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IN THIS ISSUE / DANS CE NUMÉRO

MathemAttic

No. 12

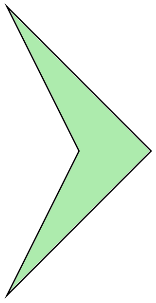
The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2020**.

MA56. For a given arithmetic series the sum of the first 50 terms is 200, and the sum of the next 50 terms is 2700. What is the first term of the series?

MA57. Define a *boomerang* as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees (see the accompanying figure). Let C be a convex polygon having s sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + \dfrac{s-2}{2}$.



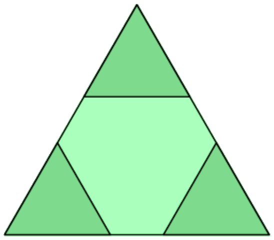
MA58. *Proposed by John McLoughlin.*

If the digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are randomly ordered to form a nine-digit number, what is the probability that the number is divisible by 99?

MA59. Find positive integer solutions of

$$x^{x^{x^{x}}} = (19 - y^x)y^{x^y} - 74.$$

MA60. Three equilateral triangles with sides of length 1 are shown shaded in a larger equilateral triangle. The total area of the three small triangles is half the area of the large triangle. What is the side-length of the larger equilateral triangle?



. . . .

Les problémes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

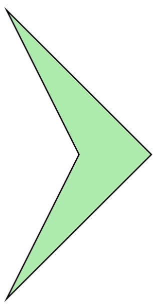
Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

MA56. Pour une certaine série arithméthique, la somme des 50 premiers termes est 200, tandis que la somme des 50 termes suivants est 2700. Déterminer le premier terme de la série.

MA57. On définit un *boomerang* comme étant un quadrilatère dont les côtés opposés ne se coupent pas et dont un des angles internes est de plus que 180 degrés, tel qu'illustré. Soit maintenant C un polygone convexe ayant s côtés. Supposer que la région interne de C est la réunion de q quadrilatères dont les intérieurs s'intersectent pas. Supposer de plus que b de ces quadrilatères sont des boomerangs. Démontrer que $q \geq b + \dfrac{s-2}{2}$.



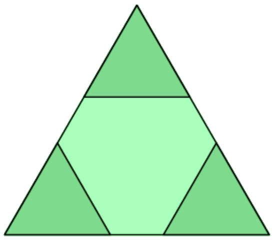
MA58. *Proposed by John McLoughlin.*

Les chiffres 1, 2, 3, 4, 5, 6, 7, 8 et 9 sont réarrangés de façon aléatoire pour former un entier à neuf chiffres. Déterminer la probabilité que cet entier soit divisible par 99.

MA59. Déterminer les solutions entières positives à l'équation

$$x^{x^{x^x}} = (19 - y^x)y^{x^y} - 74.$$

MA60. Trois triangles équilatéraux à côtés de longueur 1 sont indiqués à l'intérieur d'un plus gros triangle équilatéral. La surface totale des trois petits triangles égale la moitié de la surface du gros triangle. Déterminer la longueur du côté du gros triangle.



MathemAttic SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(7), p. 379-380.

MA31. Given that the areas of an equilateral triangle with side length t and a square with side length s are equal, determine the value of $\frac{t}{s}$.

The problem was proposed by John Grant McLoughlin.

We received 4 correct solutions. We present an amalgamation of the submitted solutions.

The formula for the area of the square is s^2 and the formula for the area of the equilateral triangle is $\frac{\sqrt{3}t^2}{4}$. We can deduce that $s^2 = \frac{\sqrt{3}t^2}{4}$, which we can rearrange to $\frac{4}{\sqrt{3}} = \frac{t^2}{s^2}$. Taking the square root of both sides and rationalizing the denominator we get $\frac{t}{s} = \frac{2\sqrt[4]{27}}{3}$.

MA32. Jack and Madeline are playing a dice game. Jack rolls a 6-sided die (numbered 1 to 6) and Madeline rolls an 8-sided die (numbered 1 to 8). The person who rolls the higher number wins the game. If Jack and Madeline roll the same number, the game is replayed. If a tie occurs a second time, then Jack is declared the winner. Which person has the better chance of winning? What are the odds in favour of this person winning the game?

Adapted from NLTA Math League Problem.

We received 2 correct submissions. We present the solution by Digby Smith, modified by the editor.

There are $6 \times 8 = 48$ possible outcomes of dice tosses. Let $P(X)$ denote the probability of X occurring. We observe the following:

1. $P(\text{Jack and Madeline roll the same number}) = \dfrac{1}{8}$

2. $P(\text{Madeline wins a toss})$
   $= P(\text{Madeline rolls 8})$
   $+ P(\text{Madeline rolls 7})$
   $+ P(\text{Madeline rolls 6, Jack rolls less than 6})$
   $+ P(\text{Madeline rolls 5, Jack rolls less than 5})$
   $+ P(\text{Madeline rolls 4, Jack rolls less than 4})$
   $+ P(\text{Madeline rolls 3, Jack rolls less than 3})$
   $+ P(\text{Madeline rolls 2, Jack rolls less than 2})$
   $= \dfrac{1}{8} + \dfrac{1}{8} + \left(\dfrac{1}{8}\right)\left(\dfrac{5}{6} + \dfrac{4}{6} + \dfrac{3}{6} + \dfrac{2}{6} + \dfrac{1}{6}\right) = \dfrac{9}{16}.$

3. $P(\text{Madeline wins on first toss}) = \dfrac{9}{16}$.

4. $P(\text{Madeline wins on second toss})$
$= P(\text{Jack and Madeline roll the same number}) \times P(\text{Madeline wins a toss})$
$= \dfrac{1}{8}\left(\dfrac{9}{16}\right) = \dfrac{9}{128}$

It follows that
$$P(\text{Madeline wins}) = \frac{9}{16} + \frac{9}{128} = \frac{81}{128}$$
and
$$P(\text{Jack wins}) = 1 - \frac{81}{128} = \frac{47}{128}.$$
We conclude that the odds in favour of Madeline are 81:47 and she has a better chance of winning.

MA33. Note that $\sqrt{2\frac{2}{3}} = 2\sqrt{\frac{2}{3}}$. Determine conditions for which $\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$, where a, b, c are positive integers.

The problem was proposed by John Grant McLoughlin.

We received 7 solutions, with varying conditions. We present the solution of the Missouri State University Problem Solving Group, which went so far as to be able to generate all such a, b, and c.

Assume that a, b, and c satisfy
$$\sqrt{a\frac{b}{c}} = a\sqrt{\frac{b}{c}}$$
or rather
$$\sqrt{\frac{ac+b}{c}} = a\sqrt{\frac{b}{c}}$$
Squaring both sides we find
$$\frac{ac+b}{c} = \frac{a^2b}{c}$$
which simplifies to $ac + b = a^2b$ and then $ac = b(a^2 - 1)$.

Now, since $\gcd(a, a^2-1) = 1$ we can see that $a \mid b(a^2-1)$ implies $a \mid b$. Let $b = ka$. Then we find that
$$ac = ka(a^2 - 1)$$
or rather $c = k(a^2 - 1)$.

It is readily verified that for any choice of $a \geq 2$ and $k \geq 1$, the triple
$$(a, b, c) = (a, ka, k(a^2 - 1))$$
will satisfy the condition.

MA34. Try to replace each $*$ with a different digit from 1 to 9 so that the multiplication is correct. (Each digit from 1 to 9 must be used once.)

$$\begin{array}{cccc} * & * & * & * \\ \times & & & * \\ \hline * & * & * & * \end{array}$$

Determine whether a solution is possible. If so, determine whether the solution is unique.

Originally from "Mathematical Puzzling" by Anthony Gardiner.

We received 1 correct solution and one incorrect solution. We present an approach for the problem and the conclusion of Doddy Kastanya.

A formal proof of the answers will just devolve into case based work. One approach is to consider the possible values for the single digit in the product. It cannot be 1 as otherwise the four digit numbers would be identical.

If the single digit were d and the four digit number in the product is n, then the resulting four digit number is dn. We can see $1234 \leq n$ and $dn \leq 9876$ so $n \leq \dfrac{9876}{d}$. Putting these together, we narrow down the possible values of n to the range $1234 \leq n \leq \dfrac{9876}{d}$. Then we just check for each possible $d = 2, \cdots, 9$ for possible n in this range with distinct digits which has dn with distinct digits, using all 9 nonzero digits once.

d	range of possible values for n	n that fit the description
2	$1234 \leq n \leq 4938$	none
3	$1234 \leq n \leq 3292$	none
4	$1234 \leq n \leq 2469$	1738, 1963
5	$1234 \leq n \leq 1975$	none
6	$1234 \leq n \leq 1646$	none
7	$1234 \leq n \leq 1410$	none
8	$1234 \leq n \leq 1234$	none
9	$1234 \leq n \leq 1097$	impossible

It takes some effort, or better yet a computer program, but we find exactly two solutions:

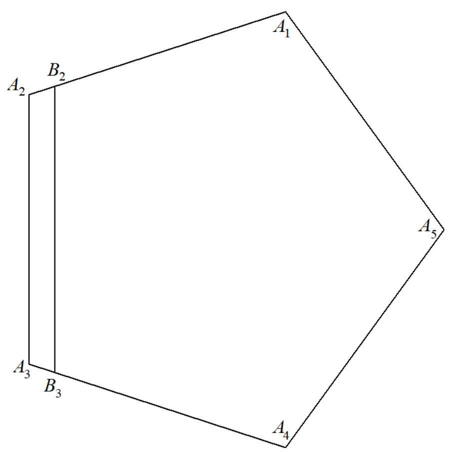
$$\begin{array}{ccccc} & 1 & 7 & 3 & 8 \\ \times & & & & 4 \\ \hline & 6 & 9 & 5 & 2 \end{array} \qquad \begin{array}{ccccc} & 1 & 9 & 6 & 3 \\ \times & & & & 4 \\ \hline & 7 & 8 & 5 & 2 \end{array}$$

MA35. A polygon has angles that are all equal. If the sides of this polygon are not all equal, show that the polygon must have an even number of sides.

Originally from "Mathematical Puzzling" by Anthony Gardiner.

We received two submissions (including the author), one proving that the claim of the author is false. We present the solution by the Missouri State University Problem Solving Group showing that the claim is false.

The claim is false. Given any regular n-gon, $n > 3$, with vertices $A_1, \ldots, A_n$, let B_2 be a point in the interior of $\overline{A_1A_2}$ and B_3 be a point in the interior of $\overline{A_3A_4}$ such that $B_2A_2 = A_3B_3$. Since $\overleftrightarrow{B_2B_3}$ is parallel to $\overleftrightarrow{A_2A_3}$, $\angle A_1B_2B_3 \cong \angle A_1A_2A_3 \cong \angle A_2A_3A_4 \cong \angle B_2B_3A_4$. Therefore the polygon with vertices $A_1, B_2, B_3, A_4, \ldots, A_n$ also has all angles congruent. However, $B_2B_3 \geq A_2A_3$ and $A_1B_2 < A_1A_2 = A_2A_3$, so all the sides are not congruent.



