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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

This term I've been lucky to teach my favourite course – 4th year History of Mathematics. As we near the end of the term, students have made themselves comfortable with exploring the human, cultural and social sides of mathematics, but back in September many of their experiences have been, as one student put it, "disorienting". The students, all math majors or minors, have been so focused on learning the tools of the mathematical trade that they have forgotten to think about the fact that math is a human endeavour, that it was developed by humans in their own time, with their own understanding, their own biases (against other humans or particular mathematical concepts, such as negative or complex numbers) and their often strong personalities.

In week 1, I presented students with the following few ancient problems and asked them to use only the tools of the time to solve them:

1. How many cattle are in a herd if 2/3 of 1/3 of them make 70, the number due as tribute to the owner? (Egypt: Rhind Papyrus (1650 BC))

2. I have a reed. I know not its dimension. I broke off from it 1 cubit and walked 60 times along its length. I restored to it what I broke off, then walked 30 times along its length. The area is 375 square cubits. What was the original length of the reed? (Babylon: Clay Tablets (2000-1000 BC))

3. A tree is 20 feet tall and has a circumference of 3 feet. There is a vine that winds seven equally spaced times around the tree and reaches the top. What is the length of the vine? (China: Nine Chapters on the Mathematical Arts (100 BC))

First of all, students found it hard to not use variables and modern algebraic techniques (try it yourself). But what was more shocking is that they didn't realize that the sources listed were original sources! They thought I'd made these problems up to be 'in the spirit of' what math could have been like in those days and that I obviously overcomplicated it. Any time labelled BC seems so long ago that surely math that ancient wasn't so sophisticated...

This brings me to 47. This is the last issue of the Volume 47 and I did spot 47 as one of my students was discussing Syrian Arab mathematician Thâbit ibn Qurra and amicable numbers. I won't ruin the surprise for you here: look it up, read about number theory developments in and around 9th century AD, be amazed. Then read this issue of ***Crux*** – how do we compare over 1000 years later?

Kseniya Garaschuk

MathemAttic

No. 30

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 28, 2022**.

MA146. *Proposed by Alex Bloom.*

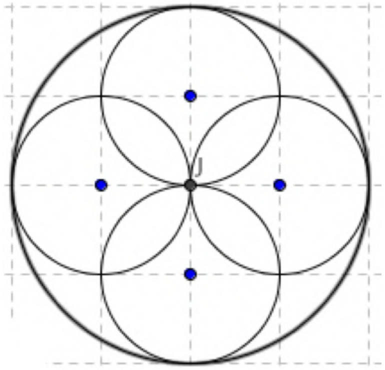
Solve the following equation for real numbers x and y:

$$(x^2+1)(y^2+1)-2(x+1)(y+1)+4=0.$$

MA147. *Proposed by Didier Pinchon.*

Let ABC be an acute triangle, D the midpoint of BC, I the center of the incircle of triangle ABD, and E the intersection between the segment AD and the circle of diameter BC. Prove that the points A, B, E and I are concyclic.

MA148. A large circle of radius 1 has centre at the point J and 4 small circles (with diameters equal to the radius of the larger circle) are drawn inside of it as shown below. Evaluate the area of the larger circle not inside any of the 4 small circles.



MA149. Calculate

$$\frac{5^2+3}{5^2-1}+\frac{7^2+3}{7^2-1}+\frac{9^2+3}{9^2-1}+\cdots+\frac{2021^2+3}{2021^2-1}.$$

MA150. Let us call a point an integer point if both its coordinates are integer numbers. For example, $(1, 2)$ and $(0, 5)$ are integer points, but $(1, 3/2)$ is not. What is the minimum number of integer points in the plane needed to guarantee that there is always a pair amongst them with an integer midpoint?

. . . .

Les problèmes dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **28 février 2022**.

MA146. *Proposeé par Alex Bloom.*

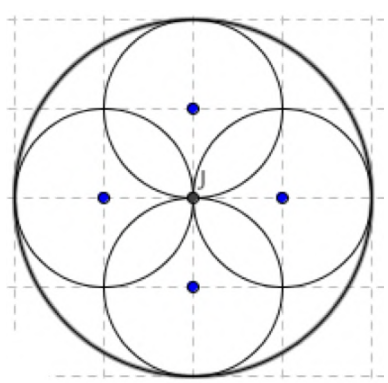
Résoudre l'équation qui suit, où x et y sont des nombres réels:

$$(x^2 + 1)(y^2 + 1) - 2(x + 1)(y + 1) + 4 = 0.$$

MA147. *Proposeé par Didier Pinchon.*

Soit ABC un triangle acutangle. Soient aussi D le point milieux de BC, I le centre du cercle inscrit du triangle ABD, puis E le point d'intersection du segment AD et du cercle de diamètre BC. Démontrer que les points A, B, E et I sont cocycliques.

MA148. Dans un grand cercle de centre J et de rayon 1, on trace 4 petits cercles de diamètres égaux au rayon du grand cercle, tel qu'indiqué ci-bas. Déterminer la surface du grand cercle se trouvant à l'intérieur d'aucun des petits cercles.



MA149. Déterminer

$$\frac{5^2+3}{5^2-1}+\frac{7^2+3}{7^2-1}+\frac{9^2+3}{9^2-1}+\cdots+\frac{2021^2+3}{2021^2-1}.$$

MA150. Un point sera dit point entier si ses coordonnées sont des nombres entiers. Par exemple, $(1,2)$ et $(0,5)$ sont des points entiers, mais $(1,3/2)$ ne l'est pas. Déterminer le nombre minimum de points entiers requis pour assurer qu'il y aura obligatoirement une paire de points dont le point milieux sera un point entier.

MathemAttic
SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(5), p. 225–226.

MA121. If $a+b+c=0$ and $abc=4$, find $a^3+b^3+c^3$.

Originally from the 21st W.J. Blundon Mathematics Contest (2004), problem 7.

We received 21 *solutions, of which* 18 *were correct. We present two solutions, each slightly modified by the editor.*

Solution 1, by Alex Bloom.

We want to find $a^3+b^3+c^3$, given $a+b+c=0$ and $abc=4$. It is a fairly well-known factorization that

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ac).$$

This claim can also be shown easily by multiplying out the right side. Therefore,

$$a^3+b^3+c^3-3\cdot 4=0\cdot(a^2+b^2+c^2-ab-ac-bc)=0,$$

so $a^3+b^3+c^3=12$.

Solution 2, by Logan Luo.

Since $a+b+c=0$, we find that $a+b=-c$. By cubing both sides of this, we get

$$\begin{aligned}(a+b)^3&=-c^3\\ a^3+3ab(a+b)+b^3&=-c^3\\ a^3+3ab(-c)+b^3&=-c^3\\ a^3+b^3+c^3&=3abc=3\cdot 4=12.\end{aligned}$$

MA122. Four people Mr Baker, Ms Carpenter, Mr Driver, and Ms Plumber are employed for four jobs as a baker, carpenter, driver, and plumber. None of them has a name identifying their occupation. They make four statements:
1. Mr Baker says he is the plumber.
2. Mr Driver says he is the baker.
3. Ms Carpenter says she is not the plumber.
4. Ms Plumber says she is not the carpenter.

Exactly  of the four statements are true. Who is the driver? (One of the editors apologizes for spilling coffee on the page, but we are sure the question used to have a unique answer!)

Adjusted version of original problem B4 from the 2012 BC Secondary School Mathematics Contest, Senior Final.

We received 4 submissions, of which 2 were mostly complete and correct. We expand on the solutions by Richard Hess and Vishak Srikanth.

Note that Mr Driver, Ms Carpenter and Ms Plumber all claim to be either the baker or the driver (Mr Driver of course cannot be the driver). Therefore if Mr Baker is the driver, exactly one statement is true (that of the baker); if Mr Baker is the carpenter, exactly two statements are true (those of the baker and the driver); and if Mr Baker is the plumber, exactly three statements are true (those of the plumber, the baker and the driver). So it is not possible that all or none of the statements are correct.

Suppose exactly two statements are true, i.e. Mr Baker is the carpenter. Suppose Mr Driver is the plumber, then either Mr Carpenter and Ms Plumber can be the driver and we do not have a unique answer. Similarly we don't get a unique answer if exactly three statements are true (Mr Baker being the plumber) and Mr Driver is the carpenter. Therefore the only possibility is that Mr Baker is the driver and exactly one of the statements is true. A possible assignment could be Mr Baker/driver, Ms Carpenter/baker, Mr Driver/plumber, Ms Plumber/carpenter.

MA123. A 12-sided polygon is inscribed in a circle of radius length l. What is the largest possible length of the shortest side of this polygon?

Originally from the Canadian National Mathematics League, Contest 6, April 1994, problem 6-5.

Four solutions were received, none of which considered the possibility that the polygon could lie inside a half-circle.

There are two possibilities, according as the centre of the circle lies in the interior of the polygon or not. In the first case, the length of each side is an increasing function of the angle (less that 180°) it subtends at the centre. Since there are twelve such angles summing to 360°, one must be no greater than 30°. The minimum can be exactly 30°, for example when the polygon is regular. The corresponding side has length

$$2l\sin 15^\circ = \frac{1}{2}(\sqrt{6}-\sqrt{2}) = (0.5176\cdots)l.$$

In the second case, one of the sides separates the rest of the polygon from the centre. The sum of the angles subtended by the remaining eleven sides at the centre does not exceed 180°. At least one of them cannot exceed $(180/11)^\circ < 30^\circ$. Thus, the bound of the first case applies here.

MA124. How many different 5-digit numbers can be formed using only the digits 1, 2, and 3, if digits placed consecutively must differ by at most 1?

Inspired by the 2014-2015 Nova Scotia Math League, Game 1, Team question 9.

We received five submissions, out of which four were correct. We present the solution by Vishak Srikanth, lightly edited.

We call a number containing only the digits 1, 2, or 3 and for which consecutive digits differ by at most 1 a *good* number. Let a_n be the number of good n-digit numbers that end on a 1 or 3 and let b_n be the number of good n-digit numbers that end on a 2. We have $a_1 = 2$ and $b_1 = 1$.

For $n \geq 2$ the first $n-1$ digits of a good n-digit number form a good $(n-1)$-digit number. Using this we can find recurrence relations for a_n and b_n:

$$\begin{aligned} a_n &= a_{n-1} + 2b_{n-1}, \\ b_n &= a_{n-1} + b_{n-1}. \end{aligned}$$

Using the recurrence relations, we calculate

$$a_2 = 4, b_2 = 3, a_3 = 10, b_3 = 7, a_4 = 24, b_4 = 17, a_5 = 58, b_5 = 41.$$

Therefore the number of 5-digit good numbers is $a_5 + b_5 = 99$.

MA125. Determine all positive integers a and b, $a < b$, so that exactly $\frac{1}{100}$ of the consecutive integers $a^2, a^2+1, a^2+2, \ldots, b^2$ are the squares of integers.

Originally from the 2009 Alberta High School Mathematics Competition, Part II, problem 2.

We received 7 solutions. We present the solution by Konstantine Zelator.

If a and b are positive integers with $1 \leq a < b$ then there are $b^2 - a^2 + 1$ consecutive integers from a^2 to b^2 (inclusive), out of which exactly $b - a + 1$ are perfect squares (namely, a^2, $(a+1)^2$, $\ldots$, $(a + (b-a))^2 = b^2$).

