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EDITORIAL

As you likely noticed when you opened this issue, ***Crux*** is boasting a new cover. In the process of moving online, the journal lost its iconic purple cover and I'm very excited to have a new beautiful design to represent ***Crux*** . The cover was designed by Rebekah Brackett and you can find more of her work on her website `https://www.rebekahbrackettart.com/`.

Rebekah is one of the people that provided inspiration and guidance in my own journey to understand and embrace First People's principles of knowing and learning. As we were organizing Fraser Valley Math Education Sq'ep (Sq'ep meaning a meeting, gathering in Halq'eméylem), we explored the connections between math, language, art, land. With the help of Tasheena Boulier and her family, consisting of the few last fluent speakers of Halq'eméylem, we created a counting booklet featuring number words in Halq'eméylem and images of the lands of the Sto:lo people. To me, number systems are fascinating as they offer a unique insight into the culture. For example, Sto:lo have different counting words depending on what is being counted, highlighting the fundamental differences between how they treat objects, animals and people. Take a look at the booklet, explore the numbers and enjoy the views of the beautiful Fraser Valley:
`https://www.ufv.ca/media/assets/mathematics/halq-booklet-j.pdf`

Pandemic has offered us an opportunity to see the importance of human connections. So where do we start in math? Veselin Jungic and I write more about our journeys in exploring Indigenous ways of knowing in mathematics in the March edition of CMS Notes: `https://cms.math.ca/publications/cms-notes/`

Let us learn together.

Kseniya Garaschuk

MathemAttic

No. 22

The problems in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 30, 2021**.

MA106. Suppose

$$N = 1 + 11 + 101 + 1001 + 10001 + \cdots + 1000\cdots01,$$

where there are 50 zeros in the last term. When N is written as a single integer in decimal form, find the sum of its digits.

MA107. A wooden cube is painted red on five of its six sides and then cut into identical small cubes, of which 52 have exactly two red sides. How many small cubes have no red sides?

MA108. Suppose that a, b, c and d are positive integers that satisfy the equations

$$ab + cd = 38, \quad ac + bd = 34, \quad ad + bc = 43.$$

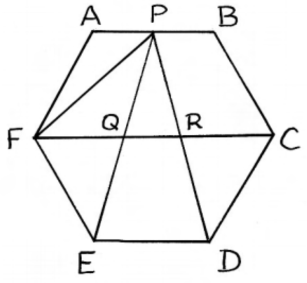
What is the value of $a + b + c + d$?

MA109. Ten equal spheres are stacked to form a regular tetrahedron. How many points of contact are there between the spheres?

MA110. In the figure, $ABCDEF$ is a regular hexagon and P is the midpoint of AB.

Find the ratio

$$\frac{Area(DEQR)}{Area(FPQ)}.$$



Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA106. Supposer que

$$N = 1 + 11 + 101 + 1001 + 10001 + \cdots + 1000\cdots01,$$

où on trouve 50 zéros dans le dernier terme. Si N est écrit en forme décimale, déterminer la somme de ses chiffres.

MA107. Un cube en bois est peint rouge sur cinq de ses six côtés et puis taillé en petits cubes identiques, dont 52 ont exactement deux faces rouges. Déterminer le nombre de petits cubes ayant aucune face rouge.

MA108. Supposer que a, b, c et d sont des entiers positifs tels que

$$ab + cd = 38, \quad ac + bd = 34, \quad ad + bc = 43.$$

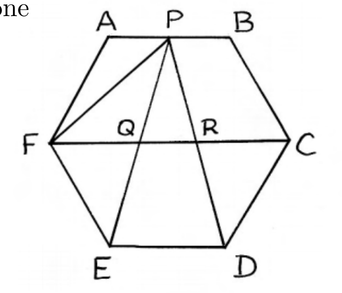
Déterminer la valeur de $a + b + c + d$.

MA109. Dix sphères de même rayon sont empilées pour former un tétraèdre. Déterminer le nombre de points de contact entre les sphères.

MA110. Dans la figure, $ABCDEF$ est un hexagone régulier et P est le point milieu de AB.

Déterminer le ratio

$$\frac{Area(DEQR)}{Area(FPQ)}.$$



MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(7), p. 285–286.

MA81. Find the sum of all positive integers smaller than 1260 which are not divisible by 2 and not divisible by 3.

Originally modified problem 8 from the 2018 Alberta High School Mathematics Competition.

We received 8 submissions, all of which were correct and complete. We present the solution by Alin Popescu and Daniel Văcaru, modified by the editor.

The sum of the first 1260 natural numbers is $\frac{(1+1260)\cdot 1260}{2} = 1261 \cdot 630$.

We want to remove the sum of the natural numbers smaller than 1260 that are divisible by 2. This sum is

$$2 \cdot 1 + 2 \cdot 2 + \ldots + 2 \cdot 630 = 2\,(1 + 2 + \ldots + 630) = 2\frac{630 \cdot 631}{2} = 630 \cdot 631.$$

We also want to remove the sum of the odd numbers under 1260 that are divisible by 3, i.e. numbers of the form $3\,(2i+1)$ for $i \leq k \in \mathbb{N}$ where the maximum such number is $3\,(2k+1) \leq 1260$. We find $2k+1 \leq 420$ or $2k \leq 419$, and since k is an integer, $k = 209$. Thus the sum of the numbers $3\,(2i+1)$ under 1260 is

$$\begin{aligned}\sum\nolimits_{i=0}^{209} 3\,(2i+1) &= \sum\nolimits_{i=0}^{209} 6i + \sum\nolimits_{i=0}^{209} 3\\ &= 6\,(1 + 2 + \ldots + 209) + 3 \cdot 210\\ &= 6\frac{209 \cdot 210}{2} + 3 \cdot 210\\ &= 630\,(209 + 1)\\ &= 630 \cdot 210.\end{aligned}$$

Thus, the desired sum is

$$1261 \cdot 630 - 630 \cdot 631 - 630 \cdot 210 = 630 \cdot (1261 - 631 - 210) = 630 \cdot 420 = 264600.$$

MA82. Let $a_n = n^2 + 2n + 50$, $n = 1, 2, \ldots$. Let d_n be the largest positive integer that is a divisor of both a_n and a_{n+1}. Find the maximum value of d_n, $n = 1, 2, \ldots$

Originally problem 12 from the 2018 Alberta High School Mathematics Competition.

We received 5 submissions, four of which were correct and complete. We present the solution by Corneliu Mănescu-Avram, lightly edited.

Since d_n divides both a_n and a_{n+1}, it also divides

$$a_{n+1} - a_n = 2n + 3$$

and

$$a_n - (2n+3) = n^2 + 47.$$

Therefore d_n divides

$$4(n^2+47) - (2n-3)(2n+3) = (4n^2+188) - (4n^2-9) = 197.$$

Since 197 is prime, $d_n \in \{1, 197\}$ for all n. For $n = 97$ we have

$$a_{97} = 49 \cdot 197, \qquad a_{98} = 50 \cdot 197.$$

Therefore the maximum value of d_n is 197.

MA83. Prove that the numbers 26^n and $26^n + 2^n$ have the same number of digits, for any non-negative integer n.

Originally problem 3 from Part II of the 2018 Alberta High School Mathematics Competition.

We received 2 solutions. We present the one by Corneliu Mănescu-Avram, modified by the editor.

The statement is easily checked for $n = 1, 2$. Let $n \geq 3$ and suppose that there exists a positive integer m such that

$$26^n < 10^m \leq 26^n + 2^n.$$

Since $n \geq 3$ and $26^3 > 10^4$, we must have $m \geq n+2$. Dividing by 2^n we obtain

$$13^n < 2^{m-n}5^m \leq 13^n + 1,$$

where $2^{m-n}5^m$ is an integer divisible by 4. Since $13^n + 1 \equiv 2 \pmod 4$, we arrive at a contradiction.

MA84. The area of the trapezoid $ABCD$ with $AB \parallel CD$, $AD \perp AB$ and $AB = 3CD$ is equal to 4. A circle inside the trapezoid is tangent to all of its sides. Find the radius of the circle.

Originally problem 15 from the 2016 Alberta High School Mathematics Competition.

We received 11 submissions, 10 of which are correct. We present the solution by Alin Popescu and Daniel Văcaru, modified by the editor.

We take $CD = b$. It follows that $AB = 3CD = 3b$. The area of the trapezoid is

$$4 = \frac{(CD + AB) \cdot h}{2} = \frac{4b \cdot h}{2},$$

which gives $bh = 2$ or

$$h = \frac{2}{b}. \tag{1}$$

However, $AD \perp AB \Rightarrow h = AD$. According to the Pitot theorem, $AD + CB = AB + CD$ which gives $h + CB = 4b$ or $CB = 4b - h$. Let $CP \perp AB$, where $\{P\} = AB \cap CP$. It follows that $AD \parallel CP$, $AB \parallel CD$ implies $AD = CP$ and $AP = CD$, so $PB = 2b$.

In the triangle CPB the angle $\angle CPB = 90^\circ$. We use the Pythagorean theorem to write $CB^2 = PB^2 + PC^2$ or $(4b - h)^2 = 4b^2 + h^2$, which gives $16b^2 - 8bh = 4b^2$ or $4b\,(4b - 2h) = 4b^2$ or $4b - 2h = b$, which finally results in

$$2h = 3b. \tag{2}$$

Together, (1) and (2) show that $2h = \frac{4}{b}$ and $3b = \frac{4}{b}$, so $b = \frac{2}{\sqrt{3}}$ and $h = \sqrt{3}$. Then the diameter of the circle is $\sqrt{3}$ and the radius is $\frac{\sqrt{3}}{2}$.

MA85. A collection of items weighing 3, 4 or 5 kg has a total weight of 120 kg. Prove that there is a subcollection of the items weighing exactly 60 kg.

Originally problem 4 from Part II of the 2018 Alberta High School Mathematics Competition.

We received 2 submissions, neither of which was fully correct and complete. You can fine the official solution at

`https://drive.google.com/file/d/0B5b6n_Nz71-rRVVxa1g5ZlBTOEdvMXNCLUFVbTczSTRWSWJr/view`

TEACHING PROBLEMS

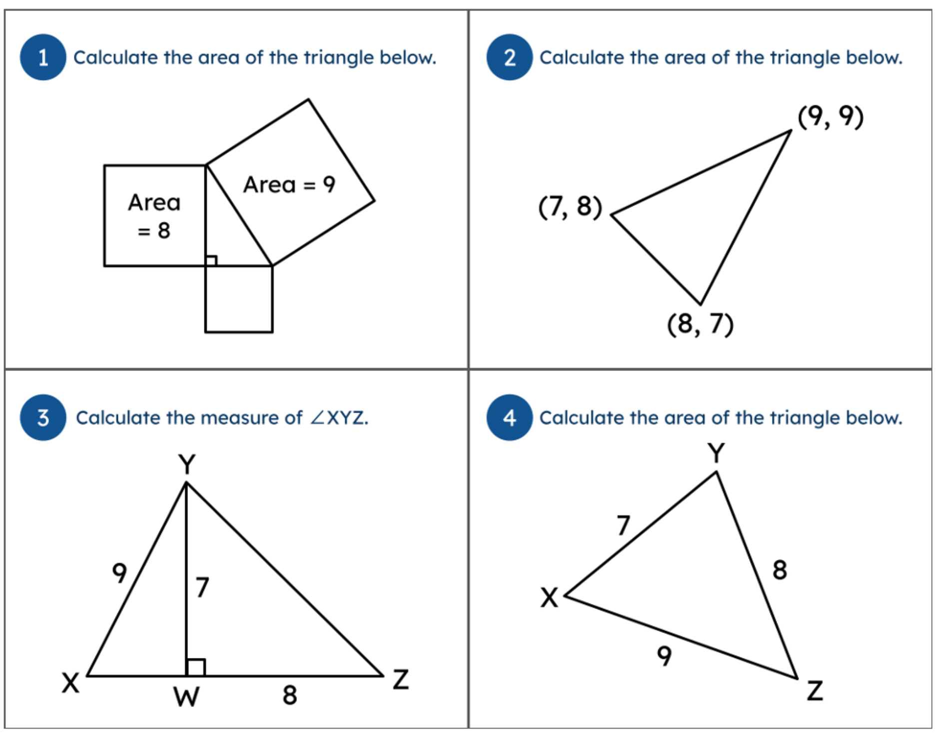
No. 13

Erick Lee

Four Triangles: An Example of Interleaved Practice

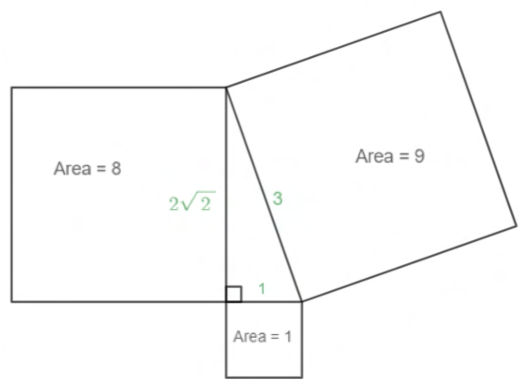
Triangles, it seems, are everywhere. You see them daily through art, architecture and nature. Triangles are common elements of school mathematics from the initial naming and categorizing of two-dimensional shapes through to deeper excursions into the realms of geometry and trigonometry. Students spend much time determining areas, side lengths and angles of triangles They should have a toolbox of strategies and techniques to solve a wide range of problems involving this basic figure; however, students often find problems with triangles to be challenging.

As an example of this unexpected complexity, I've gathered four problems involving triangles in the grid below. These questions share some commonalities on the surface. They all involve triangles and feature numbers that are alike. In order to solve these problems, students must first understand how the given information will inform the selection of a strategy. Once they have identified a strategy, they must then carry it out to determine a solution.



