos C + \cos B \cos C > 2\sqrt{\cos A \cos B \cos C}.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(6), p. 346–349.

4451. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$ with $n \geq 2$ and $0 < a < b < 1$, let

$$I(a,b) = \int_a^b \frac{(x+1)((2n-3)x^{n+1} - (2n-1)x^n + 3x - 1)}{x^2(x-1)^2}\, dx.$$

Find

$$\lim_{a \to 0^+} \left(\frac{1}{a} + \lim_{b \to 1^-} I(a,b) \right).$$

There were 15 correct solutions. We present the standard approach taken by most of the solvers.

Note that

$$(2n-3)x^{n+1} - (2n-1)x^n + 3x - 1 = (x-1)^2[(2n-3)x^{n-1} + (2n-5)x^{n-2} + \cdots + 3x^2 + x - 1],$$

and

$$\begin{aligned}&(x+1)[(2n-3)x^{n-1} + (2n-5)x^{n-2} + \cdots + 3x^2 + x - 1]\\ &= (2n-3)x^n + 4(n-2)x^{n-1} + \cdots + 4x^2 - 1.\end{aligned}$$

Therefore, the integrand is equal to

$$(2n-3)x^{n-2} + 4\sum_{k=2}^{n-1}(k-1)x^{k-2} - \frac{1}{x^2}.$$

and its antiderivative is

$$f(x) \equiv \left(\frac{2n-3}{n-1}\right) x^{n-1} + 4\sum_{k=2}^{n-1} x^{k-1} + \frac{1}{x}.$$

Now

$$\lim_{b \to 1^-} f(b) = \frac{2n-3}{n-1} + 4(n-2) + 1 = \frac{4n^2 - 9n + 4}{n-1}.$$

Therefore

$$\frac{1}{a} + \lim_{b \to 1^-} I(a,b) = \left[\frac{4n^2 - 9n + 4}{n-1}\right] - \left[\left(\frac{2n-3}{n-1}\right) a^{n-1} + 4\sum_{k=2}^{n-1} a^{n-1}\right],$$

whence

$$\lim_{a\to 0^+}\left(\frac{1}{a}+\lim_{b\to 1^-} I(a,b)\right)=\frac{4n^2-9n+4}{n-1}.$$

Comment from the editor. This is essentially the solution supplied by all the solvers. However, there were interesting aspects to the various manipulations. While most used long division to find the cofactor of $(x-1)^2$ in the factorization of

$$P_n(x)\equiv(2n-3)x^{n+1}-(2n-1)x^n+3x-1,$$

Michel Bataille and Ivko Dimitrić relied on the respective recursions

$$P_n(x)=(2n-3)x^{n-1}(x-1)^2+P_{n-1}$$

and

$$P_{n+1}(x)=xP_n(x)+(x-1)[2x(x^n-1)-(x-1)]$$

to find the cofactor by an induction argument. However, Paul Bracken did not need to bother with this, since, by some alchemy, he produced the antiderivative

$$\frac{1}{x(x-1)}\left[\left(\frac{2n-3}{n-1}\right)x^{n+1}+\left(\frac{2n-1}{n-1}\right)x^n-3x-1\right].$$