PROBLEM SOLVING VIGNETTES

No. 10

Shawn Godin

Revisiting Dirichlet

Sometimes a simple idea can be very powerful. In mathematics this happens all the time. In this issue, we revisit an idea, the pigeonhole principle, that I used to solve a problem from the course C&O 380 that I took as an undergraduate from professor Ross Honsberger [2018: 44(4), p. 157-159].

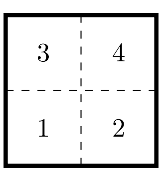
Pigeonhole principle (a.k.a. Dirichlet box principle):
If you have n pigeonholes and $m > n$ pigeons, then there must be at least one pigeonhole that contains at least two pigeons.

To see this in action, think about the following statement: if 8 people are gathered together in a room, at least two of them were born on the same day of the week. Why does this work? Let's think of the worst case scenario. Imagine the people arrive at the room one at a time, and we only let people in that were born on a different day of the week than everyone else that is already in the room. We can do this up to a point. Once we reach seven people, every day has been accounted for which means that if we allow an eighth person to enter, this person **must** have been born on the same day as somebody already present. The pigeonhole principle is an example of an **existence** theorem in mathematics. It tells us something must exist, but it doesn't really give us any idea how to find it.

The key idea for using the pigeonhole principle is to define our groups in such a way that we are forced to have an overlap. In some problems, like the example above, it is obvious. In others we have to work a bit. Let's look at problem #7 from C&O 380 [2018: 44(10), p. 419]:

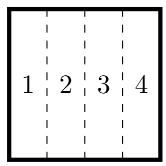
> *Of* 5 *points inside a square of unit side, show that some pair is less than* $\frac{\sqrt{2}}{2}$ *units apart.*

Since we have 5 points and we are trying to force a pair of them to have a condition, it gives us the hint that we want to work with 4 categories. It seems natural to cut the square into four congruent squares as shown below.



Thus the pigeonhole principle guarantees that at least two points will be in the same region. How far apart can they be? The furthest apart would be if they were at opposite vertices of the square, which the Pythagorean theorem tells us are $\frac{\sqrt{2}}{2}$ units apart. Since we are picking points from *inside* the square, we are guaranteed that two must be less than $\frac{\sqrt{2}}{2}$ units apart, so we are done.

Care is needed since we can break the square up into *any* four disjoint regions and the pigeonhole principle will ensure that at least two are in the same region. If we don't create our regions carefully, we may not get what we are after. For example, we could have created the four regions as shown below.

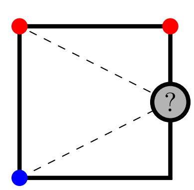


We still know for sure that at least two of the points will fall in the same rectangle, but now they can be as far apart as (almost) $\frac{\sqrt{17}}{4}$ which is larger than $\frac{\sqrt{2}}{2}$. We cannot conclude that the problem is impossible just because our configuration gives a larger number. So build your groups carefully.

Next, let's consider problem #11 from C&O 380 [2019: 45(4), p. 176]: *Prove that no matter how the points of a closed unit square are coloured red or blue, either some two red points or some two blue points are at least* $\frac{\sqrt{5}}{2}$ *units apart.*

Consider any three corners of the square. Since we have only two colours, at least two of the points have the same colour. If two of the same coloured corners are diagonally opposite of each other, they are $\sqrt{2}$ units apart which is larger than $\frac{\sqrt{5}}{2}$ and we are done.

Suppose the two points that are the same colour are not diagonally opposite. Then we can assume, without loss of generality, that two vertices on the same edge are red and the other vertex is blue, as in the diagram below. Consider the midpoint of the edge nearest one of the red vertices, but not between one of the three coloured vertices, as indicated by a "?" in the diagram. This point, taken with the further red vertex and the blue vertex, forms an isosceles triangle. Once again, we can conclude that at least two vertices will be the same colour. Since the original two vertices are coloured oppositely, the new point is the same colour as one of them. The Pythagorean theorem again yields that these points are $\frac{\sqrt{5}}{2}$ units apart, and we are done.



Notice that in the first case we were told we had 5 points which gave us the hint that we are looking at four categories. The second problem was a little different. We had to focus in on the two colours, which would be our *categories*. That meant we had to look at 3 points to force two of them to be the same colour and piece our proof together from there. Let's look at problems #21–25 from C&O 380 and choose one to focus on.

#21. Suppose each point of the plane is coloured red or blue. Show that some rectangle has its vertices all the same colour.

#22. In chess, is it possible for a knight to go from the lower left corner square of the board to the upper right corner square and in the process to land exactly once on each other square?

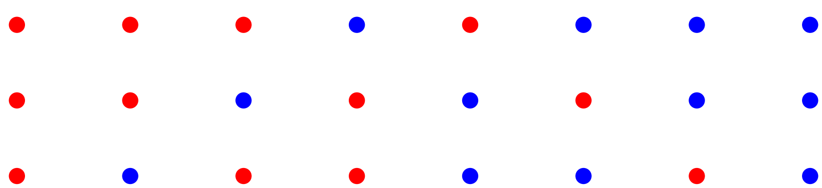
#23. Prove that the number of people at the opera next Thursday night who will shake hands an odd number of times is an even number of people.

#24. Prove that no matter what three points of a square lattice are joined, an equilateral triangle will never occur.

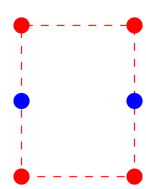
#25. Prove that

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{99}{100} < \frac{1}{10}$$

Let's looks closely at #21. Again we have two colours, so again we know that for any three points we pick on the plane, at least two will be the same colour. We need four points, all the same colour, forming a rectangle. Let's investigate all the possibilities of three coloured points. Taking into account the order of the points, there are $2 \times 2 \times 2 = 8$ possibilities, pictured below.

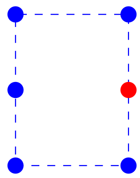


Notice that, if I have two identically coloured arrangements of three points, correctly arranged, I will have satisfied the conditions of the problem.



Thus, if I pick a 3×9 rectangular grid of points, since there are 8 possible ways to colour the points in each column, we must have at least two columns whose colourings are the same. Since there are three points in each column, at least two of them are the same, so we can create our rectangle with vertices the same colour, and we are done.

If we think a bit, we can actually refine our proof. Notice that two of the 8 colourings have all three points coloured the same. One of these three coloured configurations will match with any configuration having two points of that same colour to produce our desired rectangle.

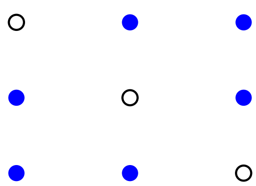


Therefore, we can classify our groups of three as one of two types: type R with *at least* 2 red points and type B with *at least* 2 blue points. Now, if we have a 3×7 rectangular grid of points, at least 4 must be of the same type R or B. This extends the original idea of the pigeonhole principle to:

Generalized pigeonhole principle:
If you have n pigeonholes and $m > kn$ pigeons, for some integer k, then there must be at least one pigeonhole that contains at least $k + 1$ pigeons.

In our case, if we look at the worst case scenario, when I have six things into two groups, I could have three of each. As soon as I add another, there must be some group with four($7 = 3 \times 2 + 1$, so we must have one with at least $3 + 1 = 4$).

So there must be at least four Rs or at least four Bs. Suppose we have four Bs. We can break the Bs up into three groups: the first and second must be blue; the first and third must be blue; and the second and the third must be blue. These possibilities are pictured below, where the "open" point could be *either* red or blue. With this classification, the configuration with three blues would fit in *any* category. Thus, since we have four Bs, but three groups, there must be at least two in the same group, guaranteeing our desired rectangle.



Knowing the nuances of a technique or theorem and how to use it in your solutions is very important. Hopefully, the examples provide some insight into using the pigeonhole principle or problem solving in general. Enjoy the rest of the problems, we may talk about some of them in a future column.

OLYMPIAD CORNER

No. 380

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **April 15, 2020**.

OC466. In a convex quadrilateral $ABCD$ the diagonals AC and BD intersect at point O. The points A_1, B_1, C_1, D_1 are respectively on the segments AO, BO, CO, DO such that $AA_1 = CC_1$ and $BB_1 = DD_1$. Let M and N be, respectively, the second intersections of the circumcircles of $\triangle AOB$ and $\triangle COD$ and the circumcircles of $\triangle AOD$ and $\triangle BOC$, and let P and Q be, respectively, the second intersections of the circumcircles of $\triangle A_1OB_1$ and $\triangle C_1OD_1$ and the circumcircles of $\triangle A_1OD_1$ and $\triangle B_1OC_1$. Prove that the points M, N, P, Q lie on a circle.

OC467. Let $p > 2$ be a prime number and let $x, y \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$. Prove that if $x\,(p-x)\,y\,(p-y)$ is a perfect square, then $x = y$.

OC468. Let $ABCD$ be a cyclic quadrilateral. The point P is chosen on the line AB such that the circle passing through C, D and P touches the line AB. Similarly, the point Q is chosen on the line CD such that the circle passing through A, B and Q touches the line CD. Prove that the distance between P and the line CD equals the distance between Q and the line AB.

OC469. Prove that for all nonnegative real numbers x, y, z satisfying $x + y + z = 1$ it holds

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

OC470. Prove that there is a natural number n having more than 2017 divisors d such that

$$\sqrt{n} \leq d < 1.01\sqrt{n}.$$

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC466. Les diagonales AC et BD du quadrilatère convexe $ABCD$ intersectent en O. Les points A_1, B_1, C_1, D_1 se trouvent sur les segments AO, BO, CO, DO, respectivement, de façon à ce que $AA_1 = CC_1$ et $BB_1 = DD_1$. Soient M et N les deuxièmes points d'intersection des cercles circonscrits de $\triangle AOB$ et $\triangle COD$, puis des cercles circonscrits de $\triangle AOD$ et $\triangle BOC$, respectivement. Aussi, soient P et Q les deuxièmes points d'intersection des cercles circonscrits de $\triangle A_1OB_1$ et $\triangle C_1OD_1$, puis des cercles circonscrits de $\triangle A_1OD_1$ et $\triangle B_1OC_1$, respectivement. Démontrer que M, N, P et Q se trouvent sur un même cercle.

OC467. Soit $p > 2$ un nombre premier et soient $x, y \in \left\{1, 2, \ldots, \frac{p-1}{2}\right\}$. Démontrer que si $x\,(p-x)\,y\,(p-y)$ est un carré parfait, alors $x = y$.

OC468. Soit $ABCD$ un quadrilatère cyclique. Le point P est choisi sur la ligne AB de façon à ce que le cercle passant par C, D et P touche la ligne AB. De façon similaire, le point Q est choisi sur la ligne CD de façon à ce que le cercle passant par A, B et Q touche la ligne CD. Démontrer que la distance entre P et la ligne CD égale la distance entre Q et la ligne AB.

OC469. Démontrer que pour tous nombres réels non négatifs x, y, z tels que $x + y + z = 1$, l'inégalité suivante tient:

$$1 \leq \frac{x}{1-yz} + \frac{y}{1-zx} + \frac{z}{1-xy} \leq \frac{9}{8}.$$

OC470. Démontrer qu'il existe un nombre naturel n ayant plus que 2017 diviseurs d tels que

$$\sqrt{n} \leq d < 1.01\sqrt{n}.$$

OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2019: 45(7), p. 395–396.