Applying the hypothesis that exactly $\frac{1}{100}$ of the integers a^2, $a^2 + 1$, $\ldots$, b^2 are perfect squares gives us the integer equation

$$\frac{1}{100}(b^2 - a^2 + 1) = b - a + 1.$$

We rewrite this as

$$(b-a)(b+a) = 100(b-a) + 99 \qquad \Leftrightarrow \qquad (b-a)(b+a-100) = 99.$$

From the assumption that a and b are integers with $1 \leq a < b$ we get that $b-a$ and $b+a-100$ are also integers, and $b-a$ is positive. Therefore, $b-a$ and $b+a-100$ are two positive integers which multiply to 99. Hence $(b-a) \in \{1, 3, 9, 11, 33, 99\}$; and since $b + a = \frac{99}{b-a} + 100$, for each possible value of $b - a$ we obtain a solution (a, b) via

$$\begin{aligned} a &= \left(\frac{99}{b-a} + 100 - (b-a)\right) \div 2 \text{ and} \\ b &= a + (b-a). \end{aligned}$$

We end up with the following 6 solutions for (a, b): $(99, 100)$, $(65, 68)$, $(51, 60)$, $(49, 60)$, $(35, 68)$ and $(1, 100)$.

Explorations in Indigenous Mathematics

No. 2

Edward Doolittle

Drum Lacing

Drums are widely used by Cree and Mohawk Indigenous people in Canada, among many others, for ceremony and social dance purposes. Typically a drum has a circular frame of some sort over which a skin is stretched. The drum skin may have holes in it, or may be secured on in a more complicated fashion, and then the drum skin is stretched tight by lacing sinew or rope across the bottom of the drum to hold the drum skin tightly to the frame. The whole endeavour of drum making involves a great deal of experience, skill, and art. We will be investigating just a small part of the process of drum making, the patterns used to lace the skin onto the frame; and of all the ways to do the lacing, we will be looking at only one simple way in this article.

Figure 1: A Water Drum (see [1] at 22:42).

A straightforward example of such a pattern can be seen in the construction of a water drum (see Figure 1). Water drums are used in my community (Six Nations) for social dances; I have participated in an Ojibway ceremony in which a water drum was used by special request of the participants; and water drums are used in the ceremonies of the Native American Church. In the case of a water drum, the frame is actually a pot which holds water. The drum skin is held in place by bunching it up with the help of pebbles at seven evenly spaced locations around the lip of the cup and then tying with rope to make "handles" between the pebbles (see Figure 1). Lacing then pulls the handles tight by following a seven-pointed star pattern as in Figure 2. Numbering the seven locations starting at 0, we have the pattern of numbers 0, 3, 6, 2, 5, 1, 4, and back to 0. In other words, we start at 0 and our first target is the position $0 + 3 = 3$, which is 3 spaces clockwise from 0. Our target from there is $3 + 3 = 6$, which is 3 more spaces clockwise from 0. Our next target is $6 + 3 \equiv 2$, which is another 3 more spaces clockwise. (See Figure 3.)

Figure 2: Star Pattern on Bottom of Water Drum (see [1] at 20:47).

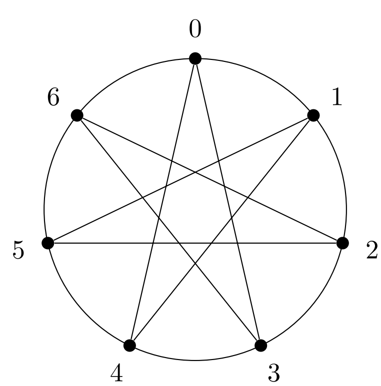


Figure 3: Number Circle mod 7 with Star Pattern $0 + 3k$ mod 7

Note that the arithmetic is a little different from the ordinary arithmetic on a line with which you are most familiar. Here we are doing "circle arithmetic," generally known in mathematics as *modular arithmetic* with the *modulus* 7. It is like ordinary arithmetic except that we take as many 7s as we can from a result before writing it down. We do not say that $6+3=2$, because $6+3=9$, but we do say $6+3=9\equiv 9-7=2$ in the modulus 7 (which is abbreviated "modulo 7" or just "mod 7"). We can do modular arithmetic in any modulus, not just 7, but for now we will be using 7 in most of our examples.

When we lace a water drum we generate an *arithmetic sequence* modulo 7. An arithmetic sequence in ordinary arithmetic (also known as "skip counting" in elementary math education) might be $0,3,6,9,12,\ldots$. In arithmetic modulo 7 we don't have $9,12,\ldots$ so the arithmetic sequence is $0,3,6,2,5,1,4,0,\ldots$ instead. Another way to think about this is that in ordinary arithmetic we are making a multiplication table for multiples of 3; so in modular arithmetic we are making a multiplication table for multiples of 3 mod 7 (see Table 1).

$\times$	0	1	2	3	4	5	6
3	0	3	6	2	5	1	4

Table 1: Multiples of 3 mod 7

We can see a few aspects of the arithmetic sequence mod 7 immediately. First, since the sequence is infinite, but there are only a finite number of positions around the circle, at least one number must show up more than once in the sequence. In fact, the first number that is repeated is 0. You should be able to show that in any arithmetic sequence in any modulus, the first number repeated is the first number in the sequence. (Hint: what happens if we reverse the sequence?) You should be able to show that no matter where we start the sequence (e.g., starting at 2 we get $2,5,1,4,0,3,6,2,\ldots$) we get a "rotation" of the original sequence, with the same numbers in the same order. Also note that our arithmetic sequence contains all the numbers around the circle (which, for historical reasons, are called "residues mod 7").

What if we change the number that we add at each stage? Let's try adding 2 instead of adding 3. You wouldn't do that when making a proper water drum because the drum head might slip off too easily, but we can imagine what the pattern would look like. We start at 0 and then move 2 places around the circle at each step to obtain $0,2,4,6,1,3,5,0,\ldots$ (see Figure 4). Note that again we have reached every position around the circle. Try skips (technically known as "common differences") other than 3 and 2. (Note that you don't have to try skips larger than 6, because a skip of 11, say, would be the same as a skip of 4.) You should find that there is only one skip that doesn't make the sequence reach every position around the circle: 0 mod 7. Using 0 as a skip generates the arithmetic sequence $0,0,0,\ldots$.

Now let's try changing the modulus to see if anything different might occur in a different modulus. The modulus 10 is interesting. I am not aware of any drums that have 10 places for the lacing around the circumference, although there may be such drums; but 10 is interesting for another reason, namely its relationship

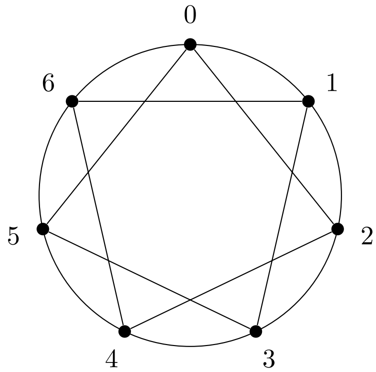


Figure 4: Arithmetic Sequence $0 + 2k$ mod 7

to ordinary arithmetic base 10. In mod 10 the arithmetic sequence with initial term 0 and skip 3 is $0, 3, 6, 9, 2, 5, 8, 1, 4, 7, 0, \ldots$, which reaches every point around the circle (see Figure 5). Also note that the relationship between multiplication in ordinary arithmetic and multiplication in modular arithmetic is very clear in mod 10 if we also keep track of the number of times we have traveled completely around the circle (see Table 2). Note how the products in ordinary arithmetic arise naturally by combining the count of the number of full rotations around the circle with the landing point. This observation is used to teach multiplication tables in Waldorf schools [2, p. 205–206].

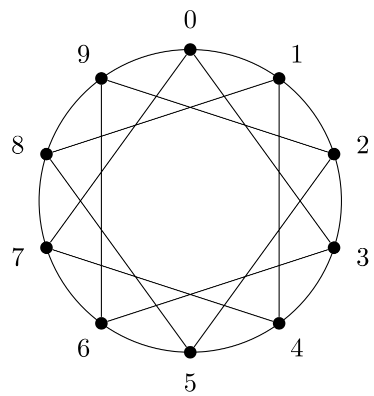


Figure 5: Arithmetic Sequence $0 + 3k$ mod 10

$3\times$	0	1	2	3	4	5	6	7	8	9
rotations	0	0	0	0	1	1	1	2	2	2
residue	0	3	6	9	2	5	8	1	4	7

Table 2: Multiples of 3 mod 10

On the other hand, the arithmetic sequence with initial term 0 and common difference 2 is $0, 2, 4, 6, 8, 0, \ldots$ which does not reach every position around the circle (see Figure 6). (That does not mean we cannot lace a drum by going two steps over each time; it means you would need two different laces, one going through the even-numbered points around the circle, and one going through the odd-numbered points.) You should identify the set S of common differences mod 10 for which the arithmetic series reaches every position around the circle (in mod 10 it is $S = \{1, 3, 7, 9\}$). What is the set S in mod 7? Try other moduli; some numbers used in drum making are 12, 16, 21, and 28. In each case, can you characterize the set S? For which moduli is the set of common differences which reach every position around the circle as large as possible (i.e., every residue except for 0)? Can you say anything about the size of the set S as a function of the modulus for specific classes of numbers? In general?

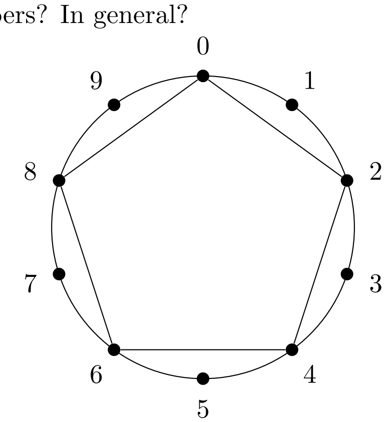


Figure 6: Arithmetic Sequence $0 + 2k \bmod 10$

Another related question is whether we can find common differences which always reach every position around the circle, and common differences which never reach every position around the circle. For example, the common difference 1 is in the first category for any modulus, and the common difference 0 is in the second category for any modulus. Can you find another common difference that is in the first category for any modulus? When making a drum, it may be important to pull the lace as close to the location directly opposite as possible, but maybe not directly opposite. Is the resulting common difference in the first category for any modulus?

We have learned a tool, modular arithmetic, to help us to understand drum making better. But there is still an enormous variety of lacing techniques that do not fit so neatly into our current framework. For example, with 12 points around the circle, some drum makers go straight across (which corresponds to a common difference of 6), and we know that means they have to use 6 separate laces. If you are interested, you can find many resources for drum making online, and you should also consult with local experts (Indigenous elders, Indigenous drum makers) to learn how drums are made in your local Indigenous communities.

Acknowledgement

I would like to thank Keith Taylor for the idea for this article, and my wife Shannon McNabb for comments and suggestions.

References

[1] Marvin Diamond and Hugh Foley. *Marvin Diamond (Otoe-Missouria): Instrumental Blessings.* YouTube. 2010. `https://www.youtube.com/watch?v=MCFriPO6Vf8` (visited on 11/14/2021).

[2] Claudia Georgia Sabba and Ubiratan D'Ambrosio. *An Ethnomathematical Perspective on the Question of the Idea of Multiplication and Learning to Multiply: The Languages and Looks Involved.* In: Teaching Multiplication with Lesson Study: Japanese and Ibero-American Theories for Mathematics Education. Ed. by Masami Isoda and Raimundo Olfos. Springer, 2021. Chap. 8, pp. 199213. doi: `https://doi.org/10.1007/978-3-030-28561-6_8`

TEACHING PROBLEMS

No. 14

John McLoughlin

An Inspiring Problem Proposal: Working from the Solutions

The following problem proposal was submitted to ***Crux*** for consideration:

> Fill in the spaces between numbers 1 through 9 with either a + or $\times$ sign so that the resulting value is a perfect square.