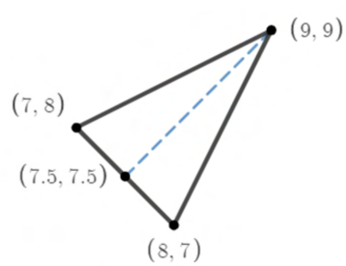
Solving the Problems

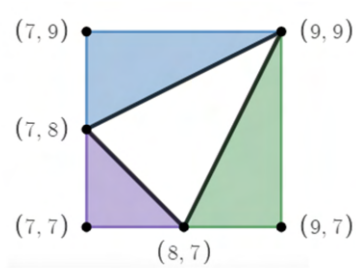
Question 1. When students see a question relating the areas of squares to the sides of a right triangle, they should think of the Pythagorean Theorem. This problem asks students to find the area of the central triangle given the areas of squares on two of its sides. The most straightforward way to solve this problem is to find the measures of the base and height of the triangle and then apply the triangle area formula. The Pythagorean Theorem states that the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares on the other two sides. The unmarked area of the square whose side is the base of the triangle is 1 since $8 + 1 = 9$. The length of the base and height are the square roots of the areas of the squares of those sides. The height of the triangle is $2\sqrt{2}$ and the base is 1. The area is therefore $\sqrt{2}$.



Question 2. This question asks students to determine the area of a triangle given the coordinates of each of its vertices. One method students might use to determine this area is to find an altitude and base of the triangle using coordinate geometry and the distance formula. If students choose the line segment between (7,8) and (8,7) as the base, the calculations will be simplified compared to other choices for the base.

A less complicated and perhaps more mathematically elegant way to find the area of this triangle is to find the area of the rectangle that encloses it and then subtract the areas of the shaded triangles shown on the right.





The area of the encompassing square is 4 square units. From this area we subtract the areas of the three shaded triangles.

$$\begin{aligned}
Area_{blue\triangle} &= \frac{1}{2} \cdot 1 \cdot 2 = 1, \\
Area_{green\triangle} &= \frac{1}{2} \cdot 2 \cdot 1 = 1, \\
Area_{purple\triangle} &= \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}, \\
Area_{\triangle} &= Area_{\square} - Area_{blue\triangle} - Area_{green\triangle} - Area_{purple\triangle} = \frac{3}{2}.
\end{aligned}$$

An alternative way to find the area of any simple polygon with vertices identified by coordinate points is to use Gauss' Area Formula, more commonly known as the Shoelace Formula. It is rarely taught in Canadian secondary schools except as an enrichment activity. If you haven't seen a description of this formula in the past, I recommend you check out James Tanton's description in his Cool Math Essay from June 2014.

Question 3. When students see right triangles and are asked about angles, they should be reminded of inverse trigonometric ratios. In this question, they can calculate the measure of $\angle XYZ$ by calculating and then adding $\angle XYW$ and $\angle WYZ$. As they have a hypotenuse and an adjacent leg of $\angle XYW$, they will need inverse cosine to find this angle. With $\angle WYZ$, students have an adjacent leg and an opposite leg and hence will need inverse tangent.

$$\begin{aligned}
\angle XYW &= \cos^{-1}\left(\frac{7}{9}\right) \approx 38.9^\circ, \\
\angle WYZ &= \tan^{-1}\left(\frac{8}{7}\right) \approx 88.5^\circ, \\
\angle XYZ &= \angle XYW + \angle WYZ \approx 127.4^\circ.
\end{aligned}$$

Question 4. Finding the sides and angles of a non-right triangle should lead students to consider the Sine and Cosine Laws. In this case, students can use the Cosine Law to find one of the angles. They can then use two sides and the contained angle to find the area of the triangle:

$$\cos X = \frac{y^2 + z^2 - x^2}{2yz} = \frac{9^2 + 7^2 - 8^2}{2 \cdot 9 \cdot 7} = \frac{66}{126},$$

so then $X = \cos^{-1}\left(\frac{66}{126}\right) \approx 58.4^\circ$.

Now that this angle is known, students can use this angle to find the area of the triangle:

$$Area = \frac{1}{2} yz \sin X = \frac{1}{2} \cdot 9 \cdot 7 \cdot \sin(58.4^\circ) \approx 26.8.$$

An alternate solution method is using Heron's Formula. This formula is rarely taught in Canadian secondary schools. The formula calculates the area of a triangle given the lengths of its three sides. With this formula, there is no need to calculate any other lengths or angles.

$$Area = \sqrt{s(s-a)(s-b)(s-b)},$$

where s is semi-perimeter of the triangle and a, b, c are side lengths. The semi-perimeter is half of the sum of the lengths of the three sides. For the given triangle this is $(7+8+9)/2 = 12$, so we have

$$Area = \sqrt{12(12-7)(12-8)(12-9)} = 12\sqrt{5} \approx 26.8.$$

The Benefits of Interleaved Practice

I created the set of problems shown at the beginning of this article to give students "interleaved" practice instead of "blocked" practice. Interleaved and blocked practice are two different methods of practicing newly acquired skills.

In "blocked" practice, a block of practice questions focused on a single skill are assigned. Blocked practice is the type of practice that is often found in textbooks. The practice questions are often subtle variations of examples that were previously demonstrated. While blocked practice gives students lots of practice on a targeted skill, it doesn't help them become better problem solvers. Because the strategy required to solve the problem is known up front, students are never challenged to analyze a problem to determine what solution strategy is needed. Students never learn to recognize the characteristics of a problem that might suggest a certain strategy.

During "interleaved" practice, students are given practice problems which require numerous different skills and strategies to solve. Since students don't know up front what strategy might be needed, they must consider various problem solving strategies and select the one that would be most useful for a given situation. This helps students become more flexible problem solvers through consideration of a wider range of strategies instead of developing an over-reliance on a specific formula or strategy. For example, when students first learn the formula for a definite integral, they sometimes jump to applying the formula to every situation. They might even rush to apply an integral to a rectangular region when its area could be calculated much more simply with length times width.

While both interleaved and blocked methods of practice are useful, many of the resources that are provided to classrooms contain predominantly blocked practice. In a recent survey of widely used US seventh grade mathematics textbooks, Dr. Doug Rohrer, et. al. found that there were more than eight blocked problems for every interleaved problem in these textbooks. Even the review assignments in each textbook were moderately blocked (see Rohrer, D., Dedrick, R. F., & Hartwig, M. K. (2020). The scarcity of interleaved practice in mathematics textbooks. *Educational Psychology Review*, 32, 873-883). While blocked practice appears to be predominant in textbooks, interleaved practice has shown significant benefits. Interested readers can see a recent published study by Rohrer, D. et al, A randomized controlled trial of interleaved mathematics practice.

Teachers that are interested in including more interleaved practice with their students don't necessarily need a new textbook. Teachers can quickly create an interleaved assignment by combining a selection of problems from several different sections throughout the textbook.

For more information and tips about interleaved practice, teachers can visit the site `https://www.retrievalpractice.org/interleaving`.

Same Surface, Different Depth

The four triangle problems at the beginning of this article are presented in a math routine useful for interleaved practice called ÒSame Surface, Different DepthÓ or SSDD. Each set of four problems is related in shape, appearance or context. While they may look similar on the surface, underneath, they require different problem solving strategies to solve.

UK mathematician Craig Barton has created a website where educators from around the world can create and share problems in this format: check it out at `https://ssddproblems.com/`. Hundreds of these problem sets have been shared which makes it easy to find one that is suitable for any secondary school mathematics outcome.

If you create an "SSDD" problem, consider sharing it with the wider mathematics education community by contributing it to the website. Criteria for submissions are included at `https://ssddproblems.com/submission-guidelines/`

Erick Lee is a Mathematics Support Consultant for the Halifax Regional Centre for Education in Dartmouth, NS. Erick blogs at `https://pbbmath.weebly.com/` and can be reached via email at `elee@hrce.ca` and on Twitter at `@TheErickLee`.

OLYMPIAD CORNER

No. 390

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 30, 2021**.

OC516. Pasha placed numbers from 1 to 100 in the cells of the square 10×10, each number exactly once. After that, Dima considered all sorts of squares, with the sides going along the grid lines, consisting of more than one cell, and painted in green the largest number in each such square (one number could be coloured many times). Is it possible that all two-digit numbers are painted green?

OC517. Denote by $\mathbb{N}$ the set of positive integers $1, 2, 3, \dots$ Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $n! + f(m)!$ divides $f(n)! + f(m!)$ for all m, $n\in\mathbb{N}$.

OC518. In a triangle ABC with $AB \neq AC$ let M be the midpoint of AB, let K be the midpoint of the arc BAC in the circumcircle of ABC, and let the perpendicular bisector of AC meet the bisector of the angle BAC at P. Prove that A, M, K, P are concyclic.

OC519. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence:

$$x, x+1, x+2, x+3, \dots$$

OC520. Larry and Rob are two robots travelling in one car from Argovia to Zillis. Both robots have control over the steering and steer according to the following algorithm: Larry makes a 90° left turn after every l kilometer driving from start; Rob makes a 90° right turn after every r kilometer driving from start, where l and r are relatively prime positive integers. In the event of both turns occurring simultaneously, the car will keep going without changing direction. Assume that the ground is flat and the car can move in any direction. Let the car start from Argovia facing towards Zillis. For which choices of the pair (l, r) is the car guaranteed to reach Zillis, regardless of how far it is from Argovia?

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC516. Pasha inscrit les nombres de 1 à 100 dans les cellules d'un grillage de taille 10 par 10, chacun exactement une fois. Par la suite, Dima considère tous les carrés contenant plus d'une cellule et dont les côtés suivent l'alignement du grillage, et puis elle colore en vert le plus grand nombre dans chaque tel carré, il étant entendu qu'un nombre pourrait bien être coloré plus d'une fois. Est-ce possible que tous les nombres à deux chiffres soient ainsi colorés vert ?

OC517. Soit $\mathbb{N}$ l'ensemble des entiers positifs $1, 2, 3, \dots$ Déterminer toutes les fonctions $f : \mathbb{N} \to \mathbb{N}$ telles que $n! + f(m)!$ divise $f(n)! + f(m!)$ pour tout m, $n \in \mathbb{N}$.

OC518. Soit M le mi point de AB dans un triangle ABC tel que $AB \neq AC$. Soit aussi K le mi point de l'arc BAC du cercle circonscrit de ABC. Enfin, supposons que la bissectrice perpendiculaire de AC rencontre la bissectrice de l'angle BAC en P. Démontrer que A, M, K, P sont cocycliques.

OC519. Démontrer que le nombre x est rationnel si et seulement si trois termes distincts en progression géométrique peuvent être choisis dans la suite suivante:

$$x, x+1, x+2, x+3, \dots$$

OC520. Laurent et Rolland sont deux robots se déplaçant en une même voiture, allant de Argovia à Zillis. Ces deux robots ont contrôle du volant et pilotent la voiture, selon les règles suivantes. À chaque l kilomètres depuis le départ, Laurent tourne à gauche par $90°$, tandis que Rolland tourne à droite par $90°$ chaque r kilomètres depuis le départ, où l et r sont des entiers relativement premiers ; advenant que les deux voudraient tourner au même moment, la voiture ne change pas de direction. Supposons que le terrain est plat et que la voiture peut se déplacer en toute direction. La voiture débute sa randonnée à Argovia, pointant directement vers Zillis. Pour quelles valeurs de la paire (l, r) est-on assuré que les robots vont se rendre à Zillis, quelle que soit sa distance de Argovia ?

OLYMPIAD CORNER SOLUTIONS

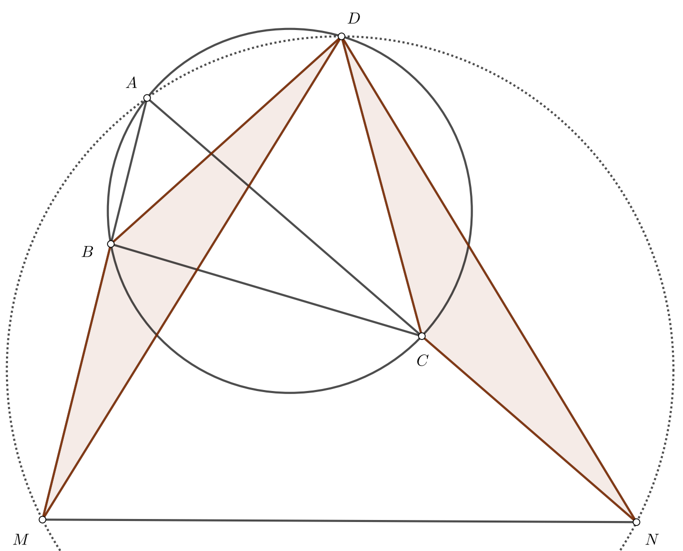
Statements of the problems in this section originally appear in 2020: 46(7), p. 294–295.

OC491. Let ABC be a triangle such that $AB \neq AC$. Prove that there exists a point $D \neq A$ on its circumcircle satisfying the following property: For any points M, N outside the circumcircle on the rays AB and AC, respectively, satisfying $BM = CN$, the circumcircle of AMN passes through D.

Originally problem 2, Grade 11-12, Day 1, Final Round of 2017 Germany Math Olympiad.

We received 12 *submissions, all correct. We present* 2 *solutions.*

Solution 1*, by UCLan Cyprus Problem Solving Group.*



Let D be the point of intersection of the perpendicular bisector of BC with the arc of BC containing A, of the circumcircle of the triangle $\triangle ABC$. Note that the definition of D is independent of the points M and N.