OC441. Let $f : [0, \infty) \to (0, \infty)$ be a continuous function.

(a) Prove that there exists a natural number n_0 such that for any natural number $n > n_0$ there exists a unique real number $x_n > 0$ for which

$$n \int_0^{x_n} f(t)\ dt = 1;$$

(b) Prove that the sequence $(nx_n)_{n\geq 1}$ is convergent and find its limit.

Originally Romania MO, 1st Problem, Grade 12, Final Round 2017.

We received 4 submissions. We present the solution by Ivko Dimitrić.

(a) The function $F(x) = \int_0^x f(t)\,dt$ is the antiderivative of f that satisfies $F(0) = 0$ and $F'(x) = f(x) > 0$. Thus, F is increasing and hence one-to-one.

Let $S = \sup_{x\geq 0} F(x)$. Then S is a positive number or ∞. If $S = \infty$, take $n_0 = 1$. Otherwise, let n_0 be the first positive integer for which $n_0 S \geq 1$. In any case, we will have $1/n_0 \leq S$. Then, for any integer $n > n_0$ we obtain

$$F(0) = 0 < \frac{1}{n} < \frac{1}{n_0} \leq S.$$

Since F is differentiable and hence continuous, by the Intermediate Value Theorem, F takes all values between 0 and S and since F is, moreover, one-to-one, there exists a *unique* number $x_n \in (0, \infty)$ such that

$$F(x_n) = \frac{1}{n}, \quad \text{i. e.} \quad n \int_0^{x_n} f(t)\,dt = 1.$$

(b) Since F is increasing and

$$F(x_n) = \frac{1}{n} > \frac{1}{n+1} = F(x_{n+1}),$$

it follows that $x_n > x_{n+1}$. So, the sequence $(x_n)_{n>n_0}$ is decreasing and being bounded by 0 from below, it must be convergent. Let $L = \lim_{n\to\infty} x_n$. By continuity of F we have then

$$F(L) = F(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \frac{1}{n} = 0 = F(0),$$

implying $L = 0$, since F is also one-to-one. By the Mean Value Theorem for the integrals, there exist a number c_n, $0 < c_n < x_n$ such that

$$\frac{1}{x_n - 0}\int_0^{x_n} f(t)\,dt = f(c_n) \quad \Longleftrightarrow \quad \frac{1}{n} = F(x_n) = x_n f(c_n),$$

so that

$$nx_n = \frac{1}{f(c_n)}.$$

Since $0 < c_n < x_n$ and $\lim_{n\to\infty} x_n = 0$, by the Squeeze Theorem we have that $\lim_{n\to\infty} c_n = 0$. Consequently, for $n > n_0$

$$\lim_{n\to\infty}(nx_n) = \frac{1}{\lim_{n\to\infty} f(c_n)} = \frac{1}{f(\lim_{n\to\infty} c_n)} = \frac{1}{f(0)}$$

by continuity of f, which shows that the sequence $(nx_n)_{n>n_0}$ is convergent with the limit of $1/f(0)$, and the same conclusion holds for the sequence $(nx_n)_{n\geq 1}$, regardless of the choice of few initial terms x_n when $n \leq n_0$.

OC442. Let $H = \{1, 2, \ldots, n\}$. Are there two disjoint subsets A and B such that $A \cup B = H$ and such that the sum of the elements in A is equal to the product of the elements in B if **(a)** $n = 2016$? **(b)** $n = 2017$?

Originally Hungary MO, 1st Problem, 2nd Category, Final Round 2017.

We received 3 *submissions. We present the solution by Nicholas Fleece, who generalized the problem.*

Rather than simply searching for a solution to these two problems, we wish to search for a method that will solve this problem for all n. We can begin this by looking at subsets for smaller n's. The first thing we note is that for $n = 3$ we have the solution:

$$A = \{1, 2\},\ B = \{3\}$$

We find no such solution for $n = 4$, but starting at $n = 5$ we begin to see a pattern. Note the following table, where in each row, the bold numbers are elements of B, and the rest are in A.

1	**2**	3	**4**	5						
1	**2**	3	4	5	**6**					
1	2	**3**	4	5	**6**	7				
1	2	**3**	4	5	6	7	**8**			
1	2	3	**4**	5	6	7	**8**	9		
1	2	3	**4**	5	6	7	8	9	**10**	
1	2	3	4	**5**	6	7	8	9	**10**	11

It is by viewing this pattern that we can arrive at the following solution.

Let $H_n = \{1, 2, ..., n\}$. Define

$$B_n = \{1, \frac{n + n(\text{mod}2)}{2} - 1, n - n(\text{mod}2)\} \quad \text{and} \quad A_n = H_n \setminus B_n.$$

Clearly, $A_n \cap B_n = \varnothing$ and $A_n \cup B_n = H_n$. It then remains to be shown that:

$$\prod_{b \in B_n} b = \sum_{a \in A_n} a. \tag{1}$$

First, note the following:

$$\sum_{h \in H_n} h = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{2}$$

In order to show that this solution holds $\forall n \geq 5$, we will break this up into two cases: one for even n, and one for odd n.

Case 1: n is odd. In this case, $B_n = \{1, \frac{n-1}{2}, n-1\}$.

In order to prove (1) holds $\forall n \geq 5$ where n is odd, we must first show that (1) holds in the base case of $n = 5$. In this case we have $A_5 = \{3, 5\}$ and $B_5 = \{1, 2, 4\}$ and

$$(1)(2)(4) = 3 + 5.$$

So (1) holds for $n = 5$. Now, inductively assume (1) holds for n.

Then we have:

$$\prod_{b \in B_n} b = \sum_{a \in A_n} a.$$

From (2) we arrive at:

$$\begin{aligned}
(1)\left(\frac{n-1}{2}\right)(n-1) &= \frac{n(n+1)}{2} - \frac{n-1}{2} - (n-1) - 1 \\
\frac{n^2 - 2n + 1}{2} &= \frac{n^2 + 1}{2} - n \\
\frac{n^2 - 2n + 1}{2} + 2n &= \frac{n^2 + 1}{2} - n + 2n \\
(1)\left(\frac{n+1}{2}\right)(n+1) &= \frac{n^2 + 2n + 1}{2} \\
(1)\left(\frac{n+1}{2}\right)(n+1) &= \frac{(n+2)(n+3) - 3n - 5}{2} \\
(1)\left(\frac{n+1}{2}\right)(n+1) &= \frac{(n+2)(n+3)}{2} - \frac{n+1}{2} - (n+1) - 1 \\
\prod_{b \in B_{n+2}} b &= \sum_{a \in A_{n+2}} a.
\end{aligned}$$

Thus, (1) is true $\forall n \geq 5$, where n is odd.

Case 2: n is even. In this case, $B_n = \{1, \frac{n-2}{2}, n\}$.

Similarly, to prove (1) holds $\forall n \geq 6$ where n is even, we must first show that (1) holds in the base case of $n = 6$. In this case we have $A_6 = \{3, 4, 5\}$ and $B_6 = \{1, 2, 6\}$ and

$$(1)(2)(6) = 3 + 4 + 5.$$

So (1) holds for $n = 6$. Again, inductively assume (1) holds for n.

Then we have:

$$\prod_{b \in B_n} b = \sum_{a \in A_n} a.$$

From (2) we arrive at:

$$\begin{aligned}
(1)\left(\frac{n-2}{2}\right) n &= \frac{n(n+1)}{2} - \frac{n-2}{2} - n - 1 \\
\frac{n^2 - 2n}{2} &= \frac{n(n+1)}{2} - \frac{n-2}{2} - n - 1 \\
\frac{n^2 - 2n}{2} + 2n &= \frac{n(n+1)}{2} - \frac{n-2}{2} - n - 1 + 2n \\
(1)\left(\frac{n}{2}\right)(n+2) &= \frac{n(n+2)}{2} \\
(1)\left(\frac{n}{2}\right)(n+2) &= \frac{(n+2)(n+3) - 3(n+2)}{2} \\
(1)\left(\frac{n}{2}\right)(n+2) &= \frac{(n+2)(n+3)}{2} - \frac{n}{2} - (n+2) - 1 \\
\prod_{b \in B_{n+2}} b &= \sum_{a \in A_{n+2}} a.
\end{aligned}$$

Thus, (1) is true $\forall n \geq 6$, where n is even.

Combining these two cases, we have that (1) is true $\forall n \geq 5$.

We then have the following solutions:

(a) $B_{2016} = \{1, 1007, 2016\}$, which is verified by:

$$\begin{aligned}
(1)(1007)(2016) &= \frac{(2016)(2017)}{2} - 2016 - 1007 - 1 \\
2030112 &= 2030112 \\
\prod_{b \in B_{2016}} b &= \sum_{a \in A_{2016}} a.
\end{aligned}$$

(b) $B_{2017} = \{1, 1008, 2016\}$, which is verified by:

$$\begin{aligned}(1)(1008)(2016) &= \frac{(2017)(2018)}{2} - 2016 - 1008 - 1\\ 2032128 &= 2032128\\ \prod_{b \in B_{2017}} b &= \sum_{a \in A_{2017}} a.\end{aligned}$$

OC443. In a triangle ABC, the foot of the altitude drawn from A is T and the angle bisector of $\angle B$ intersects side AC at D. If $\angle BDA = 45^\circ$, find $\angle DTC$.

Originally Hungary MO, 2nd Problem, 2nd Category, Final Round 2017.

We received 9 *submissions. We present* 2 *solutions.*

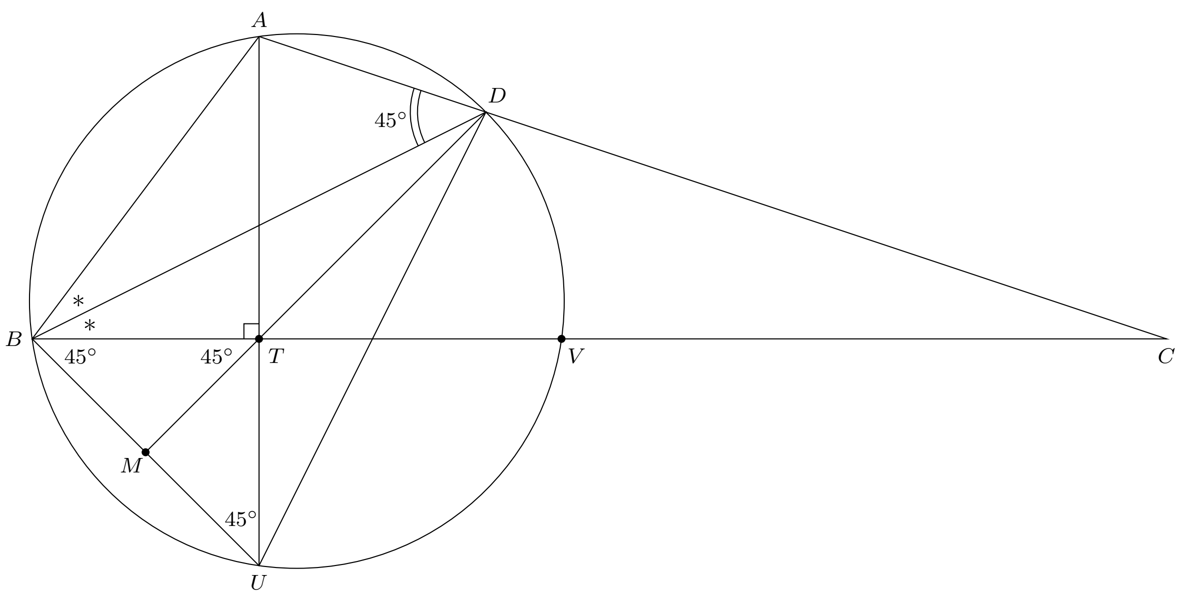
Solution 1, by Miguel Amengual Covas.