4452. *Proposed by Mihaela Berindeanu.*

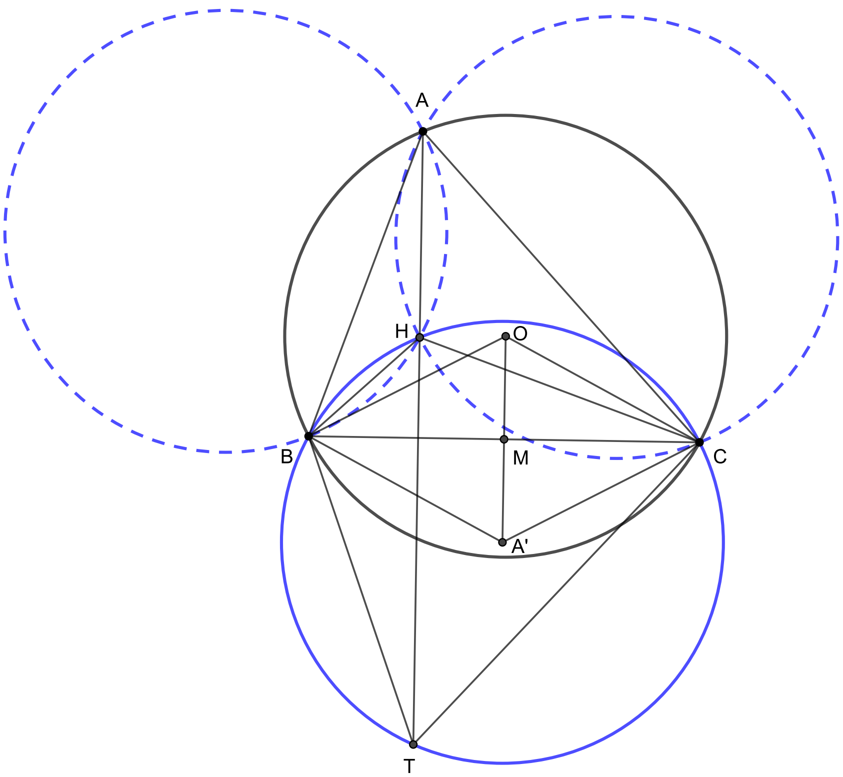
Let ABC be a triangle with orthocenter H. If A', B', C' are the circumcenters of $\triangle\ HBC, \triangle\ HAC$ and $\triangle\ HAB$, respectively, and $\overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}=\overrightarrow{0}$, show that ABC is an equilateral triangle.

We received 11 submissions, of which 9 were complete and correct. We present a solution which combines parts of those submitted by Sorin Rubinescu and Alexandru Pîrvuceanu and by Ivko Dimitrić.

Assume $\triangle ABC$ is not a right triangle, as the problem is ill-posed in such a case. Let O be the circumcenter of $\triangle ABC$ and R its circumradius (see figure below).

First we show that A' is the reflection of O over the side BC. Let M be the midpoint of BC and A'' the reflection of O with respect to M. The relationship between central and inscribed angles subtending the same arc gives us $\angle BOC = 2\angle A$. Since the diagonals in quadrilateral $BOCA''$ are perpendicular and bisect each other, $BOCA''$ is a rhombus. Thus,

$$\angle BA''C=\angle BOC=2\angle A.$$



It is easy to see (and well-known) that $\angle BHC = 180^\circ - \angle A$. Extend the segment AH until it intersects the circumcircle of $\triangle BHC$, and denote the intersection by T. Since the points B, H, C, T are concyclic, we have

$$\angle BTC = 180^\circ - \angle BHC = \angle A.$$

Thus we have shown $\angle BA''C = 2\angle BTC$, so A'' must be the centre of the circumcircle of $\triangle BHC$; that is, $A'' = A'$. In particular, $BOCA'$ is a rhombus.

We thus have
$$\overrightarrow{AA'} = \overrightarrow{AO} + \overrightarrow{OA'} = -\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

Similar arguments show that $OAC'B$ and $OCBA'$ are rhombi, and we calculate

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OA'}. \tag{1}$$

Next, we note that $OA' = 2OM = 2R\cos(\angle A)$. But from known triangle formulas we have $AH = 2R\cos(\angle A)$ as well, so $OA' = AH$. Moreover, AH and OA' are both perpendicular to BC, so $AH \parallel OA'$, which means that the quadrilateral $AHA'O$ is a parallelogram. This gives us $\overrightarrow{OA'} = \overrightarrow{AH}$, which we substitute into (1) to get

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OH}.$$

Therefore, $\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}$ is equivalent to the condition that O and H coincide, that is, the altitude of $\triangle ABC$ from any vertex is the perpendicular bisector of the opposite side i.e. the triangle ABC is equilateral.

4453. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be a triangle with no angle larger than $\frac{2\pi}{3}$ and let T be its Fermat-Torricelli point, that is the point such that the total distance from the three vertices of ABC to T is minimum possible. Suppose BT intersects AC at D and CT intersects AB at E. Prove that if $AB + AC = 4DE$, then ABC is equilateral.

The only submitted solution came from the proposers; it is presented below. In addition, we received a comment from Walther Janous.