The problem proposer, Neculai Stanciu, provides a list of 13 solutions obtained using a computer. One such solution is given here:

$$1 \times 2 + 3 + 4 + 5 \times 6 \times 7 \times 8 \times 9 = 15129 = 123^2.$$

This solution ought to clarify any concerns about the problem statement. Also, it is evident that such a solution is likely inaccessible without computer assistance or exhaustive search of the 2^8 possible cases. Likewise, a second solution on the list appears here. Unlike the first, some will recognize the result of 1225 as a perfect square:

$$1 + 2 \times 3 \times 4 \times 5 \times 6 + 7 \times 8 \times 9 = 1225 = 35^2.$$

The remainder of this article focusses attention on breaking down this problem in a manner that enriches its merits for teaching and learning about problem solving. The idea here comes from using the solutions to generate questions, observations, and guideposts that may enable finding all remaining solutions. Here we go.

Starting point

> *Observe that the sum of the digits from 1 to 9 is 45 (not a perfect square). Is it possible to change a single + to $\times$ so as to produce a perfect square?*

An important property of consecutive integers is helpful to note. Two consecutive integers always have an odd sum and an even product. Hence, changing a single + to $\times$ will result in changing the parity of the resulting value. Since 45 is odd, it is certain that the result will be even. Further, consider the largest possible increase in value. This would be $8 \times 9 - (8 + 9) = 55$. Aha! $45 + 55 = 100$ and a solution is found:

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 \times 9 = 100 = 10^2.$$

Is there another such solution? The only even perfect square between 45 and 100 is 64. A value of 64 would require an increase of 19 from the sign change. Inspection finds that 5 and 6 satisfy this requirement

$$1 + 2 + 3 + 4 + 5 \times 6 + 7 + 8 + 9 = 64 = 8^2.$$

A curious mathematical fact provides two additional solutions immediately. Observe that $1+2+3 = 1\times 2\times 3$. hence we have

$$1\times 2\times 3+4+5+6+7+8\times 9 = 100,$$
$$1\times 2\times 3+4+5\times 6+7+8+9 = 64.$$

Extending this direction of pursuit, we continue forward.

> *Is it possible to produce a perfect square by changing two plus signs to multiplication signs?*

There are two cases to consider. First, we will consider where the products are separate. Second, the case of three consecutive integers being multiplied will be examined.

Case i: Separate products.

Recall that the parity changes when a single $+$ becomes $\times$; hence, two separate products will result in the parity being unchanged. That is, the results will be odd since 45 is odd.

The smallest possible increase would be $1\times 2+3\times 4-(1+2+3+4) = 4$. the largest possible increase is $6\times 7+8\times 9-(6+7+8+9) = 84$. The result must be between $45+4$ and $45+84$ inclusive. Aha! $45+4 = 49$ and a solution is given by:

$$1\times 2+3\times 4+5+6+7+8+9 = 49.$$

The resulting odd parity and the restricted range of values makes 81 and 121 the only other candidates for resulting perfect squares. Consider the net change in value for changing $+$ to $\times$ between consecutive integers. For example, the pair $(1,2)$ gives a net change of -1 as 1×2 is less than $1+2$; in contrast, $(8,9)$ produces a gain of 55. In summary, we have

$$(1,2)\to -1,\ (2,3)\to 1,\ (3,4)\to 5,\ (4,5)\to 11,\ (5,6)\to 19,$$
$$(6,7)\to 29,\ (7,8)\to 41,\ (8,9)\to 55.$$

The desired results of 81 and 121 would require increases of 36 and 76 respectively. Since no pair of changes in the above list total either 36 or 76, these results are unattainable.

Case ii: Three consecutive integers.

Suppose that three consecutive integers are multiplied. For example, $7\times 8\times 9$ would produce an increase of $7\times 8\times 9-(7+8+9) = 480$, whereas $1\times 2\times 3$ would produce no change in value. The resulting range extends from $45+0$ to $45+480$, as in 45 to 525. The summary of changes is given by

$$(1,2,3)\to 0,\ (2,3,4)\to 15,\ (3,4,5)\to 48,\ (4,5,6)\to 105,$$
$$(5,6,7)\to 192,\ (6,7,8)\to 315,\ (7,8,9)\to 480.$$

What about the parity? The product of three consecutive integers is even, but the sum of three consecutive integers can be odd or even. Hence, all perfect squares between 45 and 525 remain plausible considerations. However, there is only one change to consider unlike the preceding case with two separate products. The question becomes "Does 45 increased by any of the changes listed above produce a perfect square?" The answer is no.

What next?

Before proceeding further, it is worth inspecting to see if 45 increased by two or more of the changes may result in a perfect square. These changes though must not involve the same digits. For example, $(2, 3, 4)$ could only work with triplets not containing any of 2, 3 or 4. I could not find any solution using exclusively triples or pairs, but the revelation of potentially mixing these became apparent as a source of solutions. For example, I noted that $45 + 315 = 360$ or 1 less than 19^2. The pair $(2, 3)$ produces a change of 1. Combining these facts gives

$$1 + 2 \times 3 + 4 + 5 + 6 \times 7 \times 8 + 9 = 361 = 19^2.$$

Let's look for more as likely some solutions can be identified. Keep in mind that the numbers in the pairs and triples cannot be duplicated. Recall the summary of the changes from above that will assist us. Note that $45 + 480 = 525$ as in $23^2 - 4$. The pairs offer $5 + (-1)$ as a representation of this change of 4:

$$1 \times 2 + 3 \times 4 + 5 + 6 + 7 \times 8 \times 9 = 529 = 23^2.$$

As a cautionary note, it appeared that $45+105+19$ would produce a solution. The digits 5 and 6 appeared in both the changes of 105 and 19, and hence no solution resulted from this arrangement. The next solution grew from an observation that $45 + 48 = 93$ and an additional 28 would make 121. Here, $28 = 29 + (-1)$.

$$1 \times 2 + 3 \times 4 \times 5 + 6 \times 7 + 8 + 9 = 121.$$

"Playing around" suggested 144 as a possible target. Note that $45 + 55 = 100$ and $29 + 15 = 44$. Rather fortuitously, there are no digits used twice, thus producing the solution

$$1 + 2 \times 3 \times 4 + 5 + 6 \times 7 + 8 \times 9 = 144.$$

It seemed that beginning with higher or lower digits, as with $(8, 9) \rightarrow 55$, left more options as the mid-range digits eliminated more possibilities due to overlap. That was an observation which grew out of understanding the problem.

Recapping to here

At this point it seemed that the playfulness was diminishing. Keep in mind that a list of solutions had accompanied the problem proposal. Aside from the two opening results of 123^2 and 35^2, all of the other solutions to this point have been found through an organized approach without checking the list of 13 solutions. In fact, 11 of these have now appeared in the article thus far.

Next steps

Before abandoning the solving process, it seems worthwhile to consider changes from using three consecutive multiplication signs as in products of four consecutive integers. For example, $1 \times 2 \times 3 \times 4 - (1 + 2 + 3 + 4) = 14$, giving $(1, 2, 3, 4) \to 14$. The summary of changes is here:

$$(1, 2, 3, 4) \to 14,\ (2, 3, 4, 5) \to 106,\ (3, 4, 5, 6) \to 342,$$
$$(4, 5, 6, 7) \to 818,\ (5, 6, 7, 8) \to 1654,\ (6, 7, 8, 9) \to 2994.$$

It seemed reasonable to check if 45+14 or 45+106 or 45+342 could be combined with changes from pairs and/or triples involving larger digits to get perfect squares. It was noted that $45 + 14 = 59$ could be combined with $(7, 8) \to 41$ to equal 100.

$$1 \times 2 \times 3 \times 4 + 5 + 6 + 7 \times 8 + 9 = 100.$$

Knowing one more solution existed and my time on the problem was winding down, I decided to peek at the list of solutions. Specifically, the glance at results showed 441 among them. Revisiting the last stage, it was noted that $342 + 45$ would get into the neighbourhood. A further increase of 54 would be needed. The 13th and final solution emerged from noting $54 = 55 + (-1)$:

$$1 \times 2 + 3 \times 4 \times 5 \times 6 + 7 + 8 \times 9 = 441 = 21^2.$$

Closing remarks

The problem proposal from Neculai Stanciu inspired me to play with mathematics while figuring a way of offering insight into the problem solving process. Beginning with the solutions, it became practical to turn this experience around to one of teaching through problem solving. The "write out loud" approach taken here has offered some lessons of its own. First, no permission is needed to play with a problem. Second, it is not necessary to exhaustively solve a problem. Mathematical experience can be gained through partially solving a problem and/or getting stuck altogether. Third, the explication of the process has ideally given some value to organized solution of a problem while informing the significance of ideas like parity. Finally, it is important to acknowledge inspiration. Upon seeing the problem proposal, it came to me that this is not to be a problem in a set but rather the basis of a richer discussion. Thank you Neculai Stanciu for bringing this proposal to my attention.

OLYMPIAD CORNER

No. 398

The problems in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **February 28, 2022**.

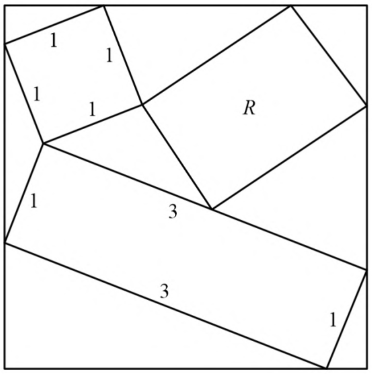
OC556. Find all integer pairs (x, y) that satisfy the equation

$$7x^2 - 4xy + 7y^2 = (|x - y| + 2)^3.$$

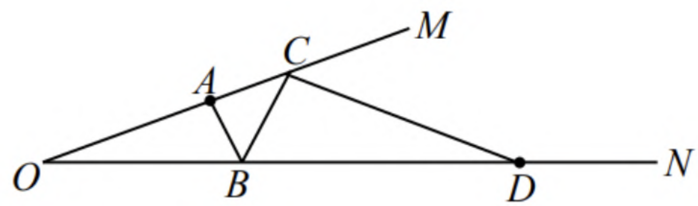
OC557. A natural number k is given. For $n \in \mathbb{N}$ we define $f_k(n)$ as the smallest integer greater than kn such that $nf_k(n)$ is a perfect square. Prove that $f_k(m) = f_k(n)$ implies $m = n$.

OC558. Anne consecutively rolls a 2020-sided die with faces labeled from 1 to 2020 and keeps track of the running sum of all her previous dice rolls. She stops rolling the first time when her running sum is greater than 2019. Let X and Y be the running sums she is most and least likely to have stopped at with non-zero probability, respectively. What is the ratio between the probabilities of stopping at Y to stopping at X?

OC559. A rectangle with side lengths 1 and 3, a square with side length 1, and a rectangle R are inscribed inside a larger square as shown. The sum of all possible values for the area of R can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?



OC560. In the figure below, points A, C are on ray OM and B, D are on ray ON. It is given that $OA = 6$cm, $OD = 16$cm and $\angle NOM = 20^\circ$. What is the minimum length, in cm, of $AB + BC + CD$?



. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **28 février 2022**.