In the accompanying figure, the segments AT (extended) and BC intersect the circumcircle of $\triangle ABD$ at U and V respectively. Quadrilateral $ABUD$ is cyclic, and on chord AB we have

$$\angle BUT = \angle BUA = \angle BDA = 45^\circ$$

in right-triangle BTU, implying

$$\text{arc } BA = \text{arc } UV \quad (= 2 \times 45^\circ = 90^\circ)\,. \tag{3}$$



Since BD bisects $\angle B$, D bisects arc AV, that is,

$$\text{arc } AD = \text{arc } DV. \tag{4}$$

Adding (1) and (2) yields

$$\text{arc } BD = \text{arc } DU$$

and hence

$$BD = DU.$$

Therefore $\triangle BDU$ is isosceles and triangles BDT and UDT are congruent (s-s-s) with $\angle BDT = \angle TDU$. Being DT the internal bisector of angle D in isosceles triangle BDU, DT is actually the perpendicular bisector of BU i.e.,

$$DT \perp BU. \tag{5}$$

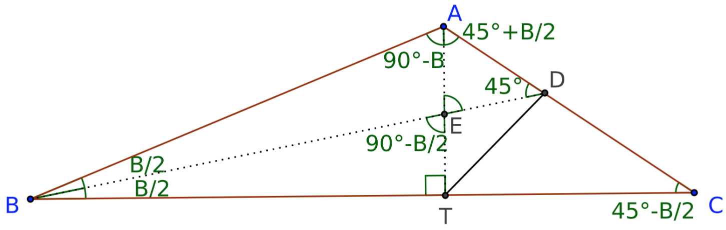
Now, if M is the midpoint of the basis BU in isosceles right-angled triangle BTU, we have

$$TM \perp BU. \tag{6}$$

From (3) and (4) we conclude that points D, T, M are collinear. Hence $\angle BTM$ and $\angle DTV$ are vertically opposite angles, and so

$$\angle DTC = \angle DTV = \angle BTM = 45^\circ.$$

Solution 2, by Oliver Geupel.



We prove that $\angle DTC = 45^\circ$. Suppose that the lines AT and BD meet at point E. Checking the sums of angles in triangles ABT, BTE, AED, and ATC, we successively find:

$$\angle BAT = 90^\circ - \angle B, \quad \angle DEA = \angle BET = 90^\circ - \angle B/2, \quad \angle TAC = 45^\circ + \angle B/2,$$

$$\angle A = \angle BAT + \angle TAC = 135^\circ - \angle B/2, \quad \angle C = 45^\circ - \angle B/2.$$

Applying the angle bisector theorem and the law of sines, we deduce

$$\begin{aligned}\frac{CD}{AD} = \frac{BC}{AB} &= \frac{\sin\angle A}{\sin\angle C} \\ &= \frac{\sin(135^\circ - \angle B/2)}{\sin(45^\circ - \angle B/2} \\ &= \frac{\cos(45^\circ - \angle B/2)}{\sin(45^\circ - \angle B/2)} \\ &= \cot\angle C \\ &= \frac{CT}{TA}.\end{aligned}$$

Thus, in triangle ATC, point D divides CA in the ratio $CT : TA$. By the converse of the angle bisector theorem, TD bisects the right angle $\angle CTA$.

OC444. We have n^2 empty boxes, each with a square bottom. The height and the width of each box are natural numbers in the set $\{1, 2, \ldots, n\}$. Each box differs from any other box in at least one of these two dimensions. We are allowed to insert a box into another if each dimension of the first box is smaller than the corresponding dimension of the second box and at least one of the dimensions is at least 2 units less than the corresponding larger box dimension. In this way, we can create a sequence of boxes inserted into each other in the same orientation (i.e. the first box is inside the second, the second box is inside the third, etc.). We store each sequence of boxes on a shelf with each shelf holding one set of nested boxes. Determine the smallest number of shelves needed to store all the n^2 boxes.

Originally Czech-Slovakia MO, 2nd Problem, Category A, Local Round, 2017.

We received no solutions to this problem.

OC445. There are 100 diamonds in a pile, of which 50 are genuine and 50 are fake. We invited a distinguished expert, who can recognize which diamonds are genuine. Each time we show him three diamonds, he chooses two of them and (truthfully) tells whether they are both genuine, one genuine or none genuine. Establish if we can guarantee to spot all the genuine diamonds no matter how the expert chooses the judged pair.

Originally Czech-Slovakia MO, 1st Problem, Final Round 2017.

We received one incomplete solution to this problem.

Applications of the Riemann integral to find limits

Robert Bosch

Sometimes the terms of a sequence can be recognized as successive Riemann sums for a function, and this can prove helpful for finding the limit of the sequence.

1 The Riemann Integral

The Riemann integral is defined as the limit of a sequence of sums:

$$\int_a^b f(x)dx = \lim_{||\mathcal{P}||\to 0} \sum_{k=1}^{n} (x_k - x_{k-1}) f(\xi_k),$$

where $||\mathcal{P}||$ is the norm of the partition $\mathcal{P}$ and $x_{k-1} \leq \xi_k \leq x_k$. The sums are called *Riemann sums.*

It was introduced by Georg Friedrich Bernhard Riemann ($1826 - 1866$) in the paper *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe* (On the representability of a function by a trigonometric series.) This paper was submitted to the University of Göttingen in 1854 as Riemann's *Habilitationsschrift* (qualification to become an instructor.) It was published in 1868 in *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (Proceedings of the Royal Philosophical Society at Göttingen,) vol. 13, pages $87 - 132$. For Riemann's definition of his integral, see section 4, "*Über den Begriff eines bestimmten Integrals und den Umfang seiner Gültigkeit*" ("On the concept of a definite integral and the extent of its validity,") pages $101 - 103$.

A partition $\mathcal{P}$ of an interval $[a, b]$ is a finite sequence of numbers of the form

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b;$$

each $[x_{k-1}, x_k]$ is called a sub-interval of the partition. The mesh or norm of a partition is defined to be the length of the longest sub-interval, that is,

$$||\mathcal{P}|| = \max\{x_k - x_{k-1} : k = 1, 2, ..., n\},$$

and f is Riemann integrable in $[a, b]$ when there exists a real number I such that, for every positive real number ϵ there is a positive real number δ such that if $||\mathcal{P}|| < \delta$, then for any Riemann sum $S(\mathcal{P}, f)$ we have $|S(\mathcal{P}, f) - I| < \epsilon$. In this article we are assuming all the functions f are continuous on $[a, b]$, so are integrable.

While the definition of a Riemann sum allows for unevenly spaced x_k and odd choices of ξ_k, but as by definition the integral is independent of these choices, it is most usual to consider uniformly spaced x_k and $\xi_k = x_k$.

2 Problems

The first and second problems are from [1]. Our solutions to both are different from the solutions in the book.

Problem 1. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{4n^2+k^2}.$$

Solution. To solve this problem the suitable partition $\mathcal{P}$, the function f and intermediate points ξ_k are

$$\begin{aligned}\mathcal{P} &= \left\{0<\frac{1}{n}<\frac{2}{n}<\cdots<\frac{n-1}{n}<1\right\},\\ f(x) &= \frac{1}{4+x^2},\\ \xi_k &= \frac{k}{n}.\end{aligned}$$

From $x_k=\dfrac{k}{n}$, we get

$$x_k-x_{k-1}=\frac{k}{n}-\frac{k-1}{n}=\frac{1}{n}.$$

Clearly $||\mathcal{P}||=\frac{1}{n}\to 0$ when $n\to\infty$. Then

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{4n^2+k^2} &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\cdot\frac{1}{4+\left(\frac{k}{n}\right)^2}=\lim_{n\to\infty}\sum_{k=1}^{n}(x_k-x_{k-1})\,f\left(\frac{k}{n}\right),\\ &= \lim_{n\to\infty}\sum_{k=1}^{n}(x_k-x_{k-1})\,f(\xi_k)=\lim_{||\mathcal{P}||\to 0}\sum_{k=1}^{n}(x_k-x_{k-1})\,f(\xi_k),\\ &= \int_0^1\frac{1}{4+x^2}dx=\frac{1}{4}\int_0^1\frac{1}{1+\left(\frac{x}{2}\right)^2}dx=\frac{1}{2}\arctan\left(\frac{x}{2}\right)\Big|_0^1=\frac{1}{2}\arctan\left(\frac{1}{2}\right).\end{aligned}$$

Problem 2. Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{n}\arctan\left(\frac{n}{n^2+k^2}\right).$$

Solution. We start with the following Lemma.

Lemma: $x-\dfrac{x^3}{3}<\arctan x<x$ for $x>0$.

Proof. The right side is equivalent to the well known inequality $\tan x>x$ for $x>0$. To prove the left side we consider the function

$$f(x)=3\arctan x+x^3-3x,$$

we will show $f(x)$ is positive for positive x. Since $f(0) = 0$, it is enough to prove that $f(x)$ is increasing on $(0, +\infty)$. The first derivative of $f(x)$ is

$$f'(x) = \frac{3}{x^2+1} + 3x^2 - 3.$$

Thus $f'(x) > 0$ if and only if $x^4 > 0$.

By the Lemma,

$$\frac{n}{n^2+k^2} - \frac{1}{3}\left(\frac{n}{n^2+k^2}\right)^3 < \arctan\left(\frac{n}{n^2+k^2}\right) < \frac{n}{n^2+k^2}.$$

Now we are ready to consider the summation from $k = 1$ to n, and later the limit when n goes to infinity, with the intention to get the initial limit proposed in Problem 2.

As with the previous problem, one will easily show

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}.$$

On the other hand, we have

$$\sum_{k=1}^{n} \left(\frac{n}{n^2+k^2}\right)^3 < \sum_{k=1}^{n} \left(\frac{n}{n^2}\right)^3 = \sum_{k=1}^{n} \frac{1}{n^3} = \frac{1}{n^2},$$

and thus

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{n}{n^2+k^2}\right)^3 = 0,$$

by the Squeeze Theorem. Another application of the Squeeze Theorem implies that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \arctan\left(\frac{n}{n^2+k^2}\right) = \frac{\pi}{4}.$$

Now, we will see two problems from The Putnam competition.