We use repeatedly the known theorem that the Fermat-Torricelli point T (of a triangle that has all angles less than $\frac{2\pi}{3}$) is the point inside ΔABC for which $\angle ATB = \angle BTC = \angle CTA = \frac{2\pi}{3}$. Set $x = AT, y = BT$, and $z = CT$. Then the cosine law applied to triangles ATB and CTA yields

$$AB = \sqrt{x^2 + xy + y^2} \quad \text{and} \quad AC = \sqrt{z^2 + zx + x^2}.$$

Furthermore, we have $\angle ATE = \frac{\pi}{3}$ (because $\angle ATE + CTA = \angle ATE + \frac{2\pi}{3} = \pi$), and (similarly) $\angle ETB = \frac{\pi}{3}$; it follows that TE is the internal bisector of the angle at T in ΔTAB, so that

$$TE = \frac{xy}{x+y}; \quad \text{similarly,} \quad TD = \frac{xz}{x+z}.$$

Therefore, in ΔTDE we have

$$DE^2 = \left(\frac{xy}{x+y}\right)^2 + \left(\frac{xy}{x+y}\right)\left(\frac{xz}{x+z}\right) + \left(\frac{xz}{x+z}\right)^2.$$

Because $\frac{\sqrt{xy}}{2} \geq \frac{xy}{x+y}$ and $\frac{\sqrt{xz}}{2} \geq \frac{xz}{x+z}$ (by the GM-HM inequality), we have

$$\frac{xy + x\sqrt{yz} + xz}{4} \geq \left(\frac{xy}{x+y}\right)^2 + \left(\frac{xy}{x+y}\right)\left(\frac{xz}{x+z}\right) + \left(\frac{xz}{x+z}\right)^2,$$

which implies that

$$2\sqrt{xy + x\sqrt{yz} + xz} \geq 4DE.$$

Moreover, because $\frac{\sqrt{xy}}{2} = \frac{xy}{x+y}$ if and only if $x = y$, and $\frac{\sqrt{xz}}{2} = \frac{xz}{x+z}$ if and only if $x = z$, we have $2\sqrt{xy + x\sqrt{yz} + xz} = 4DE$ if and only if $x = y = z$.

By the AM-GM inequality,

$$AB + AC = \sqrt{x^2 + xy + y^2} + \sqrt{z^2 + zx + x^2} \geq 2\sqrt[4]{(x^2 + xy + y^2)(z^2 + zx + x^2)},$$

while by Cauchy's inequality applied to the vectors $(x, \sqrt{xy}, y)$ and $(z, \sqrt{zx}, x)$ we have

$$(x^2 + xy + y^2)(z^2 + zx + x^2) \geq (xy + x\sqrt{yz} + xz)^2,$$

or
$$2\sqrt[4]{(x^2+xy+y^2)(z^2+zx+x^2)} \geq 2\sqrt{xy+x\sqrt{yz}+xz}.$$
Putting the pieces together, we get
$$AB+AC \geq 2\sqrt{xy+x\sqrt{yz}+xz} \geq 4DE,$$
and conclude that if $AB+AC=4AD$, then $x=y=z$. This immediately implies that ΔABC is equilateral, as desired.

Editor's comments. Walther Janous observed that our problem has the following immediate consequence:

> If T is the Fermat-Torricelli point of a triangle with no angle larger than $\frac{2\pi}{3}$, while D,E,F are the feet of the cevians through T, then the perimeter of ΔDEF equals at most the semiperimeter of ΔABC, with equality if and only if both triangles are equilateral.

4454. *Proposed by Nguyen Viet Hung.*

Prove the identity
$$\binom{4n}{0}-\binom{4n}{2}+\cdots+(-1)^n\binom{4n}{2n}=\frac{(-4)^n+(-1)^n\binom{4n}{2n}}{2}.$$

We received 25 submissions, all of which were correct and complete. We present the solution by Michel Bataille. Almost all solutions were based on the same idea.

From the binomial theorem, we have
$$(1+i)^{4n}=\sum_{k=0}^{4n}\binom{4n}{k}i^k.$$

Since for integers j we have $i^{2j}=(-1)^j$ and $i^{2j+1}=(-1)^j i$, the real part of $(1+i)^{4n}$ in the binomial expansion is
$$R=\sum_{k=0}^{2n}(-1)^k\binom{4n}{2k}=(-1)^n\binom{4n}{2n}+2\sum_{k=0}^{n-1}(-1))^k\binom{4n}{2k}.$$

The latter equality holds because
$$(-1)^k\binom{4n}{2k}=(-1)^{2n-k}\binom{4n}{2(2n-k))}$$
for $k=0,1,\ldots,n-1$. On the other hand, since
$$(1+i)^{4n}=(\sqrt{2}e^{i\pi/4})^{4n}=2^{2n}(-1))^n=(-4)^n,$$

we have $R = (-4)^n$. Comparison with the above expression for R yields

$$2\sum_{k=0}^{n}(-1))^k\binom{4n}{2k} - (-1)^n\binom{4n}{2n} = R = (-4)^n.$$

Solving for the summation in the above gives the required identity.