We claim that the triangles $\triangle DBM$ and $\triangle DCN$ are equal. Indeed it is given that $BM = CN$. We also have $DB = DC$ since D is on the perpendicular bisector of BC. Finally we have

$$\angle DBM = 180^\circ - \angle DBA = 180^\circ - \angle DCA = \angle DCN\,,$$

as D is on the circumcircle of the triangle $\triangle ABC$. (D and A are on the same arc of BC, while M, N are outside of the circumcircle on the rays AB and AC.)

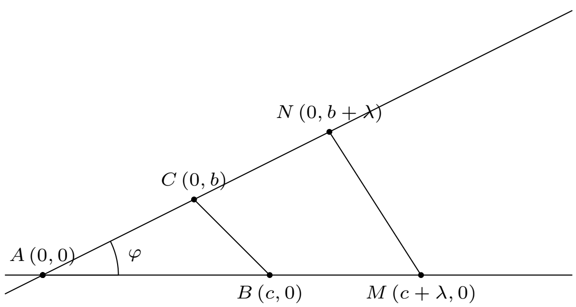
By the equality of the above triangles, we get that $\angle DMA = \angle DNA$, showing that D is on the circumcircle of the triangle $\triangle AMN$.

Solution 2, by Miguel Amengual Covas.

Let $AB = c$, $CA = b$ and $\angle CAB = \varphi$.

We consider a Cartesian coordinate system with the unit of measurement the same along both coordinate axes, the x axis along the side AB of $\triangle ABC$ and the y axis along the side CA.

The coordinates of A then are $(0, 0)$, the coordinates of B are $(c, 0)$ and those of C are $(0, b)$. Therefore M will have coordinates $(c + \lambda, 0)$ and N is at $(0, b + \lambda)$, where λ is a positive real number.



Since the general form of the equation of a circle passing through the origin is

$$x^2 + y^2 + 2xy\cos\varphi - Px - Qy = 0,$$

where P and Q are real numbers, the equations of the circumcircles of $\triangle ABC$ and AMN are

$$x^2 + y^2 + 2xy\cos\varphi - cx - by = 0 \tag{1}$$

and

$$x^2 + y^2 + 2xy\cos\varphi - (c + \lambda)\,x - (b + \lambda)\,y = 0, \tag{2}$$

respectively.

Solving simultaneously (1) and (2), we find $x = 0, y = 0$ and

$$x = \frac{-b + c}{2\,(1 - \cos\varphi)},\ \ y = \frac{b - c}{2\,(1 - \cos\varphi)}.$$

Hence

$$D\left(\frac{-b + c}{2\,(1 - \cos\varphi)}, \frac{b - c}{2\,(1 - \cos\varphi)}\right),$$

which does not depends of λ, is the required point.

OC492. Let ABC be a triangle with $AB = AC$ and let I be its incenter. Let Γ be the circumcircle of ABC. Lines BI and CI intersect Γ in two new points M and N respectively. Let D be another point on Γ lying on arc BC not containing A, and let E, F be the intersections of AD with BI and CI, respectively. Let P, Q be the intersections of DM with CI and of DN with BI respectively.

(i) Prove that D, I, P, Q lie on the same circle Ω.

(ii) Prove that lines CE and BF intersect on Ω.

Originally problem 6, Final Round of 2018 Italy Math Olympiad.

We received 4 correct submissions. We present the solution by UCLan Cyprus Problem Solving Group.

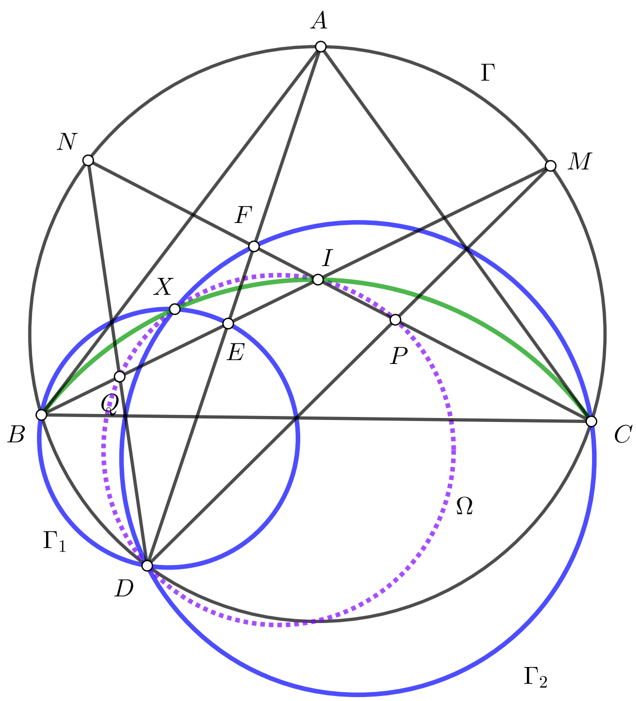
(i) We have

$$\angle QIP = \angle BIC = 180^\circ - \frac{\hat{B} + \hat{C}}{2}$$

and

$$\angle QDP = \angle NDM = \angle NDA + \angle ADM = \angle NCA + \angle ABM = \frac{\hat{B} + \hat{C}}{2}.$$

So (i) follows. Note that this holds even if the triangle $\triangle ABC$ is not isosceles.



(ii) Let Γ_1 and Γ_2 be the circumcircles of triangles $\triangle BED$ and $\triangle CFD$ respectively. Let $X \neq D$ be the other point of intersection of Γ_1 and Γ_2.

We have

$$\begin{aligned}
\angle BXC &= \angle BXD + \angle DXC \\
&= \angle BED + \angle DFC \\
&= (\angle EDM + \angle EMD) + (\angle FAC + \angle FCA) \\
&= \angle ADM + \angle BMD + \angle DAC + \angle NCA \\
&= \frac{\hat{B}}{2} + (\angle BAD + \angle DAC) + \frac{\hat{C}}{2} \\
&= \hat{A} + \frac{\hat{B}+\hat{C}}{2} = 180^\circ - \frac{\hat{B}+\hat{C}}{2} \\
&= \angle BIC
\end{aligned}$$

It follows that X belongs on the circumcircle of $\triangle BIC$.

We now have

$$\angle BXF = \angle BXC + \angle CXF = \angle BIC + \angle CDF = 180^\circ - \frac{\hat{B}+\hat{C}}{2} + \hat{B} = 180^\circ\,,$$

where here, we have for the first time used that the triangle $\triangle ABC$ is isosceles.

Thus X belongs on BF. Furthermore, we also have

$$\angle BXE = 180^\circ - \angle BDE = 180^\circ - \hat{C} = \angle BXC\,,$$

as B, X, I, C are concyclic. So X belongs on EC as well.

So to complete the proof it remains to show that $X \in \Omega$. To this end, it is enough to show that $\angle DXI + \angle IPD = 180^\circ$.

We have

$$\angle DXI = \angle DXC + \angle CXI = \angle DXE + \angle CBI = \angle DBE + \frac{\hat{B}}{2}\,.$$

So

$$\begin{aligned}
\angle DXI + \angle IPD &= \angle DBE + \angle IPD + \frac{\hat{B}}{2} \\
&= 360^\circ - \angle BIP - \angle BDP + \frac{\hat{B}}{2} \\
&= 360^\circ - (180^\circ - \hat{B}) - (\angle BDA + \angle ADM) + \frac{\hat{B}}{2} = 180^\circ\,,
\end{aligned}$$

as $\angle BDA = \angle BCA = \hat{B}$ and $\angle ADM = \angle ABM = \hat{B}/2$.

The result follows.

Editor's Comment. Sergey Sadov also proved that the first part holds true without the assumption that $AB = AC$.

OC493. Let a, b be real numbers such that $a < b$ and let $f : (a, b) \to \mathbb{R}$ be a function such that the functions $g : (a, b) \to \mathbb{R}$, $g(x) = (x - a)f(x)$ and $h : (a, b) \to \mathbb{R}$, $h(x) = (x - b)f(x)$ are increasing. Prove that the function f is continuous on (a, b).

Originally problem 4, Grade 11, District Round of 2018 Romania Math Olympiad.

We received 10 *submissions. We present the solution by Oliver Geupel.*

It is enough to show that, for every $x_0 \in (a, b)$, it holds

$$\lim_{x \nearrow x_0} f(x) = f(x_0) \qquad \text{and} \qquad \lim_{x \searrow x_0} f(x) = f(x_0). \tag{1}$$

Let $a < x < x_0 < b$. By the monotonicity of g and h, we have

$$\frac{b - x_0}{b - x} \cdot f(x_0) \leq f(x) \leq \frac{x_0 - a}{x - a} \cdot f(x_0). \tag{2}$$

Both the lower and the upper bound in (2) tend to $f(x_0)$ as $x \nearrow x_0$. Hence $f(x)$ tends to $f(x_0)$, which proves the first limit (1). The second limit (1) is analogous, using the similar relation

$$\frac{x_0 - a}{x - a} \cdot f(x_0) \leq f(x) \leq \frac{b - x_0}{b - x} \cdot f(x_0).$$

which holds for $a < x_0 < x < b$.

OC494. Let n and q be two natural numbers, $n \geq 2$, $q \geq 2$ and $q \not\equiv 1 \pmod 4$ and let K be a finite field having exactly q elements. Prove that for every $a \in K$ there exist $x, y \in K$ such that $a = x^{2^n} + y^{2^n}$.

Originally problem 4, Grade 12, District Round of 2018 Romania Math Olympiad.

We received 5 *submissions. We present the solution by Corneliu Avram Manescu.*

Let p be the characteristic of K. Then, p is a prime number and $q = p^\alpha$, where α is a positive integer. From $q \not\equiv 1 \pmod 4$, we deduce $p \not\equiv 1 \pmod 4$, therefore $p = 2$ or $p \equiv 3 \pmod 4$ and in this last case α is odd.

If $p = 2$ and $x, y \in K$ such that $x^{2^n} = y^{2^n}$, then $x = y = 0$ or $(xy^{-1})^{2^n} = 1$ for $y \neq 0$. Then, $(xy^{-1})^{2^\alpha - 1} = (xy^{-1})^{q-1} = 1$. Since $(2^n, 2^\alpha - 1) = 1$, we get that $xy^{-1} = 1$, whence $x = y$. Consequently, the function $f : K \to K$, $f(x) = x^{2^n}$ is injective, hence surjective. If $a \in K$, then there exists $x \in K$ such that $a = f(x)$, therefore $a = x^{2^n} + 0^{2^n}$.

If $p \equiv 3 \pmod 4$ and α is odd, then $q \equiv 3 \pmod 4$, i.e. $q = 4k + 3$, where k is a natural number. Define $g : K^* \to K^*$, $g(x) = x^{2^n}$ and take $x, y \in K$ for which $g(x) = g(y)$. Then

$$(xy^{-1})^{2^n} = 1,\ (xy^{-1})^{4k+2} = (xy^{-1})^{q-1} = 1$$

and since $(2^n, 4k+2) = 2$, we get $(xy^{-1})^2 = 1$. Hence, $y = \pm x$. Since $1 \neq -1$, we deduce that the image of the function g has exactly $\dfrac{q-1}{2}$ elements. Define

$$K_n = \{x^{2^n} \mid x \in K\} = \{0\} \cup \operatorname{Im} g.$$

Then $|K_n| = 1 + \dfrac{q-1}{2} = \dfrac{q+1}{2}$.

If $a \in K$, then we also have $|a - K_n| = |K_n| = \dfrac{q+1}{2}$ and so, by the Pigeonhole Principle K_n and $a - K_n$ have an element in common. So, there exists $u, v \in K_n$ such that $u = a - v$. Since $u = x^{2^n}$ for some $x \in K$ and $v = y^{2^n}$ for some $y \in K$, we conclude that $a = x^{2^n} + y^{2^n}$, where $x, y \in K$.

OC495. A box contains 2017 balls. On each ball is written exactly one integer. We randomly select two balls with replacement from the box and add the numbers written on them. Prove that the probability of getting an even sum is greater than 1/2.

Originally problem 2, First Round of 2017 Poland Math Olympiad.

We received 16 *submissions. We present the solution by UCLan Cyprus Problem Solving Group.*

Assume that m balls have an even number written on them and n balls have an odd number written on them. Then $m + n = 2017$ and therefore $m \neq n$. To get an even sum we must either pick two balls with an even number written on them, or pick two balls with an odd number written on them. So the probability that we get an even sum is

$$\frac{m^2+n^2}{2017^2} = \frac{(m+n)^2+(m-n)^2}{2 \cdot 2017^2} \geqslant \frac{2017^2+1}{2 \cdot 2017^2} > \frac{1}{2}\,.$$

Editor's Comment. Roy Barbara, Noah Garson, Kathleen E. Lewis, De Prithwijit and Jason L. Smith generalized the problem and proved the statement in the case in which the number of balls is any odd number. The proof is basically the same as the one presented.

Multifaceted Solutions to a Remarkable Geometry Puzzle

H. S. Hoffman and S. I.Warshaw

Introduction

In their book *"Mathematical Curiosities"* [1], Alfred Posamentier and Ingmar Lehmann present and solve a mathematical puzzle involving the geometrical configuration shown in Figure 1 [see 1, pp. 184–185 and pp. 237–238, respectively.]