Problem 3. [**A3, 1961**] Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n^2} \frac{n}{n^2+k^2}.$$

Solution. [2] We write the sum in the form

$$S_n = \frac{1}{n} \sum_{k=1}^{n^2} \frac{1}{1+\left(\frac{k}{n}\right)^2}.$$

Since

$$\int_{k/n}^{(k^2+1)/n} \frac{1}{1+x^2}dx < \frac{1}{n}\cdot\frac{1}{1+\left(\frac{k}{n}\right)^2} < \int_{(k-1)/n}^{k^2/n} \frac{1}{1+x^2}dx,$$

we get

$$\int_{1/n}^{(n^2+1)/n} \frac{1}{1+x^2}dx < S_n < \int_0^n \frac{1}{1+x^2}dx.$$

Now

$$\int_0^n \frac{1}{1+x^2}dx = \arctan n, \qquad \lim_{n\to\infty} \arctan n = \frac{\pi}{2},$$

$$\int_{1/n}^{n+1/n} \frac{1}{1+x^2}dx = \arctan\left(n+\frac{1}{n}\right) - \arctan\left(\frac{1}{n}\right) \to \frac{\pi}{2}.$$

So

$$\lim_{n\to\infty} S_n = \frac{\pi}{2}.$$

Problem 4. [B5, 2004] Evaluate

$$\lim_{x\to 1^-} \prod_{n=0}^{\infty} \left(\frac{1+x^{n+1}}{1+x^n}\right)^{x^n}.$$

Solution. [Greg Price, via Tony Zhang and Anders Kaseorg: from Kiran Kedlaya's Putnam archives]. By taking logarithms, we see that the desired limit is e^L, where

$$L = \lim_{x\to 1^-} \sum_{n=0}^{\infty} x^n \left(\ln(1+x^{n+1}) - \ln(1+x^n)\right).$$

Put $t_n(x) = \ln(1+x^n)$; we can then write $x^n = \exp(t_n(x)) - 1$, and

$$L = \lim_{x\to 1^-} \sum_{n=0}^{\infty} (t_n(x) - t_{n+1}(x))(1-\exp(t_n(x))).$$

The expression on the right is a Riemann sum approximating the integral

$$\int_0^{\ln 2} (1-e^t)dt,$$

over the subdivision of $[0, \ln 2)$ given by the $t_n(x)$. As $x \to 1^-$, the maximum difference between consecutive $t_n(x)$ tends to 0, so the Riemann sum tends to the value of the integral. Hence

$$L = \int_0^{\ln 2} (1-e^t)dt = \ln 2 - 1\,.$$

Problem 5. [4] Denote by G_n the geometric mean of the binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$

Prove that

$$\lim_{n\to\infty} G_n = \sqrt{e}.$$

Solution. [5]

$$\begin{aligned}\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n} &= \prod_{k=0}^{n} \frac{n!}{k!(n-k)!} = \frac{(n!)^{n+1}}{(1!2!\cdots n!)^2}\\ &= \prod_{k=1}^{n}(n+1-k)^{n+1-2k} = \prod_{k=1}^{n}\left(\frac{n+1-k}{n+1}\right)^{n+1-2k}.\end{aligned}$$

The last equality would hold for any constant denominator a because

$$\sum_{k=1}^{n}(n+1-2k) = 0,$$

whence the denominator is just $a^0 = 1$. Therefore,

$$G_n = \sqrt[n+1]{\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n}} = \prod_{k=1}^{n}\left(1-\frac{k}{n+1}\right)^{1-\frac{2k}{n+1}}.$$

Taking the natural logarithm, we obtain

$$\frac{1}{n}\ln G_n = \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{2k}{n+1}\right)\ln\left(1-\frac{k}{n+1}\right).$$

This is just a Riemann sum for the function $f(x) = (1-2x)\ln(1-x)$ over the interval $[0,1]$. Passing to the limit, we obtain

$$\lim_{n\to\infty}\frac{1}{n}\ln G_n = \int_0^1 (1-2x)\ln(1-x)dx.$$

We split the integral and compute it by parts as follows:

$$\begin{aligned}&\int_0^1 (1-2x)\ln(1-x)dx\\ &= 2\int_0^1 (1-x)\ln(1-x)dx - \int_0^1 \ln(1-x)dx\\ &= -(1-x)^2\ln(1-x)\mid_0^1 -2\int_0^1 \frac{(1-x)^2}{2}\cdot\frac{1}{1-x}dx + (1-x)\ln(1-x)\mid_0^1 + x\mid_0^1\\ &= -\int_0^1 (1-x)dx + 1 = \frac{1}{2}.\end{aligned}$$

Exponentiating back, we obtain

$$\lim_{n\to\infty} G_n = \sqrt{e}.$$

Cezar Lupu proposed problem U131, to the journal Mathematical Reflections, Volume 4, 2009:

Problem U131. Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\arctan\left(\frac{k}{n}\right)}{n+k} \cdot \frac{\varphi(k)}{k} = \frac{3\ln 2}{4\pi},$$

where φ denotes Euler's totient function.

In [7], the reader can find a solution to problem U131, and to some other ones that are similar.

3 Problems for Independent Study

After several problems with solutions, we provide the following list of problems for the reader to attempt. The requirement to find the limit as $n \to \infty$ of sums indexed from 1 to n may be taken as a hint to interpret the limit as a Riemann integral, as in Problem 1. Once an appropriate function is identified, the integral will give the limit.

In many analysis courses, where the approach is not problem-based, this method is not shown (except perhaps for the well-known Problem 6.) We hope that this article will demonstrate its utility for problem-solving.

Problem 6. Show that

$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \ln 2.$$

Problem 7. Evaluate

$$\lim_{n\to\infty} \frac{\sqrt[n]{e} + \sqrt[n]{e^2} + \cdots + \sqrt[n]{e^n}}{n}.$$

Problem 8. Evaluate

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}} \right).$$

Problem 9. Evaluate

$$\lim_{n\to\infty} \left(\frac{\sqrt[n]{2}}{n+1} + \frac{\sqrt[n]{2^2}}{n+1/2} + \cdots + \frac{\sqrt[n]{2^n}}{n+1/n} \right).$$

Problem 10. [adapted from [3]] For positive real λ evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn+\lambda}}.$$

Problem 11. Evaluate

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\sin\left(\frac{k\pi}{n}\right)}{\sqrt{n^2+k}}.$$

Problem 12. [6] Prove that

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\arctan\left(\frac{k}{n}\right)}{n+k} \cdot \frac{\varphi(k)}{k} = \frac{3\ln 2}{4\pi},$$

where φ denotes Euler's totient function. (A solution appears in [7].)

References

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **April 15, 2020**.

4511. *Proposed by Robert Frontczak.*

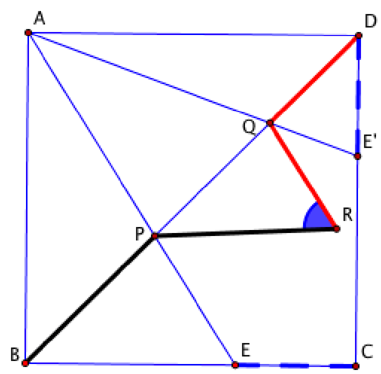
Evaluate the following sum in closed form:

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-1}\right).$$

4512. *Proposed by J. Chris Fisher.*

For any point E on the side BC of the square $ABCD$ let E' be chosen on side CD so that $DE' = CE$, and let the lines AE and AE' intersect the diagonal BD in points P and Q, respectively. If R is either point whose distance from P equals PB and from Q equals QD, then prove that $\angle QRP = 60°$.

Comment from the proposer: this problem was shown to me 15 years ago; I do not know its source.



4513. *Proposed by H. A. ShahAli.*

Let

$$a_n = \prod_{k=1}^{n}\frac{2k}{2k-1}$$

for each natural n. Prove that the number $A(m)$ of n's for which $\lfloor a_n\rfloor = m$ is non-zero and strictly increasing for any integer $m \geq 2$.

4514. *Proposed by Leonard Giugiuc.*

Let a, b and c be real numbers all greater than or equal to $\frac{1}{2}$ such that $a+b+c=3$. Prove that

$$\frac{a^2}{b+1}+\frac{b^2}{c+1}+\frac{c^2}{a+1} \geq \frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}.$$

4515. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers. Prove that

$$\frac{(2a+b)\sqrt{\frac{a}{b}}+(2b+c)\sqrt{\frac{b}{c}}+(2c+a)\sqrt{\frac{c}{a}}}{a+b+c} \geq 3.$$

4516. *Proposed by Hung Nguyen Viet.*

Find the values of:

(a) $(1-\cot 1^\circ)(1-\cot 2^\circ)(1-\cot 3^\circ)\cdots(1-\cot 44^\circ)$,

(b) $\dfrac{1}{1+\cot 1^\circ}+\dfrac{1}{1+\cot 2^\circ}+\dfrac{1}{1+\cot 3^\circ}+\cdots+\dfrac{1}{1+\cot 89^\circ}$.

4517. *Proposed by Robert Frontczak.*

Let F_n denote the n-th Fibonacci number defined by $F_{n+1}=F_n+F_{n-1}$, $F_0=0$, $F_1=1$. Further, let T_n denote the n-th triangular number, that is $T_n=\dfrac{n(n+1)}{2}$. Show that

$$\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7.$$

4518. *Proposed by Miguel Ochoa Sanchez and Leonard Giugiuc.*

If O is the circumcenter of a triangle ABC and D is any point on the line AB, let O_1 and O_2 be the respective circumcenters of triangles ADC and DBC. Prove that the orthocenter of triangle O_1DO_2 lies on the line through O that is parallel to AB.

4519. *Proposed by Leonard Giugiuc.*

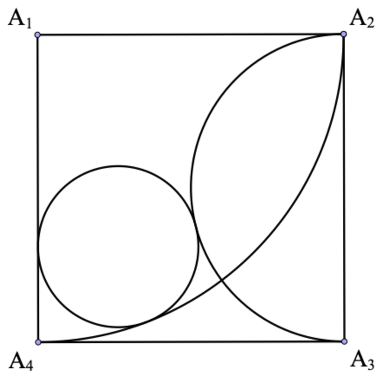
Let $ABCD$ be a non-degenerate convex quadrilateral. If

$$AB^2+CD^2+2AD\cdot BC=AC^2+BD^2,$$

prove that AD is parallel to BC.

4520. *Proposed by Arsalan Wares.*

The figure shows a square with a quarter circle, a semicircle and a circle inside it. The length of the square is the same as the length of the radius of the quarter circle which in turn is the same as the length of the diameter of the semicircle. The circle touches both the quarter circular arc (internally) and the semicircular arc (externally), and one of the sides of the square as shown. If the length of the square is 25, find the exact length of the radius of the circle.



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Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 avril 2020**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4511. *Proposée par Robert Frontczak.*

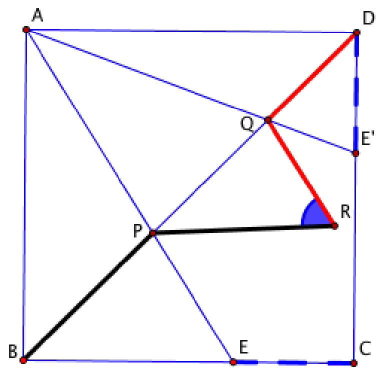
Évaluer la somme suivante en forme close:

$$\sum_{n=1}^{\infty}\left(\frac{1}{8n-7}-\frac{1}{8n-1}\right).$$

4512. *Proposée par J. Chris Fisher.*

Pour un quelconque point E sur le côté BC du carré $ABCD$, soit E' choisi sur le côté CD de façon à ce que $DE' = CE$, les lignes AE et AE' intersectant la diagonale BD en P et Q, respectivement. Si R est un des deux points dont la distance à P égale PB et la distance à Q égale QD, démontrer que $\angle QRP = 60°$.