4455. *Proposed by Marian Maciocha.*

Find all integer solutions (if any) for the equation

$$(A+3B)(5B+7C)(9C+11A) = 1357911.$$

We received 23 submissions, out of which 21 were correct and complete. We present the solution by Corneliu Manescu-Avram.

Suppose A, B, C is an integer solution to the problem. Then the numbers $A+3B$, $5B+7C$, and $9C+11A$ are all odd, since their product is odd. But their sum $4(3A+2B+4C)$ is even, which is impossible. Thus the given equation has no solutions in integers.

4456. *Proposed by Leonard Giugiuc.*

Let a, b, c be positive real numbers such that $abc = 1$. Show that

$$(a+b+c)(ab+bc+ac)+3 \geq 4(a+b+c).$$

We received 28 submissions, all correct. Most of these are similar to each other and we present the solution by Boris Čolaković.

The given inequality is equivalent to $ab+bc+ca+\frac{3}{a+b+c} \geq 4$. By the AM-GM inequality, we have

$$ab+bc+ca+\frac{3}{a+b+c} = \frac{3(ab+bc+ca)}{3} + \frac{3}{a+b+c} \geq 4\sqrt[4]{\frac{(ab+bc+ca)^3}{9(a+b+c)}}.$$

Hence it suffices to prove that

$$(ab+bc+ca)^3 \geq 9(a+b+c). \tag{1}$$

It is well known [*Ed.* and easy to show by simple algebra] that

$$(ab+bc+ca)^2 \geq 3abc(a+b+c) = 3(a+b+c). \tag{2}$$

Also,

$$ab+bc+ca \geq 3\sqrt[3]{(abc)^2} = 3. \tag{3}$$

Multiplying (2) and (3), inequality (1) follows, completing the proof.

4457. *Proposed by Hung Nguyen Viet.*

Prove that for all $-\frac{\pi}{2} < x, y < \frac{\pi}{2}$, $x \neq -y$, we have that

$$\tan^2 x + \tan^2 y + \cot^2(x+y) \geq 1.$$

There were 22 correct solutions; we present four variants here.

Solution 1, due, independently, to Michel Bataille, Ganbat Batmunkh, Martin Lukarevski, Marie-Nicole Gras, C.R. Pranesachar, Ioannis D. Sfikas, Kevin Soto Palacios, and the Missouri State University Problem Solving Group.

Let $a = \tan x$ and $b = \tan y$. Then $\cot(x+y) = (1-ab)/(a+b)$ and

$$\tan^2 x + \tan^2 y + \cot^2(x+y) - 1 = a^2 + b^2 + \frac{(1-ab)^2}{(a+b)^2} - 1.$$

Multiplying this quantity by $(a+b)^2$ yields

$$\begin{aligned}(a^4 + b^4 + 2a^3b + 2ab^3 + 3a^2b^2 + 1) - (a^2 + b^2 + 4ab)&\\ &= (a^2 + ab + b^2)^2 + 1 - (a^2 + b^2 + 4ab)\\ &= (a^2 + ab + b^2 - 1)^2 + (a-b)^2.\end{aligned}$$

Since this quantity is nonnegative, the result follows.

Equality occurs if and only if $a = b$ and $a^2+ab+b^2 = 1$, if and only if $x = y = \pm\pi/6$.

Comment from the editor. For the difference between the two sides, Devis Alvarado and Walther Janous obtained a fraction with the numerator

$$\left[\left(a+b-\tfrac{2}{\sqrt{3}}\right)^2 + (a-\tfrac{1}{\sqrt{3}})^2 + (b-\tfrac{1}{\sqrt{3}})^2\right]\left[\left(a+b+\tfrac{2}{\sqrt{3}}\right)^2 + (a+\tfrac{1}{\sqrt{3}})^2 + (b+\tfrac{1}{\sqrt{3}})^2\right].$$

Solution 2, by Digby Smith.