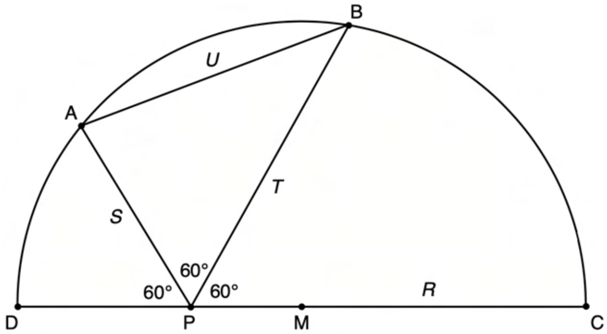


Figure 1

Triangle ABP is inscribed in a semicircle having fixed radius R and center point M, with vertices A and B on the semi-circle arc and vertex P on diameter DC, such that sides AP and BP (of length S and T respectively) make fixed angles of 60° with the diameter and with each other. The puzzle asks us to show that length U of chord AB is invariant for any position of P on the diameter between endpoints C and D, with lengths S and T correspondingly changed to accommodate different positions of P.

What is remarkable about this assertion is that we know from standard circle theorems that chord AB is invariant when apex point P with fixed subtending angle lies on the circle's circumference. Seeing this invariance when P lies on a diameter of the same circle is unexpected.

In this monograph we report different solutions to this puzzle and key properties imbedded in the configuration geometry.

Section I

We draw radii AM and BM (each of length R) from center point M to points A and B on the semi-circle of Figure 1 as shown in Figure 2, indicating the length PM by X.

Since $\cos 60^\circ = -\cos 120^\circ = 1/2$, the cosine law equations for three triangles in the figure simplify considerably. The lengths U, S and T are determined from this cosine law triad:

$$\triangle BPM : R^2 = X^2 + T^2 - XT,$$
$$\triangle APB : U^2 = S^2 + T^2 - ST,$$
$$\triangle APM : R^2 = X^2 + S^2 + XS.$$

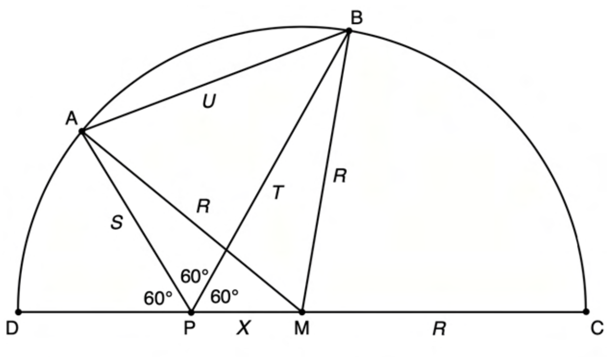


Figure 2

Combining the first and third equations, we obtain $T^2 - XT = S^2 + XS$, which becomes $T^2 - S^2 = XS + XT$, and thus $(T - S)(T + S) = X(S + T)$. This is satisfied when $T+S = 0$ and when $T-S = X$. The first condition is unacceptable, so the second one applies. On substituting $T - S$ for X in the equation for BPM (say), we obtain $R^2 = X^2 + T^2 - XT = S^2 + T^2 - ST$. This in turn is seen equal to U^2 in the equation for APB. Thus $U^2 = R^2$, and $U = R$. $\square$

Triangle ABM is seen to be equilateral with fixed side length R for any position of P on CD.

Section II

Using the proven result $U = R$, we circumscribe equilateral triangle AMB in Figure 2, with circle center at Q and radii QA, QB and QM, as in Figure 3.

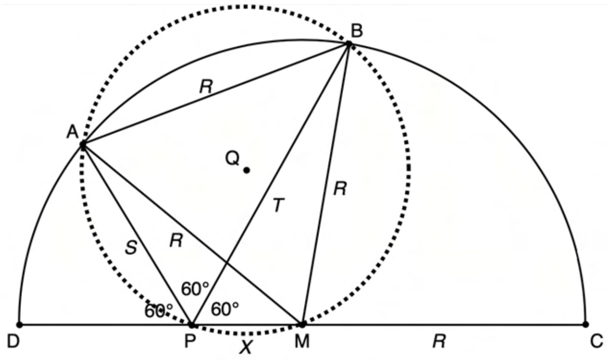


Figure 3

We find – unexpectedly – this circle now appears to include point P.

Four different approaches can be invoked to show that A, B, M and P indeed lie on the same circle, as follows.

(1) A standard circle theorem establishes that the angles subtended by a given chord of a circle from two different points on the same circle that lie on one side of the chord are equal, and its converse is also true. We have proven that ABM is equilateral, and that the angle at M that subtends chord AB is also $60°$. This equality of vertex angles APB and AMB satisfies the converse of the theorem and suffices to put P on the dotted circle.

(2) The internal angle property of a cyclic quadrilateral asserts that its opposing vertex angles are supplementary, i.e., add up to $180°$ [See 1, pp. 155 - 158]. In Figure 3, we see by inspection of $ABPM$ that the vertex angles at P and B are respectively $120°$ and $60°$; in triangle APM, the vertex angles at A and M add up to $60°$, while for equilateral triangle ABM its vertex angles at A and M are each $60°$. The vertex angle pairs at A and M add up to $180°$.

(3) Ptolemy's theorem [see 1, pp. 157 - 158 and ref. 2, pp. 42 - 49] relates the diagonals and opposing sides of a cyclic quadrilateral, and the form the Ptolemy relationship takes for $ABMP$ here is $RT = RS + XR$, which, immediately, is $T = X + S$. This expression appeared in the algebraic development in Section I. It is also known as van Schooten's theorem. [See 2 (pp. 184 - 186) and 3].

(4) Directly establishing that point P actually lies on the circumcircle of triangle AMB is equivalent to showing that the distance from P to the circumcenter Q of AMB is equal to the circumscribed circle radius, or, alternatively, to the distance between Q and any vertex at A, M or B. We provide this calculation in Section III.

Our realization that a cyclic quadrilateral might be imbedded in the puzzle first arose when we noticed the perpendicular bisector of chord PM (in Figures 1 – 3) appeared to pass through the centroid Q of triangle AMB, which itself is equilateral and the intersection of its side bisectors. Creating perpendicular bisectors of the sides of all triangles in Figure 2 with a drawing app and seeing them intersect in a single point at Q further strengthened this consideration.

Section III

We now provide solutions of the challenge proposed in item (4) of the previous section, while using the convenience of coordinate geometry. Define a coordinate system (x, y) with M as origin and x-axis on CD of Figure 3, as shown next in Figure 4.

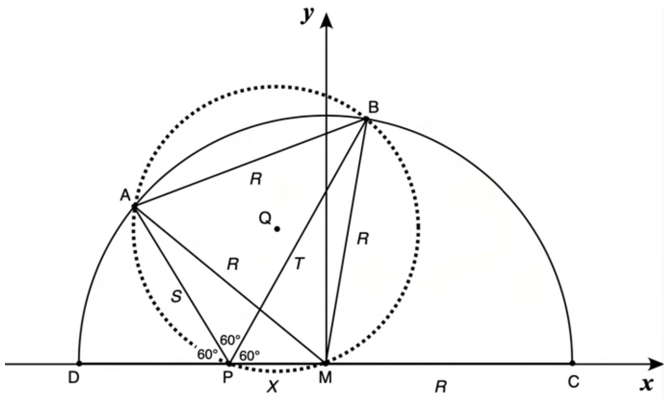


Figure 4

Because of the 60° angles at P, the x and y coordinates of the equilateral triangle vertices and point P are as follows:

Point	x	y
M	0	0
A	$-X-S/2$	$S\sqrt{3}/2$
B	$-X+T/2$	$T\sqrt{3}/2$
P	$-X$	0

We also know from Section I that $X = T - S$. The coordinates of Q (the center of equilateral triangle ABM) are also the mean values of the coordinates of vertices A, B and M, which are one-third of the sum of their respective x and y values. Thus

$$x_Q = (-2X + (T-S)/2 + 0)/3 = -X/2,$$
$$y_Q = ((S+T)\sqrt{3}/2)/3 = (S+T)/(2\sqrt{3}).$$

The value of x_Q shows that if a vertical line $x = -X/2$ passes through point Q in Figure 4, it indeed bisects the line segment PM on the semicircle diameter CD. Thus triangle PQM is isosceles and length PQ equals circle radius QM. This immediately proves that P is also a point of the circle that circumscribes triangle ABM. □

We close this section by calculating the lengths of each segment AQ, BQ, PQ and MQ from their endpoint coordinates given above, and find that these lengths all have the same squared value $(X/2)^2 + (S+T)^2/12$. If one now uses $X = T - S$, this becomes $(S^2+T^2-TS)/3$, which, from the second equation in Section I (with $U = R$), is exactly $R^2/3$. Each of these four segments thus has length $R/\sqrt{3}$.

Section IV

A mysterious aspect of the puzzle configuration as presented in Figure 1 is the triplet of adjacent 60° angles set up at point P. This angle arrangement is not as ad hoc as it appears, because it is a consequence of having four points A, B, M and P lie on a circle. We make this more explicit in Figure 5: an equilateral triangle ABM is inscribed in a circle, with a fourth point P located elsewhere on the circle rim to which lines from the triangle vertices A, B and M are drawn.

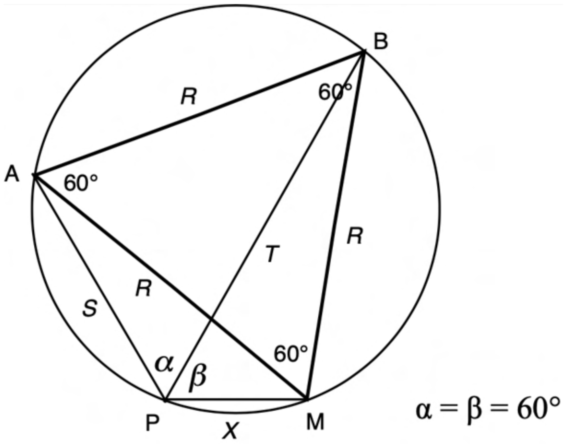


Figure 5

This arrangement of geometric elements is a copy of those shown in the preceding Figures 2, 3 and 4 without the overarching semicircle. What is now evident is that the two angles α and β in Figure 5 are each always 60° for any location of point P on the circumscribing circle. Those are seen to be the same corresponding pair of 60° angles at point P in all the previous figures.

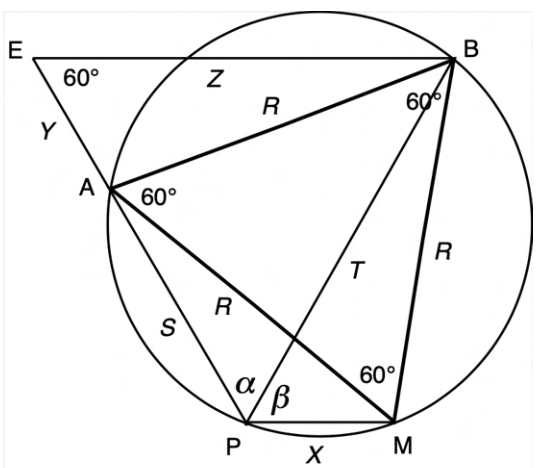


Figure 6

We can now point out that Figure 5 (and each of the prior figures) shows that van Schooten's Theorem [see 3 and pp. 184 - 186 of 2] applies, which is an interesting special case of Ptolemy's theorem [see pp. 157 - 158 of 1 and pp. 42 - 49 of 2] when three of the quadrilateral vertices form an equilateral triangle. We first draw a horizontal line through point B in Figure 5 and extend line PA until it meets this horizontal line at E. Let BE have length Z and AE have length Y as shown in Figure 6. Then $S+Y=Z=T$. Also AEB and MPB are congruent triangles. Thus $X=Y$ and $S+X=T$, which is van Schooten's theorem. [See 3 and pp. 184 - 186 of 2.]

Section V

The quantified geometry of the cyclic quadrilaterals provided so far leads one to additional and immediate geometric insights. To that end we display in Figure 7 next the actual numerical values of the vertex angles actually used to draw all Figures 1 through 4, which have been kept the same from figure to figure.

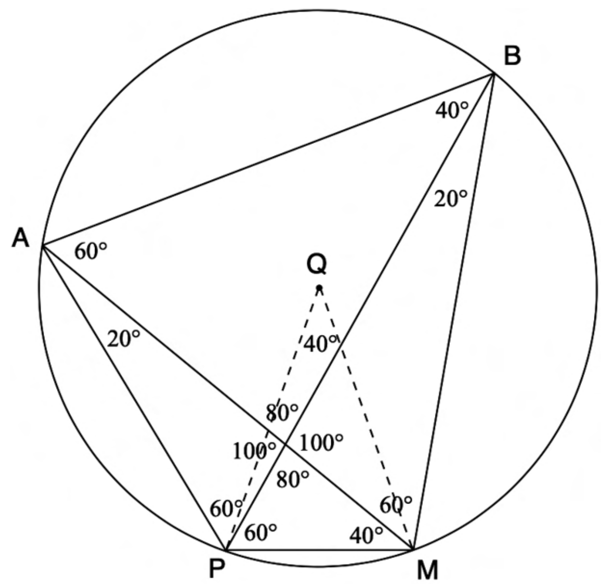


Figure 7

The length of PM (denoted by X in all earlier figures) that makes $\angle PQM = 40°$ is calculated from $X^2 = 2R^2(1-\cos 40°)/3$.

Some well-known "circle theorems" are quickly seen to be embedded in Figure 7, and the "internal angles" discussions in Section II apply. For instance, of the three angles at A, B and Q that subtend chord PM, the angle at Q is twice the angles at A and B, since Q is at the center of the circle and A and B are on its circumference. Thus vertex angles at A and B that subtend the same arc and chord PM are also equal. Numerical length values and relationships are explicitly evident: starting with equilateral triangle ABM, we have $AM = MB = BA = R$, and so $QB = QA = QM = QP = R/\sqrt{3}$.