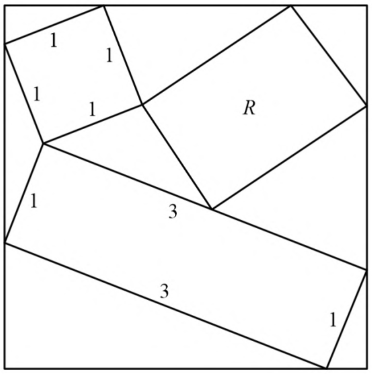
OC556. Trouvez toutes les paires (x, y) d'entiers vérifiant l'équation

$$7x^2 - 4xy + 7y^2 = (|x - y| + 2)^3.$$

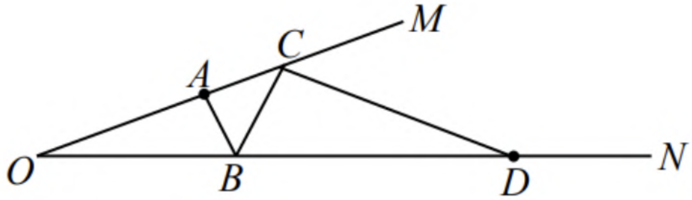
OC557. Soit k un nombre naturel. Pour un $n \in \mathbb{N}$ donné, on définit $f_k(n)$ comme étant le plus petit entier supérieur à kn pour lequel $nf_k(n)$ est un carré parfait. Montrez que $f_k(m) = f_k(n)$ implique que $m = n$.

OC558. Anne lance à répétition un dé à 2020 faces. Celles-ci sont numérotées de 1 à 2020. Anne note la somme cumulée de tous ses lancers précédents. Elle cesse de lancer le dé dès que la somme cumulée dépasse 2019. On dénote par X et Y les sommes cumulées auxquelles elle est respectivement le plus et le moins susceptible de s'être arrêtée. Quel est le rapport entre la probabilité qu'elle se soit arrêté à Y et la probabilité qu'elle se soit arrêtée à X?

OC559. Dans un certain carré sont inscrits un rectangle de côtés 1 et 3, un carré de côté 1, puis un rectangle R, tel qu'indiqué ci-bas. Or, les valeurs possibles pour la surface de R peuvent être représentées sous la forme $\frac{m}{n}$, où m et n sont des entiers positifs premiers entre eux. Déterminer $m+n$.



OC560. Comme l'illustre la figure ci-après, les points A et C sont situés sur la demi-droite OM alors que les points B et D sont quant à eux situés sur la demi-droite ON. Si $OA = 6$cm, $OD = 16$cm et $\angle NOM = 20°$, quelle est la longueur minimale, en cm, de $AB + BC + CD$?



OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2021: 47(5), p. 238–239.

OC531. Given a non-zero integer k, prove that equation

$$k = \frac{x^2 - xy + 2y^2}{x + y}$$

is satisfied by an odd number of ordered pairs of integers (x, y) if and only if k is divisible by 7.

Originally Problem 6 from the 2017 Czech-Slovakia Math Olympiad, Category A, Final Round.

We received 6 *correct solutions. We present the solution by Oliver Geupel.*

An ordered pair (x, y) of integers is a solution of the given equation if and only if

$$x \neq -y \tag{1}$$

and

$$x^2 - xy + 2y^2 - k(x + y) = 0,$$

which we rewrite in the form

$$(2x - k - y)^2 + (y - k)(7y + k) = 0. \tag{2}$$

Let L_0 be the set of solutions of (1) and (2) with $y = 0$. Let L_k be the set of solutions of (1) and (2) with $y = k$. Let $L_{-k/7}$ be the set of solutions of (1) and (2) with $y = -k/7$.

Note that $L_{-k/7} = \varnothing$ when $7 \nmid k$. Let L' be the set of solutions of (1) and (2) with $y \notin \{0, k, -k/7\}$. The total set L of solutions of (1) and (2) is then the disjoint union

$$L = L_0 \cup L_k \cup L_{-k/7} \cup L'.$$

It is straightforward to check that $L_0 = \{(k, 0)\}$ and $L_k = \{(k, k)\}$. If $7 \mid k$ then $L_{-k/7} = \{(3k/7, -k/7)\}$, otherwise $L_{-k/7} = \varnothing$.

Let $y \notin \{0, k, -k/7\}$. Let x_1 and x_2 be the two complex roots of equation (2) in the variable x. By Vieta's Theorem, we have $x_1 + x_2 = k + y$. Hence, either both numbers x_1 and x_2 are integers, or they are both non-integers. Also the hypothesis $x = -y$ leads to $y = 0$, which is impossible. Therefore, there are zero or two corresponding values of the integer x that satisfy the equation (2). Hence, $|L'|$ is even, so that $|L_0 \cup L_k \cup L'|$ is even.

It follows that $|L|$ is odd if and only if $\left|L_{-k/7}\right|$ is odd, which is satisfied exactly when k is a multiple of 7.

OC532. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$.

(a) If $A = \{m \cdot a_n \mid m, n \in \mathbb{N}^*\}$, prove that every open interval of positive real numbers contains elements of A.

(b) If for all $n \in \mathbb{N}^*$ and all $x, y \in [a, b]$ such that $|x - y| = a_n$ the following inequality holds

$$\left|\int_x^y f(t)\ dt\right| \leq |x - y|,$$

prove that

$$\left|\int_x^y f(t)\ dt\right| \leq |x - y| \qquad \forall x, y \in [a, b].$$

Originally problem 3 from the 2018 Romania Math Olympiad, Final Round.

We received 5 *correct solutions. We present the solution by Oliver Geupel.*

Let us start with part (a) of the problem. Let c and d be real numbers such that $0 < c < d$. Since the sequence (a_n) tends to zero, there is an index n with the property that $a_n < d - c$. Then,

$$1 < \frac{d}{a_n} - \frac{c}{a_n}.$$

Hence, there is a positive integer m such that

$$\frac{c}{a_n} < m < \frac{d}{a_n},$$

that is, the open interval (c, d) includes the number ma_n. This completes part (a).

Let us now turn to part (b). As a Riemann integrable function, f is bounded. Let K be a real number such that $|f(t)| < K$ whenever $a < t < b$. Let $x, y \in [a, b]$. Since the inequality to be proved is symmetric in the variables x and y, we may suppose that $x \leq y$. Let n be any positive integer and let $m = \left\lfloor \frac{y-x}{a_n} \right\rfloor$. By the triangle inequality and by the hypothesis of part (b), it holds

$$\begin{aligned}
\left|\int_x^y f(t)\ dt\right| &= \left|\left(\sum_{k=1}^{m} \int_{x+(k-1)a_n}^{x+ka_n} f(t)\ dt\right) + \int_{x+ma_n}^{y} f(t)\ dt\right| \\
&\leq \sum_{k=1}^{m} \left|\int_{x+(k-1)a_n}^{x+ka_n} f(t)\ dt\right| + \left|\int_{x+ma_n}^{y} f(t)\ dt\right| \\
&\leq ma_n + Ka_n \\
&\leq |x - y| + Ka_n.
\end{aligned}$$

By hypothesis we have $\lim\limits_{n\to\infty} Ka_n = 0$. Hence the result (b).

OC533. For $k \in \mathbb{Z}$, define the polynomial $F_k(x) = x^4 + 2(1-k)x^2 + (1+k)^2$. Find all values of k so that F_k is irreducible over $\mathbb{Z}[x]$ and reducible over $\mathbb{Z}_p[x]$ for all primes p.

Originally problem 4 from the 2018 Romania Math Olympiad, Final Round.

We received 5 *solutions, of which* 4 *were correct and complete. We present the solution by the Missouri State University Problem Solving Group.*

We claim that the result holds precisely when neither k nor $-k$ is a perfect square in $\mathbb{Z}$.

Note that if $\alpha^2 = k$ in a unitary commutative ring R, then

$$F_k(x) = \left(x^2 + 2\alpha x + (k+1)\right)\left(x^2 - 2\alpha x + (k+1)\right) \text{ in } R[x]. \tag{1}$$

Also, if $\beta^2 = -k$ in R, then

$$F_k(x) = \left(x^2 + 2\beta + (1-k)\right)\left(x^2 - 2\beta + (1-k)\right) \text{ in } R[x]. \tag{2}$$

Therefore if k or $-k$ is a perfect square in $\mathbb{Z}$, then F_k is reducible in $\mathbb{Z}[x]$. On the other hand if $-k$ is not a perfect square, then we claim F_k has no rational roots and hence no linear factor. If r were a rational root of $F_k(x)$, then r^2 would be a root of $x^2 + 2(1-k)x + (1+k)^2$, but the roots are $k - 1 \pm 2\sqrt{-k}$, and these are not rational.

If $\gamma^2 = -1$ in R, we have

$$F_k(x) = \left(x^2 + 2\gamma x - (1+k)\right)\left(x^2 - 2\gamma x - (1+k)\right) \text{ in } R[x]. \tag{3}$$

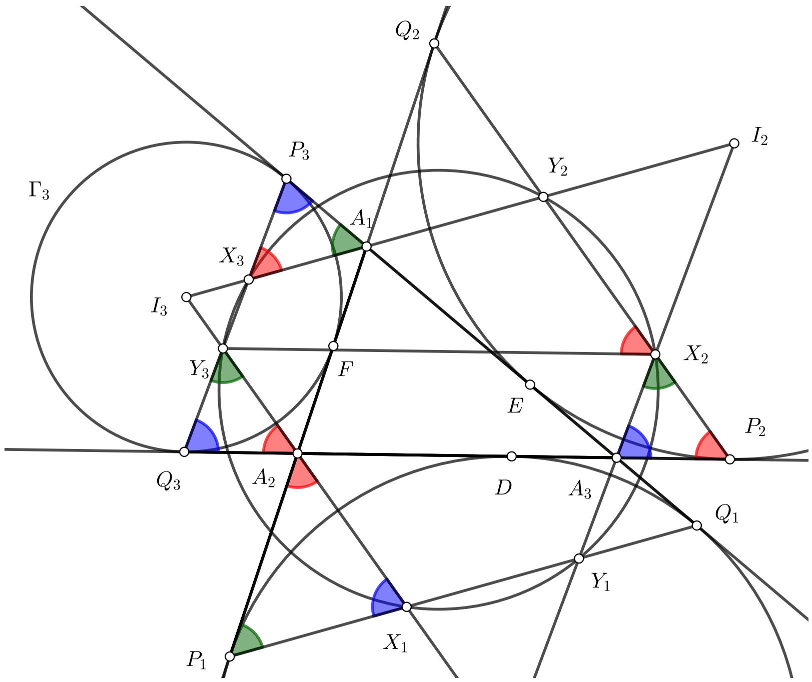
By unique factorization in $\mathbb{C}[x]$, equations (1), (2), and (3) are the only ways of factoring F_k into monic quadratics and the coefficient of x is not an integer in any of those factorizations if neither k nor $-k$ is a perfect square. Therefore F_k is irreducible in $\mathbb{Z}[x]$.

On the other hand, it is well known that in $\mathbb{Z}_p$ at least one of y, z, or yz must be a square. Therefore one of $k, -1$, or $-k$ must be a square and hence at least one of the factorizations above exists in $\mathbb{Z}_p[x]$.

OC534. The triangle $A_1A_2A_3$ is given on the plane. Assuming that $A_4 = A_1$ and $A_5 = A_2$, we define points X_t and Y_t for $t = 1, 2, 3$ as follows. Let Γ_t be the excircle of triangle $A_1A_2A_3$ tangent to the side $A_{t+1}A_{t+2}$, and let I_t be its center. Let P_t and Q_t be the points of tangency of Γ_t with the lines A_tA_{t+1} and A_tA_{t+2}, respectively. Then X_t and Y_t are the intersection points of the line P_tQ_t with the lines I_tA_{t+1} and I_tA_{t+2}, respectively. Prove that the points $X_1, Y_1, X_2, Y_2, X_3, Y_3$ lie on a circle.

Originally problem 11 from the 2018 Poland Math Olympiad, First Round.