If $|x| + |y| > \pi/2$, then at least one of $|x|$ and $|y|$ exceeds $\pi/4$ and the left side exceeds 1. Since $0 \leq |x+y| \leq |x| + |y|$, then

$$\cot^2(x+y) = \cot^2(|x+y|) \geq \cot^2(|x| + |y|).$$

Thus, we may suppose that x and y are both nonnegative. If $x+y = \pi/2$, then the left side exceeds 2. Suppose $x + y \neq \pi/2$. Then, using the arithmetic-geometric means inequality, we have that

$$\begin{aligned}&\tan^2 x + \tan^2 y + \frac{1}{\tan^2(x+y)}\\ &= \frac{1}{2}\left[(\tan^2 x + \tan^2 y) + \left(\tan^2 x + \frac{1}{\tan^2(x+y)}\right) + \left(\tan^2 y + \frac{1}{\tan^2(x+y)}\right)\right]\\ &\geq \tan x \tan y + \frac{\tan x}{\tan(x+y)} + \frac{\tan y}{\tan(x+y)} = \tan x \tan y + \frac{\tan x + \tan y}{\tan(x+y)}\\ &= \tan x \tan y + 1 - \tan x \tan y = 1.\end{aligned}$$

Solution 3, built on ideas from Sefkat Arslanagić and Roy Barbara, independently.

If $|x| + |y| > \pi/2$, then at least one of $|x|$ and $|y|$ exceeds $\pi/4$ and the left side exceeds 1. Since $0 \le |x+y| \le |x| + |y|$, then

$$\cot^2(x+y) = \cot^2(|x+y|) \ge \cot^2(|x| + |y|).$$

Thus, it suffices to establish the result when $0 \le x \le y \le x + y \le \pi/2$.

Since $\tan^2 x$ is convex on $(0, \pi/2)$,

$$\tan^2 x + \tan^2 y \ge 2\tan^2\left(\frac{x+y}{2}\right).$$

Therefore, we need only establish that

$$2\tan^2\theta + \cot^2 2\theta \ge 1$$

for $0 < \theta \le \pi/2$. Let $t = \tan^2\theta$. Then

$$2\tan^2\theta + \cot^2 2\theta - 1 = 2t + \frac{(1-t)^2}{4t} - 1 = \frac{9t^2 - 6t + 1}{4t} = \frac{(3t-1)^2}{4t} \ge 0.$$

Equality occurs if and only if $x = y = \pm\pi/6$.

Solution 4, by Vivek Mehra.

Since $\tan^2 t = \sec^2 t - 1$ and $\cot^2 t = \csc^2 t - 1$, the inequality is equivalent to

$$\frac{1}{\cos^2 x} + \frac{1}{\cos^2 y} + \frac{1}{\sin^2(x+y)} \ge 4.$$

Let $t = \cos^2 x + \cos^2 y$. Applying the AM-HM inequality leads to

$$\frac{1}{\cos^2 x} + \frac{1}{\cos^2 y} \ge \frac{4}{\cos^2 x + \cos^2 y} = \frac{4}{t}.$$

Also, from the AM-GM inequality, we have that

$$\begin{aligned}
\sin^2(x+y) &= \sin^2 x\cos^2 y + \cos^2 x\sin^2 y + 2\sin x\cos x\sin y\cos y \\
&\le \sin^2 x\cos^2 y + \cos^2 x\sin^2 y + 2|\sin x\cos x\sin y\cos y| \\
&\le \sin^2 x\cos^2 y + \cos^2 x\sin^2 y + \sin^2 x\cos^2 x + \sin^2 y\cos^2 y \\
&= (\cos^2 x + \cos^2 y)(\sin^2 x + \sin^2 y) = t(2-t).
\end{aligned}$$

Since

$$\frac{4}{t} + \frac{1}{t(2-t)} - 4 = \frac{4t^2 - 12t + 9}{t(2-t)} = \frac{(2t-3)^2}{t(2-t)} \ge 0,$$

this, along with the foregoing inequalities, yields the result. Equality occurs if and only if $x = y$ and $\cos^2 x = \cos^2 y = 3/4$, if and only if $x = y = \pm\pi/6$.

4458. *Proposed by Marian Cucoaneş and Marius Drăgan.*

Let a, b, c, d be the sides of a cyclic quadrilateral with circumradius R and lengths of diagonals d_1 and d_2. Prove that

$$\sum_{cyclic} \frac{a}{b+c+d-a} \geq \frac{4R}{\sqrt{d_1 d_2}}.$$

We received 5 submissions, of which 3 were correct and complete. We present the solution by Marie-Nicole Gras, lightly edited.