Acknowledgments

We wish to thank Alfred Posamentier and Ingmar Lehmann for graciously responding to our e- mailed inquiries about their puzzle. In particular, AP and IL encouraged SW in the observation that the problem has a circle-circumscribed quadrilateral imbedded in it, and HH was informed that copyright issues of referencing their Problem 39 can be obviated with appropriate source acknowledgement, which we have provided in the references section herewith.

We also thank Gerald Minerbo, scholar, colleague and friend, for discussions of the solutions presented, their ramifications, and pertinent editorial critiques.

References

[1] Posamentier, Alfred and Lehmann, Ingmar, *Mathematical Curiosities*, Prometheus Books, 2014. (paperback)

[2] Pritchard, Chris (Ed.), *The Changing Shape of Geometry*, Mathematical Association of America, Cambridge University Press, 2003 (paperback)

[3] Viglione, Raymond, *Proof Without Words: van Schooten's Theorem, Mathematics Magazine* 89(2):132, April 2016. Also `https://www.researchgate.net/publication/303865413`

PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 30, 2021**.

4611. *Proposed by Nguyen Viet Hung.*

Evaluate

$$\frac{1}{\sin^4 \frac{\pi}{14}} + \frac{1}{\sin^4 \frac{3\pi}{14}} + \frac{1}{\sin^4 \frac{5\pi}{14}}.$$

4612. *Proposed by Mihaela Berindeanu.*

In the convex quadrilateral $ABCD$, we have

$$\measuredangle(BAC) = \measuredangle(CAD) \quad \text{and} \quad \measuredangle(CDA) = \measuredangle(BCA).$$

Denote $O \in AC$, $X \in BC$, $Y \in CD$ such that $OA = OC$, $AX \perp BC$ and $AY \perp CD$. The perpendicular line from A to XY cuts BD at Z. Show that $\overrightarrow{OZ} = \overrightarrow{OA} + \overrightarrow{OX} + \overrightarrow{OY}$.

4613. *Proposed by Daniel Sitaru.*

Let A and B be $n \times n$ real matrices with $n \in \mathbb{N}, n \geq 2$ such that $AB = BA$. Show that

$$\det(4(A^2 + B^2) + AB + 3(A + B) + I_n) \geq 0.$$

4614. *Proposed by Florin Stanescu.*

Let k be a given natural number and let $(a_n)_{n\geq 1}$ be a sequence such that

$$\lim_{n\to\infty} \frac{1}{n^k}\left(\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right) = 1.$$

Prove that the sequence $\left(\dfrac{a_1 + a_2 + \cdots + a_n}{n^{k+1}}\right)_{n\geq 1}$ is convergent by finding its limit.

4615. *Proposed by Anthony Garcia.*

Let f be a twice differentiable function on $[0, 1]$ such that $\int_0^1 f(x)dx = \frac{f(1)}{2}$. Prove that

$$\int_0^1 (f''(x))^2 dx \geq 30(f(0))^2.$$

4616. *Proposed by Marius Drăgan, modified by the Editorial Board.*

For each suitable point N on side AC of ΔABC define P to be the point where the line parallel to AB meets the side BC, and M to be the point on side AB for which $\angle MNA = \angle B$. If the area of ΔABC equals 1, determine the maximum area of triangle MPN.

4617. *Proposed by Nermin Hodzic, Adnan Ali and Salem Malikic.*

Let a, b, c be positive real numbers such that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2.$$

Show that $\max(a, b, c) \geq \sqrt[3]{9abc}$.

4618. *Proposed by Cherng-tiao Perng.*

Let $\mathcal{C}$ be a nondegenerate conic and $\mathcal{L}$ be a line. Let O, P be two distinct points such that $O, P \notin \mathcal{L}$ and $P \in \mathcal{C}$. Denote the alternative intersection of OP and $\mathcal{C}$ by Q_0. Furthermore let P' be a point on OP such that $P' \notin \mathcal{L}$. For any Q on $\mathcal{C}$ other than Q_0, let

$$QP \cap \mathcal{L} = \{D\} \text{ and } DP' \cap QO = \{Q'\}.$$

Prove that when Q varies on $\mathcal{C}$, Q' moves on a fixed conic through P'.

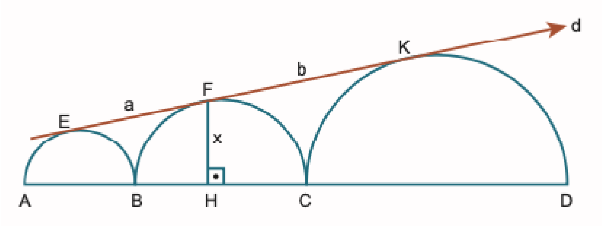
4619. *Proposed by D. M. Bătineţu-Giurgiu and Neculai Stanciu.*

Consider the sequences a_n and b_n such that $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and $b_n = \sum_{k=1}^{n} \frac{1}{(2k-1)^2}$.

Compute $\lim_{n\to\infty} \left(\frac{\pi^4}{48} - a_n b_n\right) n$.

4620. *Proposed by Alpaslan Ceran.*

Consider three semicircles in the configuration below:



Prove that $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$.

. . . .

Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **30 avril 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4611. *Proposée par Nguyen Viet Hung.*

Évaluer

$$\frac{1}{\sin^4 \frac{\pi}{14}} + \frac{1}{\sin^4 \frac{3\pi}{14}} + \frac{1}{\sin^4 \frac{5\pi}{14}}.$$

4612. *Proposée par Mihaela Berindeanu.*

Dans le quadrilatère convexe $ABCD$, $\measuredangle(BAC) = \measuredangle(CAD)$ et $\measuredangle(CDA) = \measuredangle(BCA)$. Dénoter $O \in AC,\ X \in BC,\ Y \in CD$ tels que $OA = OC,\ AX \perp BC$ et $AY \perp CD$. La ligne perpendiculaire de A vers XY intersecte BD en Z. Démontrer que $\overrightarrow{OZ} = \overrightarrow{OA} + \overrightarrow{OX} + \overrightarrow{OY}$.

4613. *Proposée par Daniel Sitaru.*

Soient A et B des matrices $n \times n$ ré elles tels que $AB = BA$, où $n \in \mathbb{N}, n \geq 2$. Démontrer que

$$\det(4(A^2 + B^2) + AB + 3(A + B) + I_n) \geq 0.$$

4614. *Proposée par Florin Stanescu.*

Soit k un nombre naturel et soit $(a_n)_{n\geq 1}$ une suite telle que

$$\lim_{n\to\infty} \frac{1}{n^k}\left(\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}\right) = 1.$$

Démontrer que la suite $\left(\dfrac{a_1 + a_2 + \cdots + a_n}{n^{k+1}}\right)_{n\geq 1}$ est convergente et calculer sa limite.

4615. *Proposée par Anthony Garcia.*

Soit fonction f qui est deux fois dérivable sur $[0, 1]$ et telle que $\int_0^1 f(x)dx = \frac{f(1)}{2}$. Démontrer que

$$\int_0^1 (f''(x))^2 dx \geq 30(f(0))^2.$$

4616. *Proposée par Marius Drăgan, modifié par le Comité de rd́action.*

Soit le point N sur le côté AC de ΔABC, soit P le point où la ligne parallèle à AB intersecte BC, et soit M le point sur le côté AB tel que $\angle MNA = \angle B$. Si la surface de ΔABC est égale à 1, déterminer la plus grande valeur possible pour la surface de ΔMPN.

4617. *Proposée par Nermin Hodzic, Adnan Ali et Salem Malikic.*

Soient a, b, c des nombres réels positifs tels que

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 2.$$

Démontrer que $\max(a, b, c) \geq \sqrt[3]{9abc}$.

4618. *Proposée par Cherng-tiao Perng.*

Soit $\mathcal{C}$ une conique non dégénérée et soit $\mathcal{L}$ une ligne. Soient O, P des points distincts tels que $O, P \notin \mathcal{L}$ et $P \in \mathcal{C}$. Soit alors Q_0 le deuxième point d'intersection de OP et $\mathcal{C}$. De plus, soit P' un point sur OP tel que $P' \notin \mathcal{L}$. Pour tout Q sur $\mathcal{C}$ autre que Q_0, soit

$$QP \cap \mathcal{L} = \{D\} \text{ et } DP' \cap QO = \{Q'\}.$$

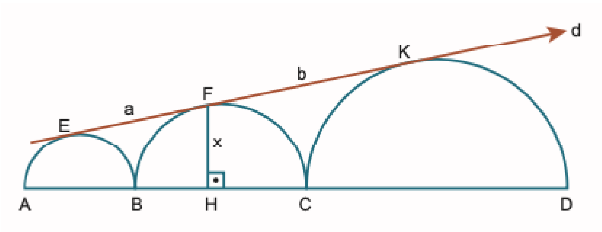
Démontrer que lorsque Q varie le long de $\mathcal{C}$, Q' se déplace sur une certaine conique passant par P'.

4619. *Proposée par D. M. Bătineţu-Giurgiu et Neculai Stanciu.*

Soient des suites a_n et b_n telles que $a_n = \sum_{k=1}^{n} \dfrac{1}{k^2}$ et $b_n = \sum_{k=1}^{n} \dfrac{1}{(2k-1)^2}$. Calculer $\displaystyle\lim_{n\to\infty} \left(\frac{\pi^4}{48} - a_n b_n\right) n$.

4620. *Proposée par Alpaslan Ceran.*

Soient trois demi cercles, tels qu'indiqués:



Démontrer que $\dfrac{1}{x} = \dfrac{1}{a} + \dfrac{1}{b}$.

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(7), p. 309–314.

4561. *Proposed by Michel Bataille.*

Let n be an integer with $n \geq 2$ and let $w_1, w_2, \ldots, w_n$ be distinct complex numbers such that $w_1 + w_2 + \cdots + w_n = 1$. For $k = 1, 2, , \ldots, n$, let $P_k(x) = \prod_{j=1, j\neq k}^{n} (x - w_j)$. If z is a complex number, evaluate

$$\sum_{k=1}^{n} \frac{w_k P_k(zw_k)}{P_k(w_k)}.$$

There were 7 correct solutions, three of which used calculus of residues. We present 4 solutions.

The sum is equal to z^{n-1}.

Solution 1, by UCLan Cyprus Problem Solving Group.

Let

$$g(z) = (z-1)\left[\sum_{k=1}^{n} \frac{w_k P_k(zw_k)}{P_k(w_k)}\right].$$

Observe that the quantity in square brackets equals 1 when $z = 1$. We will show that $g(0) = g'(0) = \cdots = g^{(n-2)}(0) = 0$ so that $g(z) = (z-1)z^{n-1}$.

Let $P(z) = \prod_{k=1}^{n}(z - w_k)$. When $z \neq 1$, for $1 \leq k \leq n$,

$$(z-1)\left[\frac{w_k P_k(zw_k)}{P_k(w_k)}\right] = \frac{(z-1)w_k P(zw_k)}{(zw_k - w_k)P_k(w_k)} = \frac{P(zw_k)}{P'(w_k)}.$$

This equation also holds for $z = 1$, so that, for $0 \leq m \leq n-1$,

$$g(z) = \sum_{k=1}^{n} \frac{P(zw_k)}{P'(w_k)} \quad \text{and} \quad g^{(m)}(z) = \sum_{k=1}^{n} \frac{w_k^m P^{(m)}(zw_k)}{P'(w_k)}.$$

Hence

$$g^{(m)}(0) = P^{(m)}(0) \sum_{k=1}^{n} \frac{w_k^m}{P'(w_k)}.$$

Let $h_m(z) = z^m/P(z)$ for $0 \leq m \leq n-2$. The function $h_m(z)$ has a simple pole at w_k with residue

$$w_k^m/P_k(w_k) = w_k^m/P'(w_k).$$

Suppose that C_R is a circle centred at the origin whose interior contains all the values w_k. Then

$$g^{(m)}(0) = \frac{P^{(m)}(0)}{2\pi i} \oint_{C_R} h_m(z)\,dz$$

and

$$|g^{(m)}(0)| \leq \frac{|P^{(m)}(0)|}{2\pi}(2\pi R)[\max_{C_R} |h_m(z)|].$$

Since the degree of $P(z)$ exceeds m by at least 2, $\lim_{R\to\infty} \max_{C_R} |h_m(z)| = 0$, and the result follows. Hence

$$\sum_{k=1}^{n} \frac{w_k P_k(zw_k)}{P_k(w_k)} = z^{n-1}.$$

Solution 2, by Madhav Modak.

Let

$$D \equiv D(w_1, \ldots, w_k, \ldots, w_n) = \prod_{1\leq i<j\leq n} (w_i - w_j),$$

and

$$f(z) = \sum_{k=1}^{m} \frac{w_k P_k(zw_k)}{P_k(w_k)} = \sum_{k=1}^{n} \frac{w_k(-1)^{i-1}D(w_1,\ldots,zw_k,\ldots,w_n)}{(-1)^{i-1}D(w_1,\ldots,w_k,\ldots w_n)}.$$

The Vandermonde matrix V whose (i,j)th element is w_j^{i-1} has determinant

$$\epsilon_n D = \sum_{i=1}^{n} w_j^{i-1} T_{ij},$$

where $\epsilon_n = (-1)^{\binom{n}{2}}$ and T_{ij} is the cofactor of w_j^{i-1} and the determinant is expanded according to the jth column.