Commentaire du proposeur : on m'a montré ce problème il y a 15 ans ; je ne connais pas son auteur.



4513. *Proposée par H. A. ShahAli.*

Soit

$$a_n = \prod_{k=1}^{n}\frac{2k}{2k-1}$$

pour tout nombre naturel n. Démontrer que le nombre $A(m)$ de n's tels que $\lfloor a_n\rfloor = m$ est non nul et strictement croissant pour les entiers $m \geq 2$.

4514. *Proposée par Leonard Giugiuc.*

Soient a, b et c des nombres réels tels que $a,\ b,\ c \geq \frac{1}{2}$ et $a+b+c=3$. Démontrer que

$$\frac{a^2}{b+1}+\frac{b^2}{c+1}+\frac{c^2}{a+1} \geq \frac{a}{b+1}+\frac{b}{c+1}+\frac{c}{a+1}.$$

4515. *Proposée par George Apostolopoulos.*

Soient a, b et c des nombres réels positifs. Démontrer que

$$\frac{(2a+b)\sqrt{\dfrac{a}{b}}+(2b+c)\sqrt{\dfrac{b}{c}}+(2c+a)\sqrt{\dfrac{c}{a}}}{a+b+c} \geq 3.$$

4516. *Proposée par Hung Nguyen Viet.*

Déterminer les valeurs de:

(a) $(1-\cot 1^\circ)(1-\cot 2^\circ)(1-\cot 3^\circ)\cdots(1-\cot 44^\circ)$,

(b) $\dfrac{1}{1+\cot 1^\circ}+\dfrac{1}{1+\cot 2^\circ}+\dfrac{1}{1+\cot 3^\circ}+\cdots+\dfrac{1}{1+\cot 89^\circ}$.

4517. *Proposée par Robert Frontczak.*

Soit F_n le $n$ième nombre de Fibonacci selon $F_{n+1}=F_n+F_{n-1}$, $F_0=0$, $F_1=1$. De plus, soit T_n le $n$ième nombre triangulaire, c'est-à-dire $T_n=\dfrac{n(n+1)}{2}$. Démontrer que

$$\sum_{n=0}^{\infty} T_n \cdot \frac{F_n}{2^{n+2}} = F_7.$$

4518. *Proposée par Miguel Ochoa Sanchez and Leonard Giugiuc.*

Si O est le centre du cercle circonscrit du triangle ABC et D est un quelconque point sur la ligne AB, soient O_1 et O_2 les centres des cercles circonscrits des triangles ADC et DBC, respectivement. Démontrer que l'orthocentre du triangle O_1DO_2 se situe sur la ligne passant par O et parallèle à AB.

4519. *Proposée par Leonard Giugiuc.*

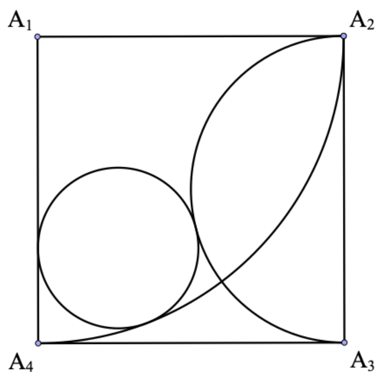
Soit $ABCD$ un quadrilatère convexe non dégénéré. Si

$$AB^2+CD^2+2AD\cdot BC=AC^2+BD^2,$$

démontrer que AD est parallèle à BC.

4520. *Proposée par Arsalan Wares.*

La figure montre un carré, un quart de cercle, un demi cercle et un cercle dans son intérieur. Noter que la longueur du côté du cercle est la même que le rayon du quart de cercle et le diamètre du demi cercle. Le cercle touche le quart de cercle à son interne, le demi cercle à son externe et un des côtés du carré, tel qu'indiqué. Si le carré est de côté 25, déterminer le rayon du cercle.



SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2019: 45(7), p. 413–416.

4461. *Proposed by Marian Dinca, Leonard Giugiuc and Daniel Sitaru.*

Let u, v and w be distinct complex numbers such that $\dfrac{w-u}{v-u}$ is not a real number. Consider a complex number $z = \alpha u + \beta v + \gamma w$, where $\alpha, \beta, \gamma > 0$ are real numbers such that $\alpha + \beta + \gamma = 1$. Prove that

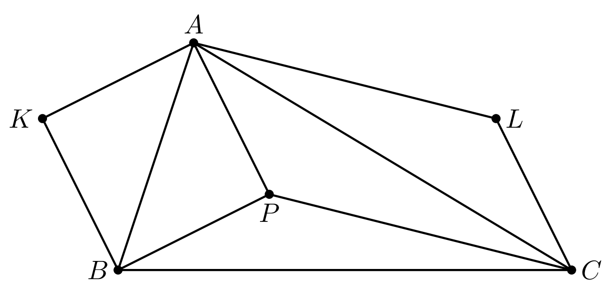
$$(|z-v| + |w-u|)^2 + (|z-w| + |u-v|)^2 > (|z-u| + |v-w|)^2.$$

We received 5 correct solutions. We present two of them.

Solution 1, by C.R. Pranesachar.

Let A, B, C, P be the points in the complex plane that correspond to u, v, w, z respectively. The conditions entail that the triangle ABC is nondegenerate and P is inside or on the triangle. It is required to show that

$$(BP + CA)^2 + (CP + AB)^2 > (AP + BC)^2.$$



Determine points K and L such that $APBK$ and $APCL$ are parallelograms. Then

$$BP + CA = KA + AC > KC$$

and

$$CP + AB = LA + BA > BL.$$

Since KB, AP and LC are all equal and parallel, $KLCB$ is a parallelogram. Hence

$$KC^2 + BL^2 = 2(BK^2 + KL^2) = 2(AP^2 + BC^2) = (AP + BC)^2 + (AP - BC)^2.$$

Therefore

$$(BP + CA)^2 + (CP + AB)^2 > KC^2 + BL^2 \geq (AP + BC)^2.$$

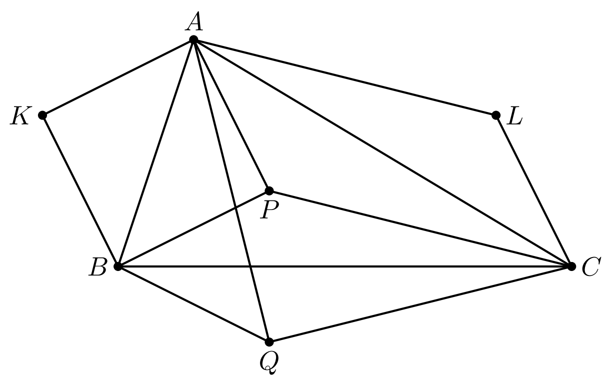
Solution 2, by Sorin Rubinescu.

Let A, B, C, P be as in the foregoing solution, and let Q be the reflection of P in the side AB. Then $ABQC$ is a convex quadrilateral to which we can apply the respective quadrilateral theorems of Euler and Ptolemy:

$$BQ^2 + CA^2 + CQ^2 + AB^2 = AQ^2 + BC^2 + 4MN^2$$

where M and N are the midpoints of BC and AQ, and

$$BQ \cdot CA + CQ \cdot AB \geq AQ \cdot BC.$$



$$\begin{aligned}
&(BP + CA)^2 + (CP + AB)^2 \\
&= (BP^2 + CA^2 + CP^2 + AB^2) + 2(BP \cdot CA + CP \cdot AB) \\
&= (BQ^2 + CA^2 + CQ^2 + AB^2) + 2(BQ \cdot CA + CQ \cdot AB) \\
&\geq AQ^2 + BC^2 + 2(AQ \cdot BC) \geq AP^2 + BC^2 + 2(AP \cdot BC) \\
&= (AP + BC)^2.
\end{aligned}$$

The second inequality is strict when P does not lie on BC. If $P = Q$, then the first inequality is strict since $M \neq N$. The result follows.

4462. *Proposed by George Apostolopoulos.*

Let a, b, c be the lengths of the sides of triangle ABC with inradius r and circumradius R. Show that

$$\frac{a^2}{b+c} + \frac{b^2}{a+c} + \frac{c^2}{a+b} \leq \frac{3\sqrt{6}R}{4r}\sqrt{R(R-r)}.$$

We received 11 submissions, all correct. We present the proof by Sorin Rubinescu.

Let s denote the semiperimeter of triangle ABC. Then

$$\begin{aligned}2s\left(\sum_{cyc}\frac{a}{b+c}\right) &= \frac{a(a+b+c)}{b+c}+\frac{b(a+b+c)}{c+a}+\frac{c(a+b+c)}{a+b}\\ &= a+b+c+\sum_{cyc}\frac{a^2}{b+c}\\ &= 2s+\sum_{cyc}\frac{a^2}{b+c}.\end{aligned}$$

Hence,

$$\sum_{cyc}\frac{a^2}{b+c} = 2s\left(\sum_{cyc}\frac{a}{b+c}-1\right). \tag{1}$$

Next, we have

$$\sum_{cyc}\frac{a}{b+c} \le 2-\frac{r}{R}, \tag{2}$$

which was problem #4212 in *Crux*, Vol. 43 (2), p. 69. [*Ed.:* A solution was published on p. 74 of *Crux*, Vol. 44 (2).]

From (1) and (2) we get

$$\sum_{cyc}\frac{a^2}{b+c} \le 2s\left(1-\frac{r}{R}\right) = \frac{2s}{R}\left(R-r\right). \tag{3}$$

By Mitrinovic's Inequality, we have

$$2s \le 3\sqrt{3}R. \tag{4}$$

[*Ed.:* See item 5.3 on p. 49 of *Geometric Inequalities* by O. Bottema et al.]

From (3) and (4) we get

$$\sum_{cyc}\frac{a^2}{b+c} \le 3\sqrt{3}(R-r).$$

But

$$\begin{aligned}3\sqrt{3}(R-r) &\le \frac{3\sqrt{6}}{4r}R\sqrt{R(R-r)}\\ \iff R-r &\le \frac{\sqrt{2}}{4r}R\sqrt{R(R-r)}\\ \iff (R-r)^2 &\le \frac{1}{8r^2}R^3(R-r)\\ \iff R-r &\le \frac{R^3}{8r^2}\\ \iff 8r^2(R-r) &\le R^3. \end{aligned} \tag{5}$$

Finally, by the AM-GM inequality and Euler's inequality, $2r \leq R$, we have

$$8r^2(R-r) = 8r \cdot r(R-r) \leq 8r\left(\frac{r+R-r}{2}\right)^2 = 2rR^2 \leq R^3,$$

so (5) holds, completing the proof.

Editor's comments. Most submitted solutions quoted various known results/inequalities, especially the Gerretsen's Inequality which states that $s^2 \leq 4R^2 + 3r^2 + 4rR$.

4463. *Proposed by Max A. Alekseyev*

For all integers $n > m \geq 0$, prove that

$$\sum_{k=0}^{n}(-1)^k \binom{2n+1}{n-k}(2k+1)^{2m+1} = 0.$$

We received 6 submissions, all of which were correct and complete. We present the solution by C.R. Pranesachar.