We received 3 *correct solutions. We present the solution by the UCLan Cyprus Problem Solving Group.*



We write α, β and γ for the angles $\angle A_3A_1A_2, \angle A_1A_2A_3$ and $\angle A_2A_3A_1$ respectively. We also write a, b, c for the side lengths A_2A_3, A_3A_1 and A_1A_2 respectively.

Since A_3Q_3 and A_3P_3 are tangent to Γ_3, we have $A_3Q_3 = A_3P_3$, which implies $\angle A_2Q_3Y_3 = 90^\circ - \gamma/2$. Since Y_3A_2 is the external angle bisector of $\hat{A}_2$, then $\angle Y_3A_2Q_3 = 90^\circ - \beta/2$. It follows that $\angle Q_3Y_3A_2 = 90^\circ - \alpha/2$.

With similar methods we can calculate the angles in the triangles $A_3X_2P_2$, $P_3A_1X_3$ and $X_1P_1A_2$ and obtain that these triangles are all similar to the triangle $Q_3Y_3A_2$

Let D, E, F be the points of tangency of Γ_1, Γ_2 and Γ_3 with the lines A_2A_3, A_3A_1 and A_1A_2 respectively.

It is well known that $A_3Q_3 = A_2P_2 = (a+b+c)/2$. Therefore $A_2Q_3 = P_2A_3$ from which we deduce that the triangles $A_3X_2P_2$ and $Q_3Y_3A_2$ are equal.

Thus Q_3Y_3 and A_3X_2 are equal and parallel and therefore $A_3Q_3Y_3X_2$ is a parallelogram.

So $\angle Y_3X_2Y_1 = \angle Y_3X_2A_3 = \angle A_2Q_3Y_3 = 90^\circ - \gamma/2 = \angle Y_3X_1P_1$. So X_1, Y_1, X_2, Y_3 are concyclic, say they belong to a circle ω_1. Analogously, X_2, Y_2, X_3, Y_1 belong to a circle ω_2 and X_3, Y_3, X_1, Y_2 belong to a circle ω_3.

We also have $\angle Y_2X_2Y_3 = \angle A_3P_2X_2 = 90^\circ - \beta/2 = \angle Y_2X_3P_3$. So X_2, Y_2, X_3, Y_3 are concyclic. It follows that Y_3 belongs to ω_2 and then that ω_2 and ω_3 coincide. Thus $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are concyclic.

OC535. The set A consists of n real numbers. For the subset $X \subseteq A$, we denote by $S(X)$ the sum of the elements of the set X, and we assume $S(\emptyset) = 0$. Let k be the number of different real numbers x such that $x = S(X)$ for some $X \subseteq A$. Let ℓ be the number of ordered pairs (X, Y) of subsets of the set A satisfying the equality $S(X) = S(Y)$. Prove that $k\ell \leq 6^n$.

Originally problem 12 from the 2018 Poland Math Olympiad, First Round.

We received only 1 *correct solution by the UCLan Cyprus Problem Solving Group.*

Let $\mathcal{B}$ be a set of subsets of A all with distinct sums and let $\mathcal{C}$ be a set of subsets of A all with the same sums. Consider the map $f : \mathcal{B} \times \mathcal{C} \to \mathcal{P}(A) \times \mathcal{P}(A)$ defined by $f(B, C) = (B \cap C, B \cup C)$. We claim that this is an injection. Indeed, suppose $B \cap C = B' \cap C'$ and $B \cup C = B' \cup C'$. Then

$$S(B)+S(C) = S(B\cup C)+S(B\cap C) = S(B'\cup C')+S(B'\cap C') = S(B')+S(C')\,.$$

Since $S(C) = S(C')$ we deduce that $S(B) = S(B')$ and by the definition of $\mathcal{B}$ we get $B = B'$. Then

$$C = ((C \cup B) \setminus B) \cup (C \cap B) = ((C' \cup B) \setminus B) \cup (C' \cap B) = C'\,.$$

Now assume that $x_1, \ldots, x_k$ are all possible sums and for each i assume that there are a_i sets achieving sum x_i. Then $\ell = a_1^2 + \cdots + a_k^2$ and $a_1 + \cdots + a_k = 2^n$.

Note that the map f of the first paragraph injects into a pair of subsets (X, Y) with $X \subseteq Y$. There are

$$\sum_{k=0}^{n} \binom{n}{k} 2^{n-k} = (1+2)^n = 3^n$$

such pairs. This is because if the set X has k elements, then we have 2^{n-k} choices for the other elements that Y is going to contain.

This shows that $ka_i \leqslant 3^n$ for each i. Therefore

$$k\ell = k(a_1^2 + \cdots + a_k^2) \leqslant 3^n(a_1 + \cdots + a_k) = 6^n\,.$$

Equations involving positive divisors of a given integer (Part II)

Salem Malikić

This article is a continuation of an article published in ***Crux*** 48(4). Throughout the article we focus only on positive divisors of a given integer, hence in most places we omit to explicitly mention the word *positive*.

Problem 1 *Let $1 = a_1 < a_2 < \dots$ and $1 = b_1 < b_2 < \dots$ be all divisors of positive integers a and b, respectively. Find all a and b such that each of them has at least 11 distinct divisors and the following system of equations is satisfied*

$$\begin{aligned} a_{10} + b_{10} &= a \\ a_{11} + b_{11} &= b. \end{aligned}$$

(Mongolia, 2017)

Solution. We will first prove that at least one of the numbers a and b has exactly 11 divisors. Assume on the contrary, that each of a and b has more than 11 divisors. Using the formula $d_i d_{\tau(n)+1-i} = n$ and the assumption that divisors are sorted in an increasing order, we have

$$\begin{aligned} a_{10} &= \frac{a}{a_{\tau(a)+1-10}} \leq \frac{a}{a_3} \leq \frac{a}{3} \\ a_{11} &= \frac{a}{a_{\tau(a)+1-11}} \leq \frac{a}{a_2} \leq \frac{a}{2} \\ b_{10} &= \frac{b}{b_{\tau(b)+1-10}} \leq \frac{b}{b_3} \leq \frac{b}{3} \\ b_{11} &= \frac{b}{b_{\tau(b)+1-11}} \leq \frac{b}{b_2} \leq \frac{b}{2} \end{aligned}$$

and adding the above inequalities implies $a_{10} + a_{11} + b_{10} + b_{11} < a + b$, which is in contradiction with the equation obtained by adding up the two equations given in the problem statement. From this contradiction it follows that at least one of the numbers a and b has exactly 11 divisors. Due to the second equation of the system, this number can not be b so a must have exactly 11 divisors. This is only possible if $a = p^{10}$, where p is a prime number.

From $a_{10} + b_{10} = a$ we now get $b_{10} = p^{10} - p^9 = p^9(p-1)$. From $a_{11} + b_{11} = b$ it follows that $b_{11} \mid a_{11}$, hence b_{11} must be a prime power of p not greater than $a_{11} = p^{10}$. Since $b_{10} = p^9(p-1) \geq p^9$, we must have $b_{11} = p^{10}$. Now, since $b_{11} = p^{10}$, from $a_{11} + b_{11} = b$ we have $b = 2 \cdot p^{10}$. From $b_{10} = p^9(p-1)$ it follows that $p - 1$ is a divisor of b. In other words, $p - 1 \mid 2 \cdot p^{10}$. As p and $p - 1$ are relatively prime, we have that $p - 1 \mid 2$. Therefore p can be either 2 or 3. For $p = 2$, we get $a = 2^{10}$ and $b = 2^{11}$ and it is easy to verify that this is a solution. For $p = 3$, we get $a = 3^{10}$ and $b = 2 \cdot 3^{10}$, and it can be easily verified that this pair does not satisfy all conditions of the problem.

In summary, $a = 2^{10}, b = 2^{11}$ is the only solution.

Problem 2 *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n such that*

$$2n = d_5^2 + d_6^2 - 1.$$

(Switzerland, 2006)

Solution. Observe first that

$$2n = d_5^2 + d_6^2 - 1 < d_6^2 + d_6^2 - 1 < 2d_6^2$$

which, together with $d_6 d_{k+1-6} = n$, implies that $k + 1 - 6 < 6$ or, equivalently, $k \le 10$. On the other hand,

$$2n = d_5^2 + d_6^2 - 1 \ge d_5^2 + (d_5 + 1)^2 - 1 > 2d_5^2$$

and the above inequality implies $n > d_5^2$. As $n = d_5 d_{k+1-5}$ we must have $k+1-5 > 5$ or, equivalently, $k \ge 10$. Now we can conclude that k must be equal to 10.

Since $k = 10$, we have $n = d_5 d_6$ and the given equation becomes

$$2d_5 d_6 = d_5^2 + d_6^2 - 1,$$

which is equivalent to

$$(d_6 - d_5)^2 = 1.$$

The last equation implies that $d_6 = d_5 + 1$, hence $n = d_5(d_5 + 1)$.

Now, as n has 10 distinct positive divisors then either $n = p^9$, for some prime p, or $n = pq^4$ for some distinct primes p and q.

We can not have $n = p^9$ because in this case d_5 and $d_5 + 1$ are each divisible by p, so p has to divide their difference, which is clearly impossible.

If $n = pq^4$, observe that $pq^4 = d_5(d_5 + 1)$ and the fact that d_5 and $d_5 + 1$ are relatively prime and both greater than 1 implies that either (i) $d_5 = p$ and $d_5 + 1 = q^4$ or (ii) $d_5 = q^4$ and $d_5 + 1 = p$. In the first case $p = q^4 - 1 = (q-1)(q+1)(q^2+1)$ must be a prime, but it is obviously a composite number so there is no solution in this case. In the second case, $q^4 + 1 = p$. Clearly, q must be even so the only possibility is $q = 2$. For $q = 2$ we have $p = 17$ and $n = 2^4 \cdot 17 = 272$. Direct verification shows that this is a solution.

In summary, the only positive integer n for which the given equation is satisfied is 272.

Problem 3 *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Determine all n such that $k \ge 22$ and*

$$d_7^2 + d_{10}^2 = \left(\frac{n}{d_{22}}\right)^2.$$

(Belarus, 1998)

Solution. First, by doing simple modular arithmetic, one can easily prove that, if integers a, b and c are such that $a^2+b^2=c^2$, then at least one of them is divisible by 3 and at least one of them is divisible by 5. Observing that, for $x \equiv 0 \pmod 4$ we have $x^2 \equiv 0 \pmod 8$, for $x \equiv 2 \pmod 4$ we have $x^2 \equiv 4 \pmod 8$, and for $x \equiv \pm 1 \pmod 4$ we have $x^2 \equiv 1 \pmod 8$, we can conclude that at least one of a, b and c must be divisible by 4.

As d_7, d_{10} and $\frac{n}{d_{22}}$ are divisors of n we conclude that n is divisible by 3, 4 and 5. Therefore $d_i = i$ for $i \in \{1,2,3,4,5,6\}$.

Since n is divisible by 2 and 5, it is also divisible by 10, hence $d_7 \leq 10$. Similarly, n is also divisible by 12, 15 and 20 implying that $d_{10} \leq 20$.

We discuss four possible cases:

1. $d_7 = 10$

   In this case the given equation becomes
   $$\left(\frac{n}{d_{22}} - d_{10}\right)\left(\frac{n}{d_{22}} + d_{10}\right) = 100.$$
   Since the numbers in the brackets on the left hand side of the last equation are of the same parity and the first is strictly smaller than the second, we have that $\frac{n}{d_{22}} - d_{10} = 2$ and $\frac{n}{d_{22}} + d_{10} = 50$, which implies $d_{10} = 24$. This is in contradiction with $d_{10} \leq 20$ so we have no solution in this case.