By replacing w_k by zw_k, we find that

$$\epsilon_n D(w_1,\ldots,zw_k,\ldots,w_n) = \sum_{i=1}^{n} z^{i-1} w_k^{i-1} T_{ik},$$

whence

$$f(z) = \frac{\epsilon_n}{D} \sum_{i=1}^{n} \left(\sum_{k=1}^{n} w_k^i T_{ik} \right) z^{i-1}.$$

The coefficient of z^{n-1} is equal to ϵ_n/D times $\sum_{k=1}^n w_k^n T_{nk}$, which is the expansion of the matrix V_n obtained from V by replacing the last row by $(w_1^n, w_2^n, \ldots, w_n^n)$. The determinant of V_n is equal to $\epsilon_n D$ multiplied by $w_1 + w_2 + \cdots + w_n = 1$. Thus the coefficient of z^{n-1} is 1.

When $i \leq n-1$, $\sum_{k=1}^n w_k^i T_{ik}$ is the expansion of the matrix V_i obtained from V by replacing the ith row by $(w_1^i, w_2^i, \ldots, w_n^i)$, making it identical to the following row. Hence the coefficient of z^{i-1} is 0 when $1 \leq i \leq n-1$. It follows that $f(z) = z^{n-1}$.

Solution 3, by C.R. Pranesachar.

Let
$$g(s) = \frac{s\prod_{k=1}^{n}(zs - w_k)}{\prod_{k=1}^{n}(s - w_k)}.$$
Since the degree of the numerator as a polynomial in s exceeds that of the denominator by 1, we can write

$$g(s) = sz^n + C + \sum_{k=1}^{n} \frac{A_k}{s - w_k},$$

where C and A_k are polynomials in z.

Hence
$$s\prod_{k=1}^{n}(zs - w_k) = (sz^n + C)\prod_{k=1}^{n}(s - w_k) + \sum_{k=1}^{n} A_k P_k(s).$$
Setting $s = w_k$, we find that

$$A_k = \frac{w_k^2(z-1)P_k(zw_k)}{P_k(w_k)}.$$

Equating the coefficients of s^n leads to

$$-z^{n-1} = z^{n-1}(-w_1 - w_2 - \cdots - w_n) = C + z^n(-w_1 - w_2 - \cdots - w_n) = C - z^n.$$

Hence
$$f(s) = sz^n + z^{n-1}(z-1) + (z-1)\sum_{k=1}^{n} \frac{w_k^2 P_k(zw)}{(s - w_k)P_k(w_k)}.$$
Setting $s = 0$ gives the required function as z^{n-1}.

Solution 4, by the proposer.

Let $P(s) = \prod_{k=1}^{n}(s - w_k) = s^n - s_{n-1} + \sum_{j=2}^{n} a_j s^{n-j}$. Fix k and define $U_m(s) = s^m + w_k s^{m-1} + \cdots + w_k^{m-1} + w_k^m$. Using the fact that

$$\frac{s^m}{s - w_k} = \frac{s^m - w_k^m}{s - w_k} + \frac{w_k^m}{s - w_k} = U_{m-1}(s) + \frac{w_k^m}{s - w_k}$$

for $1 \le m \le n+1$, we find that

$$\begin{aligned} sP_k(s) = \frac{sP(s)}{s - w_k} &= \frac{s^{n+1}}{s - w_k} - \frac{s^n}{s - w_k} + \sum_{j=2}^{n} \frac{a_j s^{n+1-j}}{s - w_k} \\ &= U_n(s) - U_{n-1}(s) + \sum_{j=2}^{n} a_j U_{n-j}(s) + \frac{w_k P(w_k)}{s - w_k} \\ &= U_n(s) - U_{n-1}(s) + \sum_{j=2}^{n} a_j U_{n-j}(s) \end{aligned}$$

Since $U_m(zw_k) = w_k^m(1 + z + \cdots + z^m)$,

$$zw_kP_k(zw_k) = (1 + z + \cdots + z^n)w_k^n - (1 + z + \cdots + z^{n-1})w_k^{n-1} + Q(w_k),$$

where $Q(s) = \sum_{j=2}^n b_j s^{n-j}$ is a polynomial whose coefficients b_j depend on z but are independent of k. Hence

$$\sum_{k=1}^n \frac{w_kP_k(zw_k)}{P_k(w_k)}$$
$$= \frac{1}{z}\left[(1 + z + \cdots + z^n)\sum_{k=1}^n \frac{w_k^n}{P_k(w_k)} - (1 + z + \cdots + z^{n-1})\sum_{k=1}^n \frac{w_k^{n-1}}{P_k(w_k)} + \sum_{j=2}^n b_j \sum_{k=1}^n \frac{w_k^{n-j}}{P_k(w_k)}\right]$$
$$= \frac{1}{z}[(1 + z + \cdots + z^n) - (1 + z + \cdots + z^{n-1}) + 0] = z^{n-1},$$

using the result established in the following appendix.

Appendix. The following result, used in Solution 4, Solution 1 and by one other solver, is of independent interest and may not be readily accessible to the reader. This proof is supplied by the proposer.

$$\sum_{k=1}^n \frac{w_k^n}{P_k(w_k)} = w_1 + w_2 + \cdots + w_n, \quad \sum_{k=1}^n \frac{w_k^{n-1}}{P_k(w_k)} = 1, \quad \sum_{k=1}^n \frac{w_k^m}{P_k(w_k)} = 0,$$

for $0 \leq m \leq n - 2$.

Proof. Let $w = w_1 + w_2 + \cdots + w_n$ and define

$$P(x) = \prod_{k=1}^n (x - w_k) = x^n - wx^{n-1} + U(x),$$

where $U(x)$ is a polynomial of degree less than $n - 1$. The Lagrange polynomial $Q(x)$ of degree less than n taking values $Q(w_k)$ when $x = w_k$ is given by

$$Q(x) = \sum_{k=1}^n \frac{Q(w_k)P_k(x)}{P'(w_k)} = \sum_{k=1}^n \frac{Q(w_k)P(x)}{P_k(w_k)(x - w_k)}.$$

Set $Q(x) \equiv 1$ to obtain

$$\frac{1}{P(x)} = \sum_{k=1}^n \frac{1}{P_k(x)(x - w_k)}.$$

Observe that $x^{n+1} = (x + w)(x^n - wx^{n-1} + U(x)) + V(x)$, with the degree of $V(x)$ less than n, so that

$$\frac{x^{n+1}}{P(x)} = x + w + \frac{V(x)}{P(x)}.$$

Also

$$\begin{aligned}\frac{x^{n+1}}{P(x)} &= \sum_{k=1}^{n} \frac{x^{n+1}}{P_k(w_k)(x-w_k)} = \sum_{k=1}^{n} \frac{x^{n+1}-w_k^{n+1}}{P_k(w_k)(x-w_k)} + \sum_{k=1}^{n} \frac{w_k^{n+1}}{P_k(w_k)(x-w_k)} \\ &= \sum_{k=1}^{n} \frac{x^n + w_k x^{n-1} + \cdots + w_k^n}{P_k(w_k)} + \sum_{k=1}^{n} \frac{w_k^{n+1}}{P_k(w_k)(x-w_k)} \\ &= \sum_{m=0}^{n} \left(\sum_{k=1}^{n} \frac{w_k^m}{P_k(w_k)} \right) x^{n-m} + \frac{R(x)}{P_k(w_k)P(x)},\end{aligned}$$

where the degree of $R(x)$ is less than n. Hence, by equating polynomials parts,

$$x + w = \sum_{m=0}^{n} \left(\sum_{k=1}^{n} \frac{w_k^m}{P_k(w_k)} \right) x^{n-m},$$

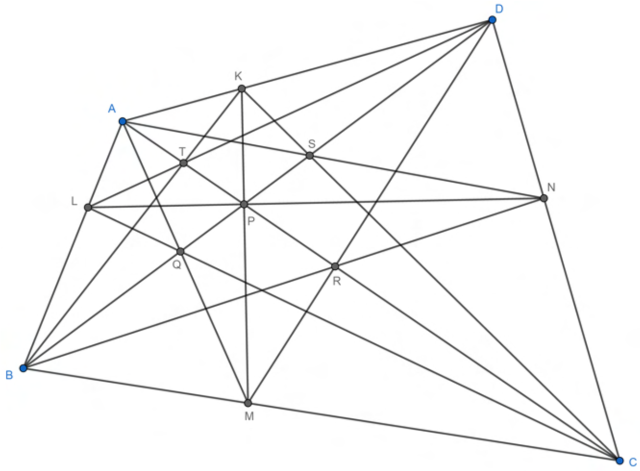
and the result follows from a comparision of coefficients.

4562. *Proposed by Pericles Papadopoulos.*

Let P be the intersection point of the diagonals AC and BD of a convex quadrilateral $ABCD$. The angle bisector of the opposite angles $\angle APD$ and $\angle BPC$ intersects AD and BC at points K and M respectively, and the angle bisector of the opposite angles $\angle APB$ and $\angle CPD$ intersects AB and DC at points L and N respectively. Show that:

(a) $(DK)(AL)(BM)(CN) = (KA)(LB)(MC)(ND)$.

(b) Cevians AM, BP, CL concur at point Q, cevians BN, CP, DM concur at point R, cevians AN, DP, CK concur at point S, and cevians DL, BK, PA concur at point T.



We received 13 solutions, all essentially the same. Here, then, is the common solution.

(a) PK, PL, PM, and PN are (interior) bisectors of the angles at P in triangles DPA, APB, BPC, and CPD respectively. Therefore,

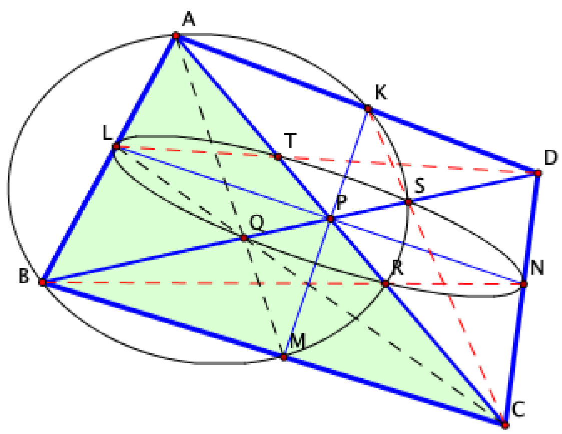
$$\frac{DK}{KA} = \frac{PD}{PA}, \quad \frac{AL}{LB} = \frac{PA}{PB}, \quad \frac{BM}{MC} = \frac{PB}{PC}, \quad \text{and} \quad \frac{CN}{ND} = \frac{PC}{PD}.$$

Multiplying these expressions, we get

$$\frac{DK}{KA} \cdot \frac{AL}{LB} \cdot \frac{BM}{MC} \cdot \frac{CN}{ND} = 1,$$

which proves part (a).

Comment by Sergey Sadov. For part (a) it is not necessary that P be the intersection point of the diagonals — the same argument (word for word) proves the identity for any point P inside a convex quadrilateral that is joined by line segments to the four vertices. Indeed, the analogous identity holds similarly for an arbitrary point P inside a convex n-gon for any $n \geq 3$.



(b) Using recriprocals of two of the equal ratios from part (a), we have

$$\frac{AP}{PC} \cdot \frac{CM}{MB} \cdot \frac{BL}{LA} = \frac{AP}{PC} \cdot \frac{PC}{PB} \cdot \frac{PB}{AP} = 1\,.$$

Because we assume here that P lies between A and C on a diagonal of the given convex quadrilateral, Ceva's theorem applied to ΔABC implies that AM, BP, CL are concurrent at a point inside the triangle. The rest of part (b) follows by a cyclic relabeling of points.

Editor's comments. Sadov describes several further properties related to the given configuration. For example, he finds that the lines AB, KM, and SR are concurrent or parallel, and those six points lie on a conic.

Other incidences arise by replacing points K, L, M, N in part (b) by K', L', M', N', where now the bisectors of angles APB and CPD meet the lines AD and BC at K' and M', while the other bisector meets AB and CD at L' and M'. He also discusses a number of other conics associated with the figure; for example, if S' is a point on the line AN and we define $Q' = PS' \cap CL$, then the points Q, Q', S, S', R, and T lie on a conic. Consequently,

- The points Q, R, S, T, L, N lie on a conic.
- The ellipse through the points Q, R, S, T that is tangent to the line AN at S is tangent to CL at Q.

4563. *Proposed by George Stoica.*

Find all perfect squares in the sequence $x_0 = 1, x_1 = 2, x_{n+1} = 4x_n - x_{n-1}, n \geq 1$.

We received 11 submissions, of which 7 were correct and complete. We present the solution by Theo Koupelis.

The characteristic equation of the given recurrence relation is $r^2 - 4r + 1 = 0$, whose solutions are $r = 2 \pm \sqrt{3}$. Therefore, the general term of the sequence is given by $x_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$, for all $n \geq 0$, where A, B are constants. Taking into account that $x_0 = 1$ and $x_1 = 2$, we find that $A = B = 1/2$, and therefore, for $n \geq 0$,

$$x_n = \frac{1}{2}\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n\right]. \tag{1}$$

Clearly $x_0 = 1$ is a perfect square. We will show that no other term x_n, where $n \geq 1$, is a perfect square.

We start by examining the terms of the sequence modulo 3 and modulo 5. Using the first two terms and the recursive definition, the residues of the sequence modulo 3 are $1, 2, 1, 2, 1, 2, 1, \cdots$; in particular, $3 \nmid x_n$ for any n. Similarly, modulo 5 the residues are $1, 2, 2, 1, 2, 2, 1, \cdots$; that is, $x_{3k} \equiv 1 \mod 3$ for k a non-negative integer and $x_n \equiv 2 \mod 5$ for n not a multiple of 3. Since 2 is not a quadratic residue $\mod 5$, if x_n is to be a perfect square, then we must have $n = 3k$ for k a non-negative integer.