Recall the following well-known identity. For any integers n, t such that $n > t \geq 0$.

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} k^t = 0.$$

This can be obtained by applying the differential operator $\frac{d}{dx}\left(x\frac{d}{dx}\right)^{t-1}$ on the binomial identity

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k,$$

and finally substituting $x = -1$. Notice that $(1+x)$ will be a factor in all terms of the lefthand side, because $t < n$. A suitable linear combination of such identities yields another identity

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k}(a+kd)^t = 0, \qquad \text{for } n > t \geq 0,$$

where a and d are fixed constants. We replace n by $2n+1$, t by $2m+1$, with $n > m \geq 0$, a by $-(2n+1)$ and d by 2, to get

$$\sum_{k=0}^{2n+1}(-1)^k \binom{2n+1}{k}\left(-(2n+1)+2k\right)^{2m+1} = 0.$$

It is easy to see that among the $2n+2$ terms in the above, the first $(n+1)$ terms are the same as the next $(n+1)$ terms, in the reverse order. That is,

$$(-1)^k \binom{2n+1}{k}\left(-(2n+1)+2k\right)^{2m+1}$$
$$= (-1)^{2n+1-k}\binom{2n+1}{2n+1-k}\left(-(2n+1)+2(2n+1-k)\right)^{2m+1},$$

for $0 \le k \le n$. Hence we have

$$2\sum_{k=n+1}^{2n+1}(-1)^k\binom{2n+1}{k}\left(2(k-n-1)+1\right)^{2m+1}=0.$$

Dividing by 2 and adjusting the index of summation $k \mapsto k+n+1$ we obtain

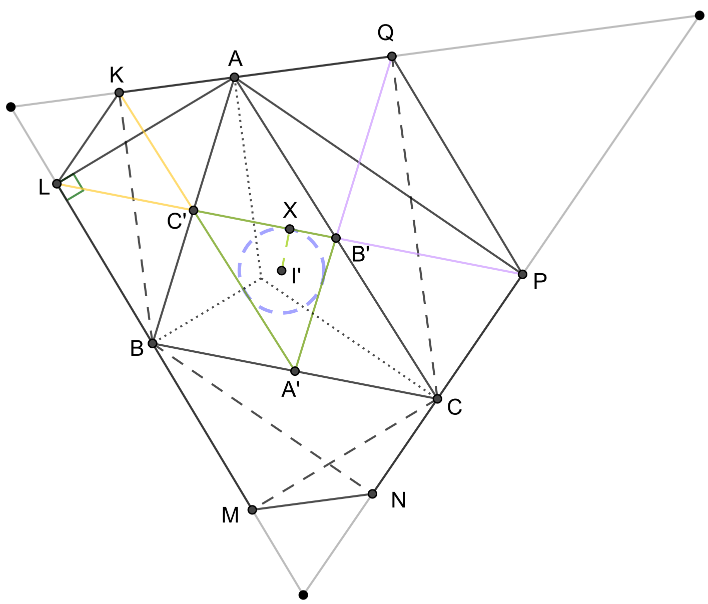
$$\sum_{k=0}^{n}(-1)^{k+n+1}\binom{2n+1}{k+n+1}(2k+1)^{2m+1}=0.$$

Since $\binom{2n+1}{k+n+1}=\binom{2n+1}{n-k}$, we divide the above by $(-1)^{n+1}$ and obtain the desired identity.

4464. *Proposed by Borislav Mirchev and Leonard Giugiuc.*

Let ABC be a triangle with external angle bisectors k, l and m to angles A, B and C, respectively. Projections of A on l and m are L and P, respectively. Similarly, projections of B on m and k are N and K and projections of C on k and l are Q and M. Show that the points M, N, P, Q, K and L are concyclic.

We received 13 submissions, of which 9 were complete and correct. The presented solution is that of Cristóbal Sánchez-Rubio, completed by the editor.



Let $BC=a$, $AC=b$, $AB=c$ and denote by s the semiperimeter of $\triangle ABC$. Let A', B', C' be the midpoints of BC, CA and AB respectively.

We will show first that L, C', B' and P are collinear and $LP=s$. Since LM is the external angle bisector of $\angle B$ we have $\angle LBC' = 90^\circ - \frac{\angle B}{2}$. LC' is a median in the right angled triangle $\triangle ALB$, so $\triangle LC'B$ is isosceles, giving us

$$LC' = C'B = \frac{c}{2} \text{ and } \angle LC'B = 180^\circ - 2\angle LBC' = \angle B.$$

As a midline in $\triangle ABC$, $C'B'||BC$, whence $\angle AC'B' = \angle B$, and so we note that $\angle AC'B' = \angle LC'B$. We can thus conclude that $\angle LC'B$ and $\angle AC'B'$ are opposite angles; that is, L, C' and B' are collinear. Similarly we show that $B'P = B'C = \frac{b}{2}$ and C', B' and P are collinear. Hence, L, C', B' and P are collinear. Finally,

$$LP = LC' + C'B' + B'P = \frac{c}{2} + \frac{a}{2} + \frac{b}{2} = s.$$

Let I' be the incentre of $\triangle A'B'C'$ and r the inradius. We will show that L and P are equidistant from I'. Denote by X the point where the incircle of $\triangle A'B'C'$ is tangent to the side $B'C'$. The perimeter of $\triangle A'B'C'$ is s and $A'C' = \frac{b}{2}$; thus $B'X = \frac{s}{2} - \frac{b}{2}$, which means

$$XP = XB' + B'P = \left(\frac{s}{2} - \frac{b}{2}\right) + \frac{b}{2} = \frac{s}{2}.$$

In other words, X is the midpoint of LP. Since $I'X \perp LP$ we get

$$I'L = I'P = \sqrt{r^2 + \frac{s^2}{4}}.$$

Repeat the same steps with K, C', A', N and then Q, B', A', M to conclude that all the points M, N, P, Q, K and L lie on the circle with centre I' and radius $\sqrt{r^2 + \frac{s^2}{4}}$.

4465. *Proposed by Nguyen Viet Hung.*

Let ABC be a triangle with centroid G and medians m_a, m_b, m_c. Rays AG,BG,CG intersect the circumcircle at A_1,B_1,C_1 respectively. Prove that

$$\frac{\text{Area}[A_1B_1C_1]}{\text{Area}[ABC]} = \frac{(a^2+b^2+c^2)^3}{(8m_am_bm_c)^2}.$$

We received 10 submissions, all of which were correct, and feature the solution by Sorin Rubinescu with references supplied by the editor.

Let a_1, b_1, c_1 be the lengths of the sides of $\triangle A_1B_1C_1$. Then,

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{a_1b_1c_1}{4R} \cdot \frac{4R}{abc} = \frac{a_1b_1c_1}{abc}$$

Because A, B_1, A_1, B are concylic, we have that $\triangle GAB \sim \triangle GB_1A_1$, so that $\frac{A_1B_1}{BA} = \frac{GB_1}{GA}$ and, therefore,

$$c_1 = c \cdot \frac{GB_1}{GA} = c \cdot \frac{GB_1 \cdot GB}{GA \cdot GB} = c \cdot \frac{R^2 - OG^2}{GA \cdot GB},$$

where $R^2 - OG^2$ is the power of G with respect to the common circumcircle of triangles ABC and $A_1B_1C_1$.

Analogously, $a_1 = a \cdot \frac{R^2 - OG^2}{GB \cdot GC}$ and $b_1 = b \cdot \frac{R^2 - OG^2}{GC \cdot GA}$.

Thus,

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{(R^2 - OG^2)^3}{(GA \cdot GB \cdot GC)^2}$$

According to references dealing with the Euler line (for example, Nathan Altshiller Court's *College Geometry*, Corollary 110 on page 71), we have that

$$\begin{aligned} R^2 - OG^2 &= \frac{1}{3}(GA^2 + GB^2 + GC^2) \\ &= \frac{1}{3}\left[\left(\frac{2}{3}m_a\right)^2 + \left(\frac{2}{3}m_b\right)^2 + \left(\frac{2}{3}m_c\right)^2\right] \\ &= \frac{4}{27}\left(m_a^2 + m_b^2 + m_c^2\right). \end{aligned}$$

However, $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ (by the same reference, Theorem 106, page 70).

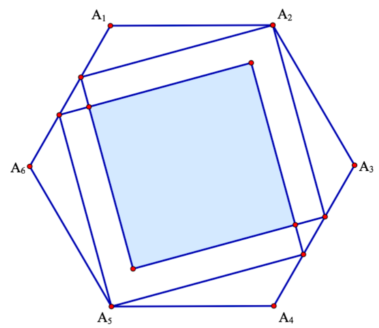
Hence, $R^2 - OG^2 = \frac{a^2+b^2+c^2}{9}$.

As a result,

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{\frac{1}{9^3}(a^2 + b^2 + c^2)^3}{\frac{1}{27^2}(8m_a m_b m_c)^2} = \frac{(a^2 + b^2 + c^2)^3}{(8m_a m_b m_c)^2}.$$

4466. *Proposed by Arsalan Wares.*

Let A be a regular hexagon with vertices $A_k, k = 1, 2, \ldots, 6$. There are two congruent overlapping squares inside A. Each of the squares shares one vertex with A and two vertices of each square lie on opposite sides of hexagon A as in the figure:



Find the exact area of the shaded region, if the length of each side of hexagon A is 2.

We received 22 submissions, out of which 20 were correct and complete. We present the solution by Richard Hess.

The distance between A_2 and A_5 is 4. The diagonal length of the large squares is the same as the distance between parallel sides of the hexagon, which equals $d = 2\sqrt{3}$. For the shaded square, we obtain a diagonal length of

$$4 - 2(4 - d) = 4(\sqrt{3} - 1),$$

leading to an area of

$$\frac{1}{2}(4(\sqrt{3} - 1))^2 = 32 - 16\sqrt{3}.$$

4467. *Proposed by Paul Bracken.*

Show that for $x > 0$,

$$\arctan x \cdot \arctan \frac{1}{x} > \frac{x}{2(x^2 + 1)}.$$

(*Ed.*: Take a look at the problem 4327.)

We received 22 submissions, all correct. Seven solvers proved a stronger inequality using, instead of 1/2, values $\pi^2/8$, $(5\pi)^2/6^3$ or $(\pi - 1)/2$. We present the solution approach taken by several solvers.

We know that $\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$. Consider the function

$$f(x) = \arctan(x) \cdot \arctan\left(\frac{1}{x}\right) - \frac{x}{2(x^2 + 1)}$$

for $x > 0$. We need to show that $f(x) > 0$ for all $x > 0$.

Since $f\left(\frac{1}{x}\right) = f(x)$, it suffices to show that $f(x) > 0$ for all $x \in (0, 1]$.

We have

$$f'(x) = \frac{\pi}{2(x^2 + 1)} - \frac{2\arctan(x)}{x^2 + 1} + \frac{x^2 - 1}{2(x^2 + 1)^2} = \frac{g(x)}{x^2 + 1},$$

where $g(x) = \dfrac{\pi}{2} - 2\arctan(x) + \dfrac{x^2 - 1}{2(x^2 + 1)}$. Since

$$g'(x) = -\frac{x^2 - x + 1}{(x^2 + 1)^2},$$

we have $g'(x) < 0$ for all $x \in (0, 1]$ and g is decreasing on $(0, 1]$.