2. $d_7 = 9$

   In this case the given equation becomes
   $$\left(\frac{n}{d_{22}} - d_{10}\right)\left(\frac{n}{d_{22}} + d_{10}\right) = 81.$$
   so either $\frac{n}{d_{22}} - d_{10} = 1$ and $\frac{n}{d_{22}} + d_{10} = 81$ or $\frac{n}{d_{22}} - d_{10} = 3$ and $\frac{n}{d_{22}} + d_{10} = 27$.
   In the first case $d_{10} = 40$, which contradicts $d_{10} \leq 20$.
   In the second case, $d_{10} = 12$ and $\frac{n}{d_{22}} = 15$. As n is not divisible by 7 (because $d_6 = 6$ and $d_7 = 9$) it is also not divisible by 14. Therefore we either have $d_{11} = 15$ or $d_{12} = 15$, implying that either $\frac{n}{d_{22}} = d_{11}$ or $\frac{n}{d_{22}} = d_{12}$. As $n = d_i \cdot d_{k+1-i}$ this would imply that k, the number of divisors of n, is either 32 or 33. On the other hand, since $d_7 = 9$ and $d_{10} = 12$, we must have $d_8 = 10$ and $d_9 = 11$. Now, observe that n is divisible by 2^2, 3^2, 5 and 11 so the number of its divisors is at least $3 \cdot 3 \cdot 2 \cdot 2 = 36$, which contradicts the previous conclusion that $k \in \{32, 33\}$. Therefore in this case we also have no solution.

3. $d_7 = 8$

   In this case the given equation becomes
   $$\left(\frac{n}{d_{22}} - d_{10}\right)\left(\frac{n}{d_{22}} + d_{10}\right) = 64.$$

so either $\frac{n}{d_{22}} - d_{10} = 2$ and $\frac{n}{d_{22}} + d_{10} = 32$ or $\frac{n}{d_{22}} - d_{10} = 4$ and $\frac{n}{d_{22}} + d_{10} = 16$. The second case obviously does not give any solution as it implies $d_{10} = 6 < d_7$.
In the first case, we have $d_{10} = 15$ and $\frac{n}{d_{22}} = 17$ implying that $\frac{n}{d_{22}} = d_{11}$ or $\frac{n}{d_{22}} = d_{12}$ so n has either 32 or 33 divisors. Also, as $d_7 = 8$ and $d_{10} = 15$, we must have $d_8 = 10$ and $d_9 = 12$.

We can now easily conclude that n can not have 33 divisors because it is divisible by 3, but not divisible by 9, hence it is not a perfect square (i.e., number of its divisors is an even number).

On the other hand, as n is already divisible by 2^3, 3^1, 5^1 and 17^1, the number of its divisors is at least $(3+1)\cdot(1+1)\cdot(1+1)\cdot(1+1) = 32$. Therefore, if the number of divisors of n is 32 then we must have $n = 2^3 \cdot 3 \cdot 5 \cdot 17 = 2040$ and direct verification shows that this is indeed a solution.

4. $d_7 = 7$

In this case the given equation becomes

$$\left(\frac{n}{d_{22}} - d_{10}\right)\left(\frac{n}{d_{22}} + d_{10}\right) = 49.$$

so we must have $\frac{n}{d_{22}} - d_{10} = 1$ and $\frac{n}{d_{22}} + d_{10} = 49$, which implies $d_{10} = 24$. As $d_{10} = 24$ contradicts the previous conclusion that $d_{10} \leq 20$, we do not have any solution in this case.

In conclusion, the only number for which all conditions of the problem are fulfilled is $n = 2040$.

Problem 4 *Find the smallest positive integer n such that*

$$d_1^2 + d_2^2 + \cdots + d_k^2 = (n+3)^2,$$

where $d_1, d_2, \ldots, d_k$ are all positive divisors of n.

(Bulgaria, 1999)

Solution. The given equation is equivalent to

$$d_2^2 + d_3^2 + \cdots + d_{k-1}^2 = 6n + 8$$

and from here it obviously follows that $k \geq 3$.

For $k \geq 8$,

$$\begin{aligned} & d_2^2 + d_3^2 + \cdots + d_{k-1}^2 \\ & \geq (d_{k-1} - d_2)^2 + (d_{k-2} - d_3)^2 + (d_{k-3} - d_4)^2 + 2d_{k-1}d_2 + 2d_{k-2}d_3 + 2d_{k-3}d_2. \end{aligned}$$

We claim that RHS of the above inequality is greater than $6n + 8$. To prove this, observe that $d_{k-3} - d_4 \geq 1$. Then $d_{k-2} - d_3 \geq (d_{k-3} + 1) - (d_4 - 1) \geq 3$. Similarly $d_{k-1} - d_2 \geq 5$. On the other hand, $d_{k-i}d_{i+1} = n$ for $i \in \{1, 2, 3\}$. Combining these equations and the above inequalities we get the desired result. This implies that $k \geq 8$ is impossible so in the rest we focus on the cases where $3 \leq k \leq 7$.

First, we prove that n can not be equal to p^α, where p is a prime number and $\alpha = k - 1 \geq 2$. Namely, in that case our equation becomes

$$p^2 + p^4 + \cdots + p^{2\alpha-2} = 6p^\alpha + 8,$$

which is equivalent to

$$p^2 \left(1 + p^2 + \cdots + p^{2\alpha-4} - 6p^{\alpha-2}\right) = 8.$$

This implies that $p^2 \mid 8$ so $p = 2$ and

$$1 + p^2 + \cdots + p^{2\alpha-4} - 6p^{\alpha-2} = 2.$$

For $\alpha \geq 5$ we have $1 + p^2 + \cdots + p^{2\alpha-4} > 1 + 2^2 + 2^{\alpha-2}2^{\alpha-2} > 2 + 6 \cdot 2^{\alpha-2}$, whereas for each $\alpha < 5$ a direct verification shows that the above equation does not hold for $p = 2$.

We can now conclude that n can not be of the form p^α, where p is a prime and α is an integer. Therefore k must be different from 3 as the only positive integers that have exactly 3 divisors are those that can be expressed as p^2 for some prime number p. Similarly k must be different from 5 and 7, so we are now left with discussing the following two cases:

- $k = 4$ and $n = pq$, where p and q are two distinct primes.

  Without loss of generality we may assume that $p < q$. The given equation becomes
  $$p^2 + q^2 = 6pq + 8.$$

  From here $q \mid p^2 - 8$. Direct inspection on p shows that the smallest prime p for which there exists a solution is $p = 7$. Then $q = 41$ and $n = 287$. For $p \geq 17$ we have $pq > p^2 > 287$. As we are interested in minimal n and have already verified that 287 satisfies all conditions of the problem, here we can conclude that it is enough to focus only on analyzing cases where $p < 17$. It is straightforward to check each of them to conclude that in this case $n = 287$ is the smallest number satisfying all conditions of the problem.

- $k = 6$ and $n = pq^2$, where p and q are two distinct primes.

  The given equation becomes

  $$p^2 + q^2 + p^2q^2 + q^4 = 6pq^2 + 8$$

  We can not have $p \geq 7$ as in that case $p^2q^2 > 6pq^2$ and $q^4 > 8$ so the left hand side of the above equation would be greater than its right hand side. For discussing cases $p \leq 5$, first observe that $p^2 - 8 = q^2(6p - q^2 - p^2 - 1)$, hence $q^2 \mid p^2 - 8$. If $p = 2$ then $q^2 \mid -4$ so q must be equal to 2, which contradicts the assumption that p and q are distinct. For $p = 3$ we get that q^2 divides 1 and for $p = 5$ we get that q^2 divides 17. Both of these are obviously impossible so we can conclude that there is no solution in this case.

In summary, the smallest positive integer n satisfying all conditions of the problem is 287.

Problem 5 *Let* $1 = d_1 < d_2 < \cdots < d_k = n$ *denote all divisors of an integer* n*. Find all positive integers* n *such that* $k \geq 7$ *and*

$$n = d_6^2 + d_7^2 - 1.$$

(IMO Shortlist, 1984)

Solution. First, observe that d_6 and d_7 are relatively prime. Namely, if d is a positive integer which divides each of d_6 and d_7 then d divides n as well. But then from the given equation we have that $d \mid 1$, hence $d = 1$. Second, observe that d_6 and d_7 can not each have at least four distinct divisors. Namely, if this is the case then d_6 has two divisors, u and v such that $1 < u < v < d_6$ and $d_6 = uv$. Similarly d_7 has two divisors, w and s, such that $1 < w < s < d_7$ and $d_7 = ws$. Since $d_6 < d_7$ we have $uw < \sqrt{d_6}\sqrt{d_7} < d_7$. But then 1, u, v, w, s, uv and uw are divisors of n and each of them is smaller than d_7. Furthermore, since d_6 and d_7 are relatively prime, all of these 7 divisors must be distinct. This implies that we have found 7 distinct divisors of n, each less than d_7, which is impossible. Therefore at least one of the numbers d_6 and d_7 has less than 4 divisors so it must be equal to either p or p^2, where p is some prime number. As $d_7 > d_6 \geq 6$, we must have $p > 2$.

Let x denote the number among d_6 and d_7 which equals p or p^2 for some prime $p > 2$ and let y denote the other of these two numbers (i.e., $y \in \{d_6, d_7\}$ and $y \neq x$). Adopting this notation, the given equation becomes $n = x^2 + y^2 - 1$. As $x \mid n$ and $y \mid n$ we conclude that $x \mid y^2 - 1$ and $y \mid x^2 - 1$. As (i) $x \mid (y-1)(y+1)$ and (ii) x is a power of an odd prime and (iii) $\gcd(y-1, y+1) \leq 2$, we can conclude that either $x \mid y - 1$ or $x \mid y + 1$. Therefore we discuss the following two cases:

1. $x \mid y - 1$

   In this case, $y - 1 = xa$ and $x^2 - 1 = yb = (xa + 1)b$, for some positive integers a and b. The last equation can be rewritten as

   $$x(x - ab) = b + 1.$$

   As $x \mid b + 1$ we have that $b + 1 \geq x$. But then

   $$x^2 - 1 = (xa + 1)b \geq (x + 1)(x - 1) = x^2 - 1$$

   with the equality only if $a = 1$. Therefore $y - 1 = x$, implying that $x = d_6$, $y = d_7$ and $d_7 = d_6 + 1$.

2. $x \mid y+1$

   In this case, $y+1 = xa$ and $x^2-1 = yb = (xa-1)b$, for some positive integers a and b. The last equation can be rewritten as

   $$x(ab-x) = b-1.$$

   If $b=1$ then $ab=x$, which implies $a=x$, and $y = xa-1 = x^2-1 = (x-1)(x+1)$. Clearly x and y must be equal to d_6 and d_7, respectively. From $d_7 = (d_6-1)(d_6+1)$ we conclude that $d_6+1 \mid d_7$. As $d_6 > 2$ then $d_7 = (d_6-1)(d_6+1) > d_6+1$ so n has a divisor d_6+1 which is greater than d_6 and smaller than d_7. This is clearly impossible so $b>1$. Now, as $x \mid b-1$ and $b>1$ we have that $b-1 \geq x$. Then

   $$x^2-1 = (xa-1)b \geq (x-1)(x+1) = x^2-1$$

   with the equality only if $a=1$. In that case $y = xa-1 = x-1$ so $y = d_6$, $x = d_7$ and $d_7 = d_6+1$.

From the above we conclude that $d_7 = d_6+1$. Combined with the equation $n = d_6^2 + d_7^2 - 1$, this implies that $n = 2d_6(d_6+1)$. We discuss the following four possible cases:

1. $d_6 = p$, where p is an odd prime number.