From (1) and the fact that $(2+\sqrt{3})(2-\sqrt{3}) = 1$ we have

$$x_{3k} = \frac{1}{2}\left\{\left[(2+\sqrt{3})^k + (2-\sqrt{3})^k\right]^3 - 3\left[(2+\sqrt{3})^k + (2-\sqrt{3})^k\right]\right\},$$

or

$$x_{3k} = x_k \cdot (4x_k^2 - 3).$$

We know that $x_0 = 1$ is a perfect square while $x_3 = 26$ is not. Now let $n = 3k$ be the smallest positive integer for which x_{3k} is a perfect square. Then if $d = \gcd(x_k, 4x_k^2 - 3)$, then $d|3$ and therefore $d = 3$ or $d = 1$. But $d \neq 3$ because

$3 \nmid x_n$. Therefore $d = 1$. But if x_{3k} is a perfect square, and it is a product of two terms that have no common divisor, each term must be a perfect square. Thus x_k must be a perfect square, which contradicts the assumption that $n = 3k$ was the smallest positive integer for which x_n is a perfect square. Therefore the only perfect square in the given sequence is $x_0 = 1$.

4564. *Proposed by Alijadallah Belabess.*

Let a, b, c and d be non-negative real numbers with $ab + bc + cd + da = 4$. Prove that:

$$a^3 + b^3 + c^3 + d^3 + 4abcd \geq 8$$

We received 10 submissions, all correct. We present the solution by Marie-Nicole Gras, slightly modified by the editor.

By hypothesis, we have $ab + bc + cd + da = (a + c)(b + d) = 4$. Without loss of generality, we may assume that $a + c \geq 2$.

The given inequality is equivalent to

$$F := 4(a + c)^3 \big[a^3 + b^3 + c^3 + d^3 + 4abcd - 8\big] \geq 0,$$

or $F = 4(a+c)^3\big[(a+c)^3 + (b+d)^3 - 3ac(a+c) - 3bd(b+d) + 4abcd - 8\big]$. Since $(a + c)(b + d) = 4$, and $4ac = (a + c)^2 - (a - c)^2$, we can write $F = G + H$ where

$$\begin{aligned} G &= 4(a+c)^6 + 256 - 3(a+c)^4\big[(a+c)^2 - (a-c)^2\big] - 32(a+c)^3 \\ &= (a+c)^6 - 32(a+c)^3 + 256 + 3(a+c)^4(a-c)^2 \\ &= \big[(a+c)^3 - 16\big]^2 + 3(a+c)^4(a-c)^2, \qquad \text{and} \\ H &= -48bd(a+c)^2 + 16abcd(a+c)^3 \\ &= -48\big[b(a+c)\big]\big[d(a+c)\big] + 16ac(a+c)\big[b(a+c)\big]\big[d(a+c)\big]. \end{aligned}$$

Set $x = b(a+c)$. Then by $(a+c)(b+d) = 4$, we have $d(a+c) = 4 - b(a+c) = 4 - x$, $x(4 - x) \geq 0$, and

$$H = -48x(4-x) + 16ac(a+c)x(4-x) = 16x(4-x)\big[ac(a+c) - 3\big]. \qquad (1)$$

Then $F = \big[(a+c)^3 - 16\big] + 3(a+c)^4(a-c)^2 + 16x(4-x)[ac(a+c) - 3]$. Hence $F \geq 0$ if $ac(a + c) \geq 3$.

If $ac(a + c) < 3$, then by (1) we can write

$$\begin{aligned} F = H + G &= 16x(4-x)\big[ac(a+c) - 3\big] + G \\ &= 16(x-2)^2\big[3 - ac(a+c)\big] + 64\big[ac(a+c) - 3\big] + G. \end{aligned} \qquad (2)$$

Let $T = 64\big[ac(a+c)-3\big]+G$. Then by (2) we get

$$\begin{aligned}T &= 16(a+c)\big[(a+c)^2-(a-c)^2\big]-192+(a+c)^6-32(a+c)^3+256+3(a+c)^4(a-c)^2\\ &= (a+c)^6-16(a+c)^3+64+3(a+c)^4(a-c)^2-16(a+c)(a-c)^2\\ &= \big[(a+c)^3-8\big]^2+(a+c)(a-c)^2\big[3(a+c)^3-16\big]. \qquad (3)\end{aligned}$$

Since $a+c \geq 2$, we see from (3) that $T \geq 0$, so finally we have from (2) that $F = 16(x-2)^2\left[3-ac(a+c)\right]+T \geq 0$, completing the proof.

Editor's comment. Out of the ten solvers, four of them also showed that equality holds if and only if $(a,b,c,d) = (1,1,1,1)$ or $(2^{1/3}, 2^{2/3}, 2^{1/3}, 0)$ together with all its cyclic permutations.

4565. *Proposed by Daniel Sitaru.*

Let m_a, m_b and m_c be the lengths of the medians of a triangle ABC. Prove that

$$4(am_bm_c+bm_cm_a+cm_am_b) \geq 9abc.$$

We received 11 solutions, one of which was incorrect. We present the solution by Sergey Sadov.

Consider the triangle in the complex plane. Let the origin (complex zero) be at the center of mass of the triangle and u, v, w be the complex coordinates of the midponts of the sides a, b, and c, respectively. Then

$$m_a = 3|u| \quad m_b = 3|v|, \quad m_c = 3|w|,$$

and

$$a = 2|v-w|, \quad b = 2|w-u|, \quad c = 2|u-v|.$$

Put

$$\begin{aligned}\xi &= \frac{4}{9}\cdot\frac{m_a}{a}\cdot\frac{m_b}{b} = \frac{u}{v-w}\cdot\frac{v}{w-u},\\ \eta &= \frac{4}{9}\cdot\frac{m_b}{b}\cdot\frac{m_c}{c} = \frac{v}{w-u}\cdot\frac{w}{u-v},\\ \zeta &= \frac{4}{9}\cdot\frac{m_c}{c}\cdot\frac{m_a}{a} = \frac{w}{u-v}\cdot\frac{u}{v-w}.\end{aligned}$$

The required identity takes the form $|\xi|+|\eta|+|\zeta| \geq 1$, and it follows, by the triangle inequality, from the identity $\xi+\eta+\zeta = -1$, which we are about to prove.

Equivalently, we want to prove that

$$(u-v)(v-w)(w-u)+uv(u-v)+vw(v-w)+wu(w-u) = 0.$$

Consider the coefficients at powers of u:

$$\begin{array}{ll} u^2: & (w-v)+v-w=0, \\ u^1: & (v-w)(v+w)-v^2+w^2=0, \\ u^0: & vw(w-v)+vw(v-w)=0. \end{array}$$

The proof is finished.

A generalization. In the above proof we did not use the relation $u+v+w=0$. Therefore we have in fact proved a more general fact:

> Let D be any point in the plane of triangle ABC. Then
>
> $$AD \cdot BD \cdot c + BD \cdot CD \cdot a + CD \cdot AD \cdot b \geq abc.$$

The given problem is equivalent to the particular case of this proposition with D being the center of mass.

Case of equality. A natural question to ask is: when, in the described generalization, does the inequality turn to equality. I will show that this happens if and only if D is the orthocenter. As a corollary, in the original problem the equality takes place only for the equilateral triangle.

For the equality

$$|-1| = |\xi+\eta+\zeta| = |\xi|+\eta|+\zeta|$$

to hold, it is necessary and sufficient that ξ, η, ζ be real and nonpositive. At least one of them is nonzero. Suppose $\xi \neq 0$ and consider the condition $\xi < 0$. It means that

$$\frac{w-v}{v} \cdot \frac{w-u}{u} > 0.$$

Hence the arguments of the complex numbers $(w-v)/v$ and $(w-u)/u$ have equal magnitudes and opposite signs. Geometrically it means that the signed magnitudes of the angles DBA and ACD (considering the counterclockwise direction as positive) are equal.

Denote the unsigned magnitude of the angles as $\angle DBA = \angle DCA = \alpha'$, $\angle DAB = \angle DCB = \beta'$ and $\angle DAC = \angle DBC = \gamma'$. Then

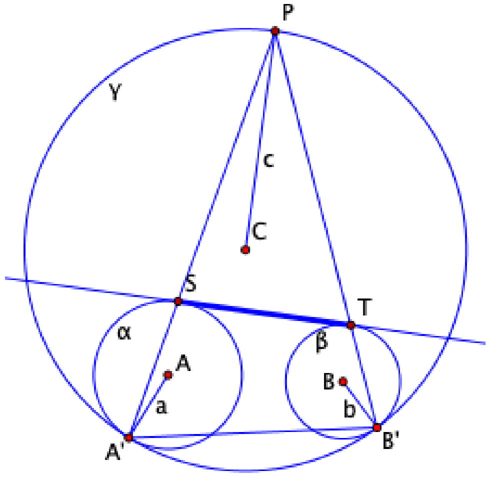
$$\beta'+\gamma' = \alpha \ (=\angle A), \ \alpha'+\beta' = \gamma, \ \gamma'+\alpha' = \beta, \ 2(\alpha'+\beta'+\gamma') = \pi.$$

It follows that $\alpha' = \pi/2 - \alpha$ etc. This condition defines the orthocenter.

Editor's note. Several other solvers (Gayen, Giugiuc, Janous, Văcaru, and the proposer) also asserted the generalization of the inequality to an arbitrary point in the plane of the triangle, with Gayen and Giugiuc citing this generalization as Hyashi's Inequality and Janous indicating it as a generalization of Murray Klamkin's "Polar Moment of Inertia" Inequality.

4566. *Proposed by J. Chris Fisher.*

Given three circles, α, β, and γ with centers A, B, C and radii a, b, c, respectively, where γ is tangent to α at A' and to β at B', either both circles externally or both internally. One exterior common tangent line is tangent to α at S and to β at T.



(a) Prove that the lines $A'S$ and $B'T$ intersect at a point of γ.
(b) Show that

$$(A'B')^2 = \frac{c^2 \cdot ST^2}{(c \pm a)(c \pm b)},$$

where the plus signs are used when α and β are externally tangent to γ, and the negative signs when internally tangent to γ.

Comment. Part (b) is problem 1.2.8 on page 5 of H. Fukagawa and D. Pedoe, *Japanese Temple Geometry Problems, San Gaku* (The Charles Babbage Research Centre, 1989). Instead of a proof, the authors provide (on page 82) a reference to a 19th century Japanese geometry text together with the comment, "Called 'Three Circles and Tangent Problem', or 'Sanen Bousha', and applied in the solution to many problems."

All 6 submissions were complete and correct. We have selected a different correspondent for each part.

(a) Solution by Sergey Sadov.

Let P_1 be the second point of intersection of the line $A'S$ with circle γ. Similarly, let P_2 be the second point of intersection of the line $B'T$ with circle γ. We will prove that $P_1 = P_2$.

The circles α and γ are homothetic with homothety centre A'. Under this homothety, the triangle $A'AS$ corresponds to the triangle $A'CP_1$. It follows that $CP_1 || AS$. Similarly, $CP_2 || BT$.

Since $AS \perp ST$ and $BT \perp ST$, we conclude that $AS || BT$, hence $CP_1 || CP_2$; hence, $P_1 = P_2$.

(b) *Solution by Marie-Nicole Gras.*

Let L be the point of AS such that $AL = BT$. The quadrilateral $ABTL$ is a parallelogram so that $TL = AB$; in the right-angled $\triangle LST$, we have

$$AB^2 = LT^2 = ST^2 + (a-b)^2.$$

The cosine law applied to the isosceles $\triangle A'CB'$ gives us $A'B'^2 = 2c^2(1-\cos(\angle C))$, and in $\triangle ACB$, we have

$$\begin{aligned}\cos(\angle C) &= \frac{CA^2 + CB^2 - AB^2}{2\,CA{\cdot}CB},\\ 1-\cos(\angle C) &= \frac{2\,CA{\cdot}CB - CA^2 - CB^2 + AB^2}{2\,CA{\cdot}CB} = \frac{AB^2 - (CA-CB)^2}{2\,CA{\cdot}CB}\,.\end{aligned}$$

When α and β are internally tangent to γ,

$$CA = c-a,\ CB = c-b,\ \text{and } (CA-CB)^2 = (a-b)^2.$$

When α and β are externally tangent to γ,

$$CA = c+a,\ CB = c+b,\ \text{and } (CA-CB)^2 = (a-b)^2.$$

We deduce that

$$A'B'^2 = c^2\,\frac{ST^2 + (a-b)^2 - (a-b)^2}{CA{\cdot}CB} = \frac{c^2{\cdot}ST^2}{(c\pm a)(c\pm b)}\,.$$

4567. *Proposed by Paul Bracken.*

Prove that for any $n \in \{0,1,2,3,\ldots\}$, the following holds

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{n-k}(2k+1)^{2n+1} = (-1)^n 2^{2n}(2n+1)!$$

We received 10 submissions and they were all correct. We present 3 solutions.

The first step is a reduction. We can transform the left-hand side in order to obtain a sum over all values of k from 0 to $2n+1$, which was observed by all solvers. Let S_n be the left-hand side of the equation. Note that

$$\begin{aligned}S_n &= \sum_{k=0}^{n}(-1)^{n-k}\binom{2n+1}{k}(2n+1-2k)^{2n+1}\\ &= \sum_{k=0}^{n}(-1)^{n-k}\binom{2n+1}{2n+1-k}(2n+1-2(2n+1-k))^{2n+1}(-1)^{2n+1}\\ &= \sum_{m=n+1}^{2n+1}(-1)^{n-m}\binom{2n+1}{m}(2n+1-2m)^{2n+1},\end{aligned}$$

where we change to the variable $m = 2n + 1 - k$ in the last step. So it suffices to show that

$$2S_n = \sum_{k=0}^{2n+1} (-1)^{n-k} \binom{2n+1}{k} (2n + 1 - 2k)^{2n+1} = (-1)^n 2^{2n+1} (2n + 1)!. \qquad (1)$$

The second step is to prove the binomial identity (1).