Denote by s the semiperimeter of the quadrilateral. In a cyclic quadrilateral we can make use of the well-known formulas

$$R = \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}} \quad \text{and} \quad d_1 d_2 = ac+bd.$$

The inequality we need to prove is thus equivalent to

$$\frac{1}{2}\cdot\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+\frac{d}{s-d}\right) \geq \sqrt{\frac{(ab+cd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}. \qquad (1)$$

In order to simplify our calculations, we let $x = s-a$, $y = s-b$, $z = s-c$ and $t = s-d$; then $x, y, z, t > 0$. Note that $x+y+z+t = 2s$. On the right hand side of (1), we have

$$\begin{aligned} ab+cd &= (s-x)(s-y)+(s-z)(s-t) \\ &= 2s^2 - sx - sy - sz - st + xy + zt \\ &= xy + zt, \end{aligned}$$

and similarly $ad+bc = xt+yz$. Thus, the expression under the square root becomes

$$\frac{(xy+zt)(xt+yz)}{xyzt} = \frac{x^2yt+xy^2z+yz^2t+xzt^2}{xyzt} = \frac{x}{z}+\frac{y}{t}+\frac{z}{x}+\frac{t}{y}.$$

Let $F = \frac{x}{z}+\frac{y}{t}+\frac{z}{x}+\frac{t}{y}$, so the right hand side of (1) is simply $\sqrt{F}$.

On the left hand side of (1), we have

$$\frac{a}{2(s-a)} = \frac{s-x}{2x} = \frac{-x+y+z+t}{4x} = -\frac{1}{4}+\frac{1}{4}\cdot\left(\frac{y}{x}+\frac{z}{x}+\frac{t}{x}\right),$$

and similarly for the remaining terms, so that the left hand side of (1) becomes

$$-1+\frac{1}{4}\cdot\left(\frac{y}{x}+\frac{z}{x}+\frac{t}{x}+\frac{x}{y}+\frac{z}{y}+\frac{t}{y}+\frac{x}{z}+\frac{y}{z}+\frac{t}{z}+\frac{x}{t}+\frac{y}{t}+\frac{z}{t}\right).$$

Moreover, we have that $\frac{y}{x}+\frac{x}{y} \geq 2$, $\frac{t}{x}+\frac{x}{t} \geq 2$, $\frac{z}{y}+\frac{y}{z} \geq 2$, $\frac{t}{z}+\frac{z}{t} \geq 2$, so in order to show (1) it suffices to prove that

$$-1+\frac{1}{4}(8+F) \geq \sqrt{F}.$$

Finally, observe that $F+8=(\sqrt{F}-2)^2+4+4\sqrt{F}\geq 4+4\sqrt{F}$ to conclude the proof.

4459. *Proposed by Leonard Giugiuc and Miguel Ochoa Sanchez.*

Let ABC be an isosceles triangle with $AB=AC$. For a point P on side AB let Q be a point of the extension of AC beyond C for which the midpoint N of PQ lies on the segment BC; similarly, for a point R on side AC let S be a point of the extension of AB beyond B for which the midpoint M of RS lies on the segment BC. Prove that

$$\frac{PQ}{RS}=\frac{\cos\angle RMN}{\cos\angle PNM}.$$

We received 11 submissions, all of which were correct; we will sample two of the variety of solutions.

Solution 1 is a composite of almost identical solutions submitted (independently) by Marie-Nicole Gras, C.R. Pranesachar, and Titu Zvonaru.

Define P' and Q' to be the feet of the perpendiculars from P and Q, respectively, to the line BC. Since $PN=NQ$, it follows that the right triangles PNP' and QNQ' are congruent and, hence, $PP'=QQ'$ and $P'N=Q'N$. The former implies that the right triangles $PP'B$ and $QQ'C$ are also congruent, because we have, in addition, $\angle PBP'=\angle ABC=\angle ACB=\angle QCQ'$. Thus $BP'=CQ'$; consequently, the translation that takes B to P' will take C to Q', whence $P'Q'=BC$. It follows that

$$\cos\angle PNM=\frac{P'N}{PN}=\frac{NQ'}{NQ}=\frac{P'Q'}{PQ}=\frac{BC}{PQ},$$

and, therefore,

$$PQ\cos\angle PNM=BC.$$

Similarly, we obtain $RS\cos\angle RMN=BC$, and the desired conclusion follows.