   It is obvious that $p \geq 7$. Observe that $d_7 = d_6+1$ is even and does not have two or more odd prime divisors. Namely, an even number having at least two odd prime divisors has at least 8 divisors. On the other hand, each divisor of d_7 is also a divisor of n, so d_7 can not have more than 7 divisors. Therefore $d_7 = 2^{\alpha+1}q^{\beta}$ for some odd prime q and non-negative integers α and β. Combined with $n = 2d_6d_7$, we get that $n = 4 \cdot 2^{\alpha} \cdot p \cdot q^{\beta}$. It is impossible that $\beta \geq 2$ because in that case 1, 2, 4, p, q, $2q$, q^2 are 7 distinct divisors of n, all smaller than d_7. If $\beta = 1$ then $n = 4 \cdot 2^{\alpha} \cdot p \cdot q$. We can not have $\alpha \geq 1$ because in such a case $d_7 \geq 4q$ so 1, 2, 4, 8, p, q and $2q$ are 7 distinct divisors of n, all smaller than d_7. Therefore either $\beta = 0$ or $(\alpha, \beta) = (0,1)$. In the latter case $n = 4pq$ so n has 12 divisors. Then $n = d_6d_7$ and our original equation becomes $d_6d_7 = d_6^2 + d_7^2 - 1$, which does not have any solution as the right hand side is strictly greater than the left hand side. So we must have $\beta = 0$ and this implies that $d_7 = 2^{\alpha+1}$. As any divisor of d_7 is also a divisor of n and p is an odd divisor of n smaller than d_7, $2^{\alpha+1}$ can have at most 6 distinct divisors implying that $\alpha \leq 4$. By checking each of the possible values of α we can easily conclude that only $\alpha = 4$ yields solution $n = 1984$.

2. $d_7 = p$, where p is an odd prime number.

   Analogously as in the previous case, we first prove that $d_6 = p-1$ can not have two or more odd prime divisors implying that $d_6 = 2^{\alpha+1}q^{\beta}$, for some odd prime q and non-negative integers α and β. This also implies that $n = 2d_6d_7 = 4 \cdot 2^{\alpha} \cdot p \cdot q^{\beta}$. If $\alpha \geq 1$ and $\beta \geq 1$ then $4q \leq d_6 < d_7$ and 1, 2, 4, 8, q, $2q$, $4q$ are 7 distinct divisors of n, each smaller than d_7, which is impossible. Therefore either $\alpha = 0$ or $\beta = 0$ and below we discuss each of these cases separately.

For the case when $\alpha = 0$, we have $d_6 = 2q^\beta$. Since d_6 has at most 6 distinct divisors, we must have $\beta \leq 2$. The case $\beta = 0$ will be discussed below. For $\beta = 1$ we have $n = 4pq$, which has been discussed above and does not yield any solution. If $\beta = 2$, then $d_6 = 2q^2$ and $p = d_7 = d_6 + 1 = 2q^2 + 1$. If $q > 3$ the number $2q^2 + 1$ is divisible by 3 and greater than 3, hence it can not be equal to a prime number p. Therefore we must have $q = 3$ implying that $d_6 = 18$ and $n = 2 \cdot 18 \cdot 19 = 684$. However, since the sixth smallest divisor of 684 is less than 18 this is not a solution.

For the case when $\beta = 0$, we have $d_6 = 2^{\alpha+1}$ and it suffices to check all values $\alpha \leq 4$. It is easy to verify that none of them yields a solution.

3. $d_6 = p^2$, where p is an odd prime number.

   Observe that $n = 2d_6d_7$ has three divisors that do not divide $d_7 = p^2+1$ and are smaller than d_7, namely p, $2p$ and p^2. Therefore d_7 is an even number which has at most 4 distinct divisors. If d_7 is a power of 2 then $d_7 \leq 2^3$. On the other hand, $d_7 \geq 3^2 + 1 = 10$ so there are no solutions in this case. Assume now that d_7 is not a power of 2. Due to the upper bound on its number of divisors, we must have $d_7 = 2q$, where q is an odd prime number. In this case $n = 4p^2q$. If $p > 3$, then 1, 2, 4, p, $2p$ and $4p$ are six positive divisors of n and each of them is smaller than d_6, which is impossible. Direct verification shows that $p = 3$ does not yield a solution.

4. $d_7 = p^2$, where p is an odd prime number.

   In this case $d_6 = p^2 - 1 = (p-1)(p+1)$. As p is an odd prime, $p^2 - 1 \equiv 0 \pmod 4$ so $4 \mid n$. If $p \geq 11$ then 1, 2, 4, $\frac{p-1}{2}$, $p-1$ and $p+1$ are six distinct divisors of n, all less than d_6, which is impossible. We directly check each of the remaining values of p ($p = 3$, $p = 5$ and $p = 7$) and find solution $n = 144$ for $p = 3$.

In summary, $n = 144$ and $n = 1984$ are the only solutions.

Problems for self study

Problem 6 *Let n be a positive integer with divisors $1 = d_1 < d_2 < \cdots < d_k = n$. Determine all values of n for which both of the following equalities hold*

$$\begin{aligned} d_5 - d_3 &= 50 \\ 11d_5 + 8d_7 &= 3n. \end{aligned}$$

(Czech-Slovak, 2014)

Problem 7 *Let $1 = d_1 < d_2 < \ldots d_k = n$ be all divisors of a positive integer n. Find all n such that*

$$n = d_4^2 + d_5^2 - 1.$$

(www.artofproblemsolving.com)

Problem 8 *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Given that $n = d_2d_3 + d_2d_5 + d_3d_5$, find all possible values of k.*

(Belarus, 2017)

Problem 9 *Consider the set A of positive integers n such that*

$$n = d_i^4 + d_j^4 + d_k^4 + d_l^4 + d_t^4$$

where $d_i < d_j < d_k < d_l < d_t$ are some positive divisors of n

a) Prove that all elements of A are divisible by 5.

b) Does A contain only finitely many elements?

(Belarus, 2017)

Problem 10 *Find all positive integers n such that*

$$d_1^4 + d_2^4 + \cdots + d_k^4 = n^4 + n^3 + n^2 + n + 1$$

where $1 = d_1 < d_2 < \cdots < d_k = n$ are all divisors of n.

(Switzerland, 2013)

Problem 11 *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n such that $k \geq 6$ and*

$$n = d_5^2 + d_6^2.$$

(Czech-Polish-Slovak Match, 2019)

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PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 28, 2022**.

4691. *Proposed by Michel Bataille.*

Let ABC be a triangle inscribed in a circle Γ and let U_1, U_2, U_3 be distinct points of Γ. Let σ_i be the Simson line of U_i $(i = 1, 2, 3)$ and let V_k be the point of intersection of σ_i and σ_j $(\{i, j, k\} = \{1, 2, 3\})$. Given that $\Delta V_1V_2V_3$ is congruent to $\Delta U_1U_2U_3$, prove that $\Delta V_1V_2V_3$ and $\Delta U_1U_2U_3$ are symmetrical about a point and identify this point.

4692. *Proposed by Todor Zaharinov.*

Let a, b and c be nonzero real numbers such that $a^3 + b^3 + c^3 = 0$. Find the minimum possible value of

$$(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

and determine where the minimum holds.

4693. *Proposed by Michel Bataille.*

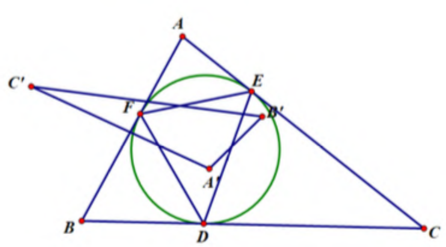
Prove that

$$\sum_{k=1}^{n} \sec^4 \frac{k\pi}{2n+1} = \frac{8n(n+1)(n^2+n+1)}{3}$$

for any positive integer n.

4694. *Proposed by Chen Jiahao.*

In triangle ABC, the inscribed circle touches side BC, CA and AB at D, E and F, respectively. Let A', B' and C' be the reflection of A, B and C in line EF, DF and DE, respectively. Show that the area of triangle DEF equals the area of triangle $A'B'C'$.



4695. *Proposed by George Apostolopoulos.*

Let triangle ABC have sides $BC = a, CA = b$ and $AB = c$ and circumradius R. Equilateral triangles A_1BC, B_1CA and C_1AB are drawn externally to triangle ABC. Let K, L and M be the centroids of the equilateral triangles, respectively. Prove that

$$[ALM] + [BMK] + [CKL] \leq \frac{3\sqrt{3}}{4}R^2,$$

where $[\cdot]$ denotes the area of the corresponding triangle.

4696. *Proposed by Elena Corobea.*

Find the following limit:

$$\lim_{n\to\infty}\int_0^1 \frac{\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^{n-1}}{n-1}\right)^{n+1}}{\left(1+x+\frac{x^2}{2}+\cdots+\frac{x^n}{n}\right)^n}dx.$$

4697. *Proposed by Amit Kumar Basistha.*

Let $f : \mathbb{N} \to \mathbb{N}$ with $f(1) = 1, f(2) = a$ for some $a \in \mathbb{N}$ and, for each positive integer $n \geq 3$, $f(n)$ is the smallest value not assumed at lower integers that is coprime with $f(n-1)$. Prove that f is onto.

4698. *Proposed by Goran Conar.*

Let $x_1, \ldots, x_n > 0$ be real numbers such that $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$\sum_{i=1}^n x_i \ln(1+x_i) < \ln 2\,.$$

4699. *Proposed by Mihaela Berindeanu.*

Let ABC be a non isosceles triangle, D and E be two points outside the triangle and F be the foot of the altitude from A. Show that if

$$\measuredangle(EAC) = \measuredangle(ECA) = \measuredangle(ABC)\,, \quad \measuredangle(DAB) = \measuredangle(ABD) = \measuredangle(BCA)$$

and $BE \cap CD \cap AF = \{X\}$, then $AX = XF$.

4700. *Proposed by Hung Nguyen Viet.*

Let $ABCD$ be a unit square. The points M and N lie on the sides BC and CD respectively such that $\angle MAN = 45°$. Prove that

$$MN + BM \cdot DN = 1.$$

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Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **28 février 2022**.

4691. *Proposé par Michel Bataille.*

Soit ABC un triangle inscrit dans le cercle Γ et soient U_1, U_2, U_3 des points distincts appartenant à Γ. Pour $i = 1, 2, 3$, soit σ_i la droite de Simson de U_i; pour $\{i, j, k\} = \{1, 2, 3\}$, le point d'intersection de σ_i et σ_j est dénoté V_k. Étant donné que $\Delta V_1V_2V_3$ est congru à $\Delta U_1U_2U_3$, démontrer que $\Delta V_1V_2V_3$ est symétrique à $\Delta U_1U_2U_3$ par rapport à un certain point; aussi, identifier ce point.

4692. *Proposé par Todor Zaharinov.*

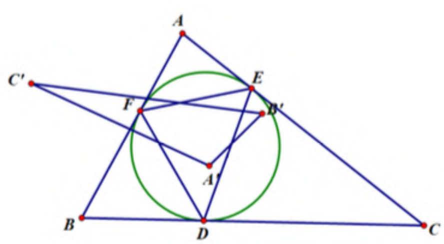
Soient a, b et c des nombres réels non nuls tels que $a^3 + b^3 + c^3 = 0$. Déterminer la valeur minimale de
$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$
et identifier les valeurs de a, b et c produisant ce minimum.

4693. *Proposé par Michel Bataille.*

Démontrer que
$$\sum_{k=1}^{n} \sec^4 \frac{k\pi}{2n+1} = \frac{8n(n+1)(n^2+n+1)}{3}$$
pour tout entier positif n.