Solution 1, by Seán M. Stewart, simplified by the editor.

We have the following well-known identity: for any $n \in \mathbb{N} \cup \{0\}$,

$$\sin^{2n+1} x = \frac{(-1)^n}{2^{2n+1} i} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{i(2n-2k+1)x}. \qquad (2)$$

Indeed, by Euler's formula and the binomial theorem, we have

$$\sin^{2n+1} x = \left(\frac{e^{ix} - e^{-ix}}{2i} \right)^{2n+1} = \frac{(-1)^n}{2^{2n+1} i} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} e^{i(2n-2k+1)x}.$$

Now we differentiate both sides of equation (2) with respect to x by $2n + 1$ times and then evaluate the result at $x = 0$. We start by differentiating the left-hand side using the general Leibniz rule:

$$(\sin^{2n+1} x)^{(2n+1)} = \sum_{k_1+k_2+\cdots+k_{2n+1}=2n+1} \frac{(2n+1)!}{k_1! k_2! \cdots k_{2n+1}!} \prod_{j=1}^{2n+1} (\sin x)^{(k_j)}. \qquad (3)$$

Note that $\sin 0 = 0$, so when we are evaluating at $x = 0$, on the right-hand side of the equation (3), the only term that survives is the term with $k_1 = k_2 = \cdots = k_{2n+1} = 1$. So when $x = 0$, equation (3) can be simplified to

$$\left(\sin^{2n+1} x\right)^{(2n+1)} (0) = (2n + 1)!(\sin'(0))^{2n+1} = (2n + 1)!.$$

For the right-hand side of equation (2), the $(2n + 1)$-st order derivative is

$$\frac{(-1)^n}{2^{2n+1} i} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} i^{2k+1} (2n - 2k + 1)^{2n+1} e^{i(2n-2k+1)x} \bigg|_{x=0}$$
$$= \frac{(-1)^n}{2^{2n+1}} \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} (2n - 2k + 1)^{2n+1}.$$

Combining the two ways of the computation, we get (1), as required.

Solution 2, by Sergey Sadov.

For any polynomial $P(x)$ the expression

$$\nabla^r P(x) = \sum_{m=0}^{r} (-1)^m \binom{r}{m} P(x-m)$$

is known as the (backward) finite difference of order r of $P(\cdot)$ at the point x.

We will employ the known fact that if $P(x) = ax^r +$ (terms of degree $< r$), then $\nabla^r x^r = ar!$ (the constant function). Applying this formula to $P(x) = (2x+1)^{2n+1}$, we get

$$2S_n = (-1)^n \nabla^{2n+1} P(x)|_{x=2n+1} = (-1)^n \cdot 2^{2n+1}(2n+1)! = (-1)^n 2^{2n}(2n+1)!.$$

Solution 3, by the majority of the solvers, slightly modified by the editor.

We will show that for any positive integer m and an integer t, we have

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} j^t = \begin{cases} 0, & \text{if } 0 \le t \le m-1, \\ (-1)^m m!, & \text{if } t = m. \end{cases} \tag{4}$$

By the binomial theorem, it is clear that equation (4) implies that

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} (a+bj)^m = (-b)^m \cdot m!. \tag{5}$$

Taking $m = a = 2n+1$ and $b = -2$ in equation (5), we obtain (1).

To prove (4), we apply t times the differential operator $x\dfrac{d}{dx}$ on the binomial identity

$$(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j.$$

Note that the right-hand side becomes

$$\sum_{j=1}^{m} \binom{m}{j} j^m x^{j-1}.$$

If $t \le m-1$, then $(1+x)$ is a factor of all terms of the left-hand side, and if $t = m$, the left-hand side is

$$(1+x)Q(x) + m!\, x^m$$

for some polynomial $Q(x)$. Then we substitute $x = -1$ and obtain identity (4).

Editor's Comment. As pointed out by Marie-Nicole Gras, this problem is similar to Problem 4463 in ***Crux*** Vol. 46 (2).

4568. *Proposed by Song Qing, Leonard Giugiuc and Michael Rozenberg.*

Let k be a fixed positive real number. Consider positive real numbers x, y and z such that

$$xy + yz + zx = 1 \quad \text{and} \quad (1 + y^2)(1 + z^2) = k^2(1 + x^2).$$

Express the maximum value of the product xyz as a function of k.

We received 9 submissions of which 6 were correct and complete. We present the solution by Arkady Alt, slightly modified.

Since $xy + yz + zx = 1$, the equation $(1 + y^2)(1 + z^2) = k^2(1 + x^2)$ is equivalent to each of:

$$\begin{aligned}
&\left(xy + yz + zx + y^2\right)\left(xy + yz + zx + z^2\right) = k^2\left(xy + yz + zx + x^2\right),\\
&(y + z)(x + y)(y + z)(x + z) = k^2(x + z)(x + y),\\
&(y + z)^2 = k^2,\\
&y + z = k.
\end{aligned}$$

Let $t = xyz$, then since $y + z = k$, we have

$$1 = (xy + zx) + yz = kx + \frac{t}{x}$$

and so $t = x(1 - kx)$. Thus, $yz = 1 - kx$ and $y + z = k$, so, by the AM-GM inequality,

$$1 - kx = yz \leqslant \frac{(y + z)^2}{4} = \frac{k^2}{4}.$$

Hence, $x \geqslant \frac{1}{k} - \frac{k}{4}$ and we are to maximize $h(x) = x(1 - kx)$ when $x \geqslant \frac{1}{k} - \frac{k}{4}$.

Since $h'(x) = 1 - 2kx$, $h(x)$ is decreasing when $\dfrac{1}{2k} < \dfrac{1}{k} - \dfrac{k}{4}$. That is, when $0 < k < \sqrt{2}$. For such k,

$$\max t = h\left(\frac{1}{k} - \frac{k}{4}\right) = \left(\frac{1}{k} - \frac{k}{4}\right)\left(1 - k\left(\frac{1}{k} - \frac{k}{4}\right)\right) = \frac{k\left(4 - k^2\right)}{16}.$$

Likewise, if $k \geqslant \sqrt{2}$, then $\dfrac{1}{k} - \dfrac{k}{4} \leqslant \dfrac{1}{2k}$ so $\frac{1}{2k}$ is in the domain of $h(x)$ and

$$\max t = h\left(\frac{1}{2k}\right) = \frac{1}{2k}\left(1 - k \cdot \frac{1}{2k}\right) = \frac{1}{4k}.$$

Thus, $\max(xyz) = \begin{cases} \dfrac{k\left(4 - k^2\right)}{16} \text{ if } k \in \left(0, \sqrt{2}\right) \\ \dfrac{1}{4k} \text{ if } k \geq \sqrt{2} \end{cases}$

4569. *Proposed by Nguyen Viet Hung.*

Solve the following equation in the set of real numbers

$$8^x + 27^{\frac{1}{x}} + 2^{x+1} \cdot 3^{\frac{x+1}{x}} + 2^x \cdot 3^{\frac{2x+1}{x}} = 125.$$

We received 20 submissions, of which 18 were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group.

Let $a = 2^x$ and $b = 3^{1/x}$. Then $a^3 + b^3 + 6ab + 9ab = 125$ and therefore

$$0 = a^3 + b^3 + (-5)^3 - 3ab(-5) = \frac{1}{2}(a + b - 5)((a - b)^2 + (a + 5)^2 + (b + 5)^2).$$

It follows that $a+b = 5$ or $a = b = -5$. However, $a = b = -5$ gives us no solutions for x.

Consider the equation $a + b = 5$, where $a = 2^x$ and $b = 3^{1/x}$. Observe that

$$x = \frac{\log a}{\log 2} = \frac{\log 3}{\log b}.$$

Thus $a + b = 5$ and $\log a \log b = \log 2 \log 3$. Two obvious solutions are $a = 2, b = 3$ and $a = 3, b = 2$ which give the solutions $x = 1$ and $x = \log_2 3$ respectively.

We will show that there are no more solutions. Note that $3^{1/x}$ is undefined at $x = 0$. Moreover, we must have $x > 0$ since otherwise $2^x + 3^{1/x} < 1 + 1 = 2$.

We consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined by $f(x) = 2^x + 3^{1/x}$. We have

$$f'(x) = 2^x \log 2 - \frac{3^{1/x} \log 3}{x^2} \quad \text{and}$$

$$f''(x) = 2^x (\log 2)^2 + \frac{3^{1/x}(\log 3)^2}{x^4} + \frac{2 \cdot 3^{1/x} \log 3}{x^3}.$$

It is clear that $f''(x) > 0$ for $x > 0$. So f is strictly convex and therefore the equation $f(x) = 5$ can have at most two solutions as claimed.

4570. *Proposed by Lorian Saceanu.*

If ABC is an acute angled triangle, then

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \le \frac{9}{\sqrt{11 + \tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}}} \le \frac{3\sqrt{3}}{2}.$$

We received 6 solutions, one of which was incorrect. We present the solution by Walther Janous, condensed by the editor.

Left-hand inequality

Letting

$$x = \cos\left(\frac{A}{2}\right), y = \cos\left(\frac{B}{2}\right), z = \cos\left(\frac{C}{2}\right)$$

the inequality becomes (upon squaring and clearing fractions)

$$(x+y+z)^2 \cdot \left(* + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) \leq 80$$

In all triangles, however,

$$x + y + z \leq \frac{3\sqrt{3}}{2},$$

and since we are here concerned only with acute triangles, we have $x, y, z > \sqrt{2}/2$. Thus, there exist positive real numbers ξ, η, and ψ such that

$$x = \frac{\sqrt{2}}{2} + \xi, y = \frac{\sqrt{2}}{2} + \eta, z = \frac{\sqrt{2}}{2} + \psi.$$

Setting

$$\Sigma = \xi + \eta + \psi,$$

we then have

$$\Sigma \leq \frac{3\left(\sqrt{3} - \sqrt{2}\right)}{2}. \tag{1}$$

Our goal is to prove the inequality

$$\left(\Sigma + \frac{3\sqrt{2}}{2}\right)^2 \cdot \left[\frac{2}{\left(\sqrt{2}\xi + 1\right)^2} + \frac{2}{\left(\sqrt{2}\eta + 1\right)^2} + \frac{2}{\left(\sqrt{2}\psi + 1\right)^2} + 8\right] \leq 81.$$

Keeping Σ fixed, we find the maximum of the second left-hand factor. Letting

$$\Phi = \frac{2}{\left(\sqrt{2}\xi + 1\right)^2} + \frac{2}{\left(\sqrt{2}\eta + 1\right)^2} + \frac{2}{\left(\sqrt{2}\psi + 1\right)^2} + 8 - \lambda \cdot (\xi + \eta + \psi),$$

we get

$$\frac{d}{d\xi}\Phi = 0 \Leftrightarrow \frac{4\sqrt{2}}{\left(\sqrt{2}\xi + 1\right)^3} - \lambda = 0,$$

and two similar expressions for η and ψ. This gives $\xi = \eta = \psi = \Sigma/3$ as the only stationary point of Φ in the interior of $B = \{\xi + \eta + \psi = \Sigma\}$. But the required inequality is then

$$\left(\Sigma + \frac{3\sqrt{2}}{2}\right)^2 \cdot \left[\frac{6}{\left(\sqrt{2} \cdot \frac{\Sigma}{3} + 1\right)^2} + 8\right] \leq 81,$$

which is succesively equivalent to

$$\left(\Sigma + \frac{3\sqrt{2}}{2}\right)^2 \leq \frac{27}{4}$$
$$4 \cdot \Sigma^2 + 12\sqrt{2} \cdot \Sigma - 9 = 0$$
$$-\frac{3\left(\sqrt{3}+\sqrt{2}\right)}{2} \leq \Sigma \leq \frac{3\left(\sqrt{3}-\sqrt{2}\right)}{2},$$

the last of which holds by (1).

We now consider the boundary of B. If say, $\psi = 0$, we have $C = \pi/2$. The inequality then becomes

$$\left[\cos\left(\frac{A}{2}\right) + \cos\left(\frac{\pi}{4} - \frac{A}{2}\right) + \frac{\sqrt{2}}{2}\right]^2 \cdot \left[12 + \tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{\pi}{4} - \frac{A}{2}\right)\right] \leq 81.$$

Setting $w = \tan\left(\frac{A}{2}\right)$, the inequality is

$$\left[\frac{1}{\sqrt{w^2+1}} + \frac{\sqrt{2}}{2} \cdot \frac{w+1}{\sqrt{w^2+1}} + \frac{\sqrt{2}}{2}\right]^2 \cdot \left[12 + w^2 + \left(\frac{1-w}{1+w}\right)^2\right] \leq 81,$$

which is equivalent to

$$\left[2 + \sqrt{2}(w+1) + \sqrt{2} \cdot \sqrt{w^2+1}\right]^2 \cdot \left[\left(12+w^2\right)(1+w)^2 + (1-w)^2\right]$$
$$\leq 324\left(w^2+1\right)(1+w)^2.$$

It is tedious but straightforward to show that this inequality holds for $w \in [1, 2]$.

Right-hand inequality

This inequality is equivalent to

$$\tan^2\left(\frac{A}{2}\right) + \tan^2\left(\frac{B}{2}\right) + \tan^2\left(\frac{C}{2}\right) \leq 1,$$

which holds by the convexity of the function $f(x) = \tan^2\left(\frac{x}{2}\right)$.