Finally, since

$$g(0) = \frac{\pi}{2} - \frac{1}{2} > 0 \text{ and } g(1) = \frac{\pi}{2} - 2\frac{\pi}{4} = 0,$$

we get that $g(x) > 0$ for all $x \in (0,1)$, so f is strictly increasing on $(0,1]$ with $\lim_{x\to 0^+} f(x) = 0$. So we conclude that $f(x) > 0$ for all $x \in (0,1]$ and consequently $f(x) > 0$ for all $x > 0$.

4468. *Proposed by Florin Stanescu.*

Let $f : [0,1] \to \mathbb{R}$ be a differentiable function such that f' is continuous and $f(0) + f'(0) = f(1)$. Show that there exists $c \in (0,1)$ such that

$$\frac{c}{2} f(c) = \int_0^c f(x)dx.$$

We received four submissions to the problem, out of which three correctly identified that there was a mistake in the problem. We present the submission by Alexandru Daniel Pîrvuceanu.

Let $a, b \in \mathbb{R}^*$ and consider the function $f : [0,1] \to \mathbb{R}, f(x) = ax + b$. Then $f(0) = b$, $f'(0) = a$ and $f(1) = a + b$, so $f(0) + f'(0) = f(1)$. Let's assume that there exists $c \in (0,1)$ with the required property. Then

$$\frac{ac^2 + bc}{2} = \frac{ac^2 + 2bc}{2} \iff bc = 0,$$

which is impossible since both b and c are nonzero.

Editor's Comment. It appears that the problem intended to ask for the existence of c such that $cf(\frac{c}{2}) = \int_0^c f(x)dx$. Please submit your solutions to the corrected version directly to `crux.eic@gmail.com`

4469. *Proposed by Leonard Giugiuc and Dan-Stefan Marinescu.*

Let ABC be a triangle and let P be an interior point of ABC. Denote by R_a, R_b, R_c the circumradii of the triangles PBC, PCA and PAB, respectively. Prove that $R_a R_b R_c \geq PA \cdot PB \cdot PC$.

We received 10 submissions, all correct. We present the solution by Michel Bataille.

Let $\alpha = \angle BPC$, $\beta = \angle CPA$, and $\gamma = \angle APB$, so $\alpha + \beta + \gamma = 360°$.

Let $\alpha_1 = \angle PBC$ and $\alpha_2 = \angle PCB$. Then $\alpha + \alpha_1 + \alpha_2 = 180°$ so from the Extended Law of Sines, we have

$$\frac{PB}{\sin\alpha_2} = \frac{PC}{\sin\alpha_2} = 2R_a.$$

It follows that

$$\begin{aligned} PB \cdot PC &= 4R_a^2 \sin\alpha_1 \sin\alpha_2 \\ &= 2R_a^2(\cos(\alpha_1 - \alpha_2) - \cos(180° - \alpha)) \\ &= 2R_a^2(\cos(\alpha_1 - \alpha_2) + \cos\alpha), \end{aligned}$$

and hence,

$$PB \cdot PC \leq 2R_a^2(1+\cos\alpha) = 4R_a^2 \cos^2 \frac{\alpha}{2}.$$

Similarly, $PC \cdot PA \leq 4R_b^2 \cos^2 \frac{\beta}{2}$ and $PA \cdot PB \leq 4R_c^2 \cos^2 \frac{\gamma}{2}$. Therefore,

$$PA^2 \cdot PB^2 \cdot PC^2 \leq 64R_a^2R_b^2R_c^2 \cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} \cos^2 \frac{\gamma}{2},$$

so

$$PA \cdot PB \cdot PC \leq 8R_aR_bR_c \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}. \tag{1}$$

Since $\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 180^\circ$, $\frac{\alpha}{2}$, $\frac{\beta}{2}$, and $\frac{\gamma}{2}$ are the interior angles of a triangle so by a well-known inequality [*Ed:* Cf. e.g., item #2.23 on p. 25 of *Geometric Inequalities* by O. Bottema et al] we have

$$\cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \leq \frac{1}{8}. \tag{2}$$

From (1) and (2), $R_aR_bR_c \geq PA \cdot PB \cdot PC$ follows.

4470. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be three distinct complex numbers such that $|a| = |b| = |c| = 1$ and $|a+b+c| \leq 1$. Prove that $|a^2 + bc| \geq |b+c|$.

There were 15 correct solutions. We feature 5 solutions.

Solution 1, by Oliver Geupel.

The circumcentre of the triangle with vertices at a, b, c in the complex plane is at 0; the orthocentre of this triangle at $a+b+c$ is contained in this circle, and so the triangle has all angles acute.

Since the problem is symmetric in b and c, we may assume that a, b, c are in counterclockwise order, so that $b = ae^{2\gamma i}$ and $a = ce^{2\beta i}$ where β and γ are the respective angles at B and C and $0 < \beta, \gamma \leq (\pi/2)$. Then, since $a\bar{a} = b\bar{b} = c\bar{c} = 1$,

$$\begin{aligned}
|a^2+bc|^2 - |b+c|^2 &= (a^2+bc)\left(\frac{1}{a^2} + \frac{1}{bc}\right) - (b+c)\left(\frac{1}{b} + \frac{1}{c}\right) \\
&= \left(\frac{a}{b} - \frac{b}{a}\right)\left(\frac{a}{c} - \frac{c}{a}\right) \\
&= -4\sin(-2\gamma)\sin 2\beta \\
&= 4\sin 2\gamma \sin 2\beta.
\end{aligned}$$

Since $0 < 2\beta, 2\gamma \leq \pi$, then $\sin 2\gamma \sin 2\beta \geq 0$ and the result follows.

Solution 2, by Marie-Nicole Gras and C.R. Pranesachar, independently.

Let $a = e^{i\alpha}$, $b = e^{i\beta}$, and $c = e^{i\gamma}$. Then

$$\begin{aligned} 1 &\geq |a+b+c| \\ &= (e^{i\alpha}+e^{i\beta}+e^{i\gamma})(e^{-i\alpha}+e^{-i\beta}+e^{-i\gamma}) \\ &= 3+2[\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)], \end{aligned}$$

whence

$$\begin{aligned} 0 &\geq 1+\cos(\alpha-\beta)+\cos(\gamma-\beta)+\cos(\gamma-\alpha) \\ &= 1+\cos(\alpha-\beta)+\cos((\alpha-\beta)+(\gamma-\alpha))+\cos(\gamma-\alpha) \\ &= (1+\cos(\alpha-\beta))(1+\cos(\gamma-\alpha))-\sin(\alpha-\beta)\sin(\gamma-\alpha), \end{aligned}$$

so that

$$\sin(\alpha-\beta)\sin(\gamma-\alpha) \geq (1+\cos(\alpha-\beta))(1+\cos(\gamma-\alpha)) \geq 0.$$

By similar manipulations,

$$|a^2+bc|^2 = 2(1+\cos(2\alpha-\beta-\gamma)) \quad \text{and} \quad |b+c|^2 = 2(1+\cos(\beta-\gamma)).$$

Therefore

$$\begin{aligned} |a^2+bc|^2-|b+c|^2 &= 2[\cos(2\alpha-\beta-\gamma)-\cos(\gamma-\beta)] \\ &= 2[\cos((\alpha-\beta)-(\gamma-\alpha))-\cos((\alpha-\beta)+(\gamma-\alpha))] \\ &= 4\sin(\alpha-\beta)\sin(\gamma-\alpha) \geq 0. \end{aligned}$$

Editor's comment. Prithwijit De and Florentin Visescu took a similar approach, but made use of the fact that $1+\cos 2x+\cos 2y+\cos 2z = 4\cos x\cos y\cos z$ when $x+y+z=0$.

Solution 3, by Missouri State University Problem Solving Group.

We may suppose that $a = e^{i\alpha}$, $b = e^{-i\beta}$ and $c = -e^{i\beta}$ where $0 \leq \beta \leq \pi/2$. Then $|b+c| = 2\sin\beta$ and $|a^2+bc| = |a^2-b(-c)| = |a^2-1| = |a-1||a+1|$. Since $|a-1|$ and $|a+1|$ are the two legs of a right triangle with vertices at -1, a and 1, then $|a-1||a+1| = 2A$, twice the area of the triangle. This triangle has base 2 and height $|\mathfrak{Im}(a)| = |\sin\alpha|$. Thus

$$|a^2+bc| = |a-1||a+1| = 2A = 2|\sin\alpha|.$$

Since $a+b+c = \cos\alpha + i(\sin\alpha - 2\sin\beta)$, the condition $|a+b+c|^2 \leq 1$ implies that $1-4\sin\alpha\sin\beta+4\sin^2\beta \leq 1$ or

$$\sin\beta(\sin\alpha-\sin\beta) \geq 0.$$

If $\sin\beta = 0$, then the result holds trivially. If $0 < \sin\beta \leq \sin\alpha$, then the result still holds.

Solution 4, by Panagiotis Antonopoulos.

Inequality $|a+b+c| \leq 1$ is equivalent to $\mathfrak{Re}(b\bar{c}) + \mathfrak{Re}(a\bar{b}) + \mathfrak{Re}(c\bar{a}) \leq -1$. Let $a\bar{b} = x + yi$ and $c\bar{a} = u + vi$. Then $b\bar{c} = (b\bar{a})(a\bar{c}) = (xu - yv) - (xv + yu)i$. Thus, $xu - yv + x + u \leq -1$, whence $(x+1)(u+1) \leq yv$. Since $|x| \leq |a\bar{b}| = 1$ and $|u| \leq 1$, it follow that $yv \geq 0$.

Now

$$\begin{aligned} |a^2 + bc|^2 - |b+c|^2 &= 2(\mathfrak{Re}(a^2\bar{b}\bar{c} - b\bar{c})) \\ &= 2(\mathfrak{Re}(a\bar{c}(a\bar{b} - \bar{a}b))) \\ &= 4yv \geq 0, \end{aligned}$$

yielding the desired result.

Solution 5, by the proposer.

Suppose that $u = b/a$ and $v = c/a$. Then $|u| = |v| = 1$, and we have to prove that $|1 + uv| \geq |u + v|$ subject to $|1 + u + v| \leq 1$.

Suppose, first, that $uv = -1$. Then $v = -\bar{u}$. Then

$$|1 + u + v|^2 = (1 + (u - \bar{u}))(1 - (u - \bar{u})) = 1 - (u - \bar{u})^2.$$

Since $u - \bar{u}$ is a real multiple of i whose square is nonpositive, $|1 + u + v| \leq 1$ implies that $|u + v|^2 = (u - \bar{u})^2 = 0$ and the result follows.

When $uv \neq -1$, the quantity $(u + v)(1 + uv)^{-1}$ is well-defined and equal to its complex conjugate. Hence

$$\frac{(1 + u + v) - 1}{1 - (-uv)} = \frac{u + v}{1 + uv}$$

is real. Therefore 1, $-uv$ and $1+u+v$ are collinear in the complex plane. However, 1 and $-uv$ lie on the circumference of the unit circle while $1 + u + v$ lies within it. Hence $|u + v|$, the distance between $1 + u + v$ and 1, does not exceed $|1 + uv|$, the distance between $-uv$ and 1.