Solution 2, by Walther Janous.

We place the origin of a vector space at the midpoint of BC and denote the vector from the origin to a generic point X by $\vec{X}$. Without loss of generality we set

$$\vec{A}=(0,t),\quad \vec{B}=(-1,0),\quad \vec{C}=(1,0),$$

where $t>0$. We are given a point P on side AB, which means that

$$P=\lambda\vec{A}+(1-\lambda)\vec{B}=(\lambda-1,t\lambda)\quad\text{and}\quad Q=\vec{C}+(\vec{C}-\vec{A})s=(s+1,-st),$$

for $0<\lambda<1$ and s to be determined. Specifically, $\vec{N}=\frac{1}{2}(\vec{P}+\vec{Q})=\left(\frac{s+\lambda}{2},\ \frac{t(\lambda-s)}{2}\right)$, so that N is on BC if and only if $s=\lambda$. Consequently, we have

$$N=(\lambda,0)\quad\text{and}\quad \vec{P}-\vec{N}=(-1,t\lambda).$$

Similarly, when $\vec{R} = \mu\vec{A} + (1-\mu)\vec{C} = (1-\mu, t\mu)$, with $0 < \mu < 1$, we have

$$\vec{M} = (-\mu, 0),$$

so that

$$\vec{R} - \vec{M} = (1, t\mu) \quad \text{and} \quad \vec{M} - \vec{N} = (-\lambda - \mu,\ 0).$$

Finally, we must verify that

$$|\vec{P} - \vec{Q}| \frac{(\vec{P} - \vec{N}) \cdot (\vec{M} - \vec{N})}{|\vec{P} - \vec{N}| \cdot |\vec{M} - \vec{N}|} = |\vec{R} - \vec{S}| \frac{(\vec{R} - \vec{M}) \cdot (\vec{N} - \vec{M})}{|\vec{R} - \vec{M}| \cdot |\vec{N} - \vec{M}|},$$

which is easy because

$$|\vec{P} - \vec{Q}| = 2|\vec{P} - \vec{N}|, \quad |\vec{R} - \vec{S}| = 2|\vec{R} - \vec{M}|, \quad |\vec{M} - \vec{N}| = |\vec{N} - \vec{M}| \neq 0,$$

and

$$(\vec{P} - \vec{N}) \cdot (\vec{M} - \vec{N}) = (\vec{R} - \vec{M}) \cdot (\vec{N} - \vec{M}) = \lambda + \mu.$$

Editor's comments. Note that the restriction of the points P and R to the sides AB and AC can be omitted — Janous's argument shows that the result continues to hold starting with any point P on the line AB and any point $R \neq Q$ on the line AC (because λ and μ are free to be assigned any real values as long as $\lambda + \mu \neq 0$).

4460. *Proposed by Gantumur Choijilsuren and Leonard Giugiuc.*

Let $(x_n)_{n\geq 1}$ be a sequence of real numbers such that $(3x_{n+1} - 2x_n)_{n\geq 1}$ is convergent. Show that $(x_n)_{n\geq 1}$ is convergent.

We received 16 submissions of which 15 were correct. We present the solution by Ángel Plaza.

Let $y_n = 3x_{n+1} - 2x_n$. Then

$$\begin{aligned} x_n &= \frac{1}{3}y_{n-1} + \frac{2}{3}x_{n-1} \\ &= \frac{1}{3}y_{n-1} + \frac{2}{9}y_{n-2} + \frac{4}{9}x_{n-2} \\ &= \frac{1}{3}y_{n-1} + \frac{2}{9}y_{n-2} + \frac{4}{27}y_{n-3} + \frac{8}{27}x_{n-3} \\ &\cdots \\ &= \sum_{k=0}^{n-2} \frac{1}{3}\left(\frac{2}{3}\right)^k y_{n-1-k} + \left(\frac{2}{3}\right)^{n-1} x_1. \end{aligned}$$

Since

$$\sum_{k=0}^{n-2} \frac{1}{3}\left(\frac{2}{3}\right)^k = \frac{\frac{1}{3} - \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}}{1 - \frac{2}{3}} \to 1$$

and $(y_n)_{n\geq 1}$ converges, then $(x_n)_{n\geq 1}$ converges as well.