4694. *Proposé par Chen Jiahao.*

Pour le triangle ABC, le cercle inscrit touche les côtés BC, CA et AB en D, E et F, respectivement. Soient A', B' et C' les réflexions de A, B et C par rapport aux lignes EF, DF et DE, respectivement. Démontrer que les surfaces de $\triangle DEF$ et $\triangle A'B'C'$ sont égales.



4695. *Proposé par George Apostolopoulos.*

Le triangle ABC a des côtés de longueurs données par $BC = a, CA = b$ et $AB = c$; le rayon du cercle circonscrit est dénoté R. Des triangles équilatéraux A_1BC, B_1CA et C_1AB sont tracés à l'extérieur du triangle ABC. Enfin, soient K, L et M les centrodes de ces triangles équilatéraux, respectivement. Démontrer que

$$[ALM] + [BMK] + [CKL] \leq \frac{3\sqrt{3}}{4}R^2,$$

où $[\cdot]$ dénote la surface du triangle en question.

4696. *Proposé par Elena Corobea.*

Déterminer la limite suivante:

$$\lim_{n\to\infty} \int_0^1 \frac{\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^{n-1}}{n-1}\right)^{n+1}}{\left(1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}\right)^n} dx.$$

4697. *Proposé par Amit Kumar Basistha.*

Soit $f : \mathbb{N} \to \mathbb{N}$ une fonction telle que $f(1) = 1$ et $f(2) = a$ pour un certain $a \in \mathbb{N}$; pour $n \geq 3$, $f(n)$ est défini comme étant le plus petit entier positif copremier avec $f(n-1)$, puis distinct de $f(1), f(2), \ldots, f(n-1)$. Démontrer que f est surjective.

4698. *Proposé par Goran Conar.*

Soient $x_1, \ldots, x_n > 0$ des nombres réels tels que $x_1 + x_2 + \cdots + x_n = 1$. Démontrer que

$$\sum_{i=1}^{n} x_i \ln(1 + x_i) < \ln 2\,.$$

4699. *Proposé par Mihaela Berindeanu.*

Soit ABC un triangle non isocèle et soient D et E deux points à l'extérieur du triangle; aussi, soit F le pied de l'altitude émanant de A. Démontrer que si

$$\measuredangle(EAC) = \measuredangle(ECA) = \measuredangle(ABC)\,, \quad \measuredangle(DAB) = \measuredangle(ABD) = \measuredangle(BCA)$$

et $BE \cap CD \cap AF = \{X\}$, alors $AX = XF$.

4700. *Proposé par Hung Nguyen Viet.*

Soit $ABCD$ un carré de côté 1. Les points M et N se trouvent sur les côtés BC et CD, respectivement, de façon à ce que $\angle MAN = 45°$. Démontrer que

$$MN + BM \cdot DN = 1.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2021: 47(5), p. 255–258.

4641. *Proposed by Al Şeymanur.*

Let K, L, and M be the midpoints of the sides BC, CA, and AB, respectively, of an acute triangle ABC. Denote by

A', L', M''	the reflections of A, L, M in the line BC,
B', M', K''	the reflections of B, M, K in the line CA,
C', K', L''	the reflections of C, K, L in the line AB.

Using square brackets to denote areas, we set $T = [ABC]$, $T' = [A'B'C']$, and $H = [K'M''L'K''M'L'']$. Prove that

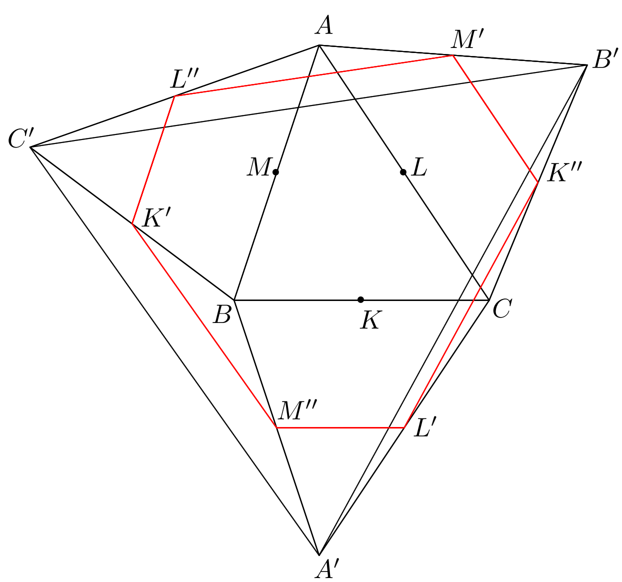
$$4H - T' = 9T, \quad T' \leq 4T \quad \text{and} \quad H \leq \frac{13}{4}T.$$

We received 7 correct solutions. We present 3 solutions.

Solution 1, by Marie-Nicole Gras.

Let $a = BC$, $b = CA$, $c = AB$ be the sides of $\triangle ABC$; the labels A, B, C denote the angles $\angle BAC$, $\angle CBA$ and $\angle ACB$, respectively.

Points M'', L', K'', M', L'' and K' are the middle of $A'B, A'C, B'C, B'A, C'A$ and $C'B$, respectively. The area of polygon $A'CB'AC'B$ is equal to $4T$.



Since $\triangle ABC$ is acute, we have $3A < \frac{3\pi}{2}$; we put $\varepsilon_1 = 1$ if $3A < \pi$, that is to say the vertex A is outside $\triangle A'B'C'$, and $\varepsilon_1 = -1$, otherwise; we define ε_2 and ε_3, analogously.

Let $S = \varepsilon_1\big[AB'C'\big] + \varepsilon_2\big[BC'A'\big] + \varepsilon_3\big[CA'B'\big]$; then

$$S = \frac{1}{2}bc\sin(3A) + \frac{1}{2}ca\sin(3B) + \frac{1}{2}ab\sin(3C).$$

We deduce that

$$T' = 4T - S. \tag{1}$$

Now, we compute H and note that

$$\big[A'M''L'\big] = \big[B'K''M'\big] = \big[C'L''K'\big] = \frac{1}{4}T,$$
$$\big[AM'L''\big] + \big[BK'M''\big] + \big[CL'K''\big] = \frac{1}{4}S;$$

We deduce that

$$H = 4T - \frac{3}{4}T - \frac{1}{4}S = \frac{13}{4}T - \frac{1}{4}S; \tag{2}$$

It follows

$$4H = 13T - (4T - T') = 9T + T'. \tag{3}$$

To prove the inequalities, it remains, using (1) and (2), to show that $S \geq 0$. We have, (R is the circumradius):

$$\begin{aligned} 2S &= ab\sin C(1 - 4\sin^2 C) + bc\sin A(1 - 4\sin^2 A) + ac\sin B(1 - 4\sin^2 B) \\ &= \frac{abc}{2R}\Big(3 - 4(\sin^2 A + \sin^2 B + \sin^2 C)\Big); \end{aligned}$$

Since it is known [for example, O. Bottema et al, *Geometric Inequalities* (1968), item 2.3] that $\sin^2 A + \sin^2 B + \sin^2 C \leq \frac{3}{4}$, we deduce $S \geq 0$, whence the proof.

Solution 2, by Theo Koupelis.

Let a, b, c be the side lengths of triangle ABC and R its circumradius. Clearly

$$[AML] = [MBK] = [KCL] = [KLM] = \frac{1}{4}T.$$

Triangles $AL''M$, $ML''K'$, $MK'B$ are the reflections of triangles ALM, MLK, and MKB, respectively, about the line AB, and thus $[AL''K'B] = \frac{3}{4}T$, with

$$\angle MAL'' = \angle A \quad \text{and} \quad \angle MBK' = \angle B.$$

Similarly, $[BM''L'C] = \frac{3}{4}T$, with

$$\angle KBM'' = \angle B \quad \text{and} \quad \angle KCL' = \angle C,$$

and $[CK''M'A] = \frac{3}{4}T$, with $\angle LCK'' = \angle C$ and $\angle LAM' = \angle A$. Finally, we have

$$[M'AL''] = \frac{1}{2} \cdot AM' \cdot AL'' \cdot \sin(\angle M'AL'') = \frac{bc}{8} \cdot \sin(\angle M'AL'').$$

We note that if $180^\circ \le 3\angle A < 270^\circ$, then $\angle M'AL'' = 360^\circ - 3\angle A < 180^\circ$, and triangle $M'AL''$ is inside the polygon $K'M''L'K''M'L''$; otherwise, if $3\angle A < 180^\circ$, then triangle $M'AL''$ is outside the polygon. Similarly for triangles $K'BM''$ and $L'CK''$. Therefore,

$$H = T + \frac{9}{4}T - \frac{1}{8}\left[ab \cdot \sin(3\angle C) + ac \cdot \sin(3\angle B) + bc \cdot \sin(3\angle A)\right].$$

Using the law of sines $\dfrac{\sin \angle A}{a} = \dfrac{\sin \angle B}{b} = \dfrac{\sin \angle C}{c} = \dfrac{1}{2R}$ and the identity $\sin(3x) = 3\sin x - 4\sin^3 x$, we rewrite the above expression as

$$H = \frac{13}{4}T - \frac{abc}{16R^3}\left[9R^2 - (a^2 + b^2 + c^2)\right]. \tag{4}$$

Similarly, triangles $AC'B$, $BA'C$, and $CB'A$ are reflections of triangle ABC about its corresponding sides. Therefore $[AC'B] = [BA'C] = [CB'A] = T$. Also, $[B'AC'] = \dfrac{1}{2} \cdot AB' \cdot AC' \sin \angle B'AC' = \dfrac{bc}{2} \cdot \sin \angle M'AL''$, because the points A, M', B' are collinear, and so are the points A, L'', C'. Similarly for triangles $C'BA'$ and $A'CB'$. Therefore, as above, we get

$$T' = T + 3T - \frac{1}{2}\left[ab \cdot \sin(3\angle C) + ac \cdot \sin(3\angle B) + bc \cdot \sin(3\angle A)\right],$$

or

$$T' = 4T - \frac{abc}{16R^3}\left[9R^2 - (a^2 + b^2 + c^2)\right]. \tag{5}$$

From (4) and (5) we see that $4H - T' = 9T$. Also, it is well-known that for an acute triangle we have $a^2 + b^2 + c^2 \le 9R^2$, with equality if the triangle is equilateral. Therefore we also have $T' \le 4T$ and $H \le \frac{13}{4}T$.

Solution 3, by Sorin Rubinescu.

Because $\triangle ABC$ is acute angled, one of the triangles' $AB'C', BC'A', CB'A'$ interior and ABC's interior are not disjoint. Let this triangle be $CB'A'$.

The areas of the congruent triangles ABC, ABC', ACB', BCA are equal to T and the area of the triangle $A'B'C$ is equal to:

$$\begin{aligned} T' &= 4T + [BC'A'] + [AB'C'] - [CB'A'] \\ &= 4T + \frac{bc \cdot \sin(2\pi - 3A)}{2} + \frac{ac \cdot \sin(2\pi - 3B)}{2} - \frac{ab \cdot \sin 3C}{2} \\ &= 4T - \frac{1}{2}(ab \cdot \sin 3C + ac \cdot \sin + bc \cdot \sin 3A) \le 4T, \end{aligned}$$

where a, b, c are the lengths of the sides of the triangle ABC.

Thus,

$$T' \leq 4T. \tag{6}$$

We have

$$\begin{aligned} H &= [K'L"AB] + [K'BM'] + [BM'L'C] + [L"AM'] + [ACK'M'] - [CK"L'] + [ABC] \\ &= \frac{3}{4}T + \frac{1}{4}[BA'C'] + \frac{3}{4}T + \frac{1}{4}[AB'C'] + \frac{3}{4}T - \frac{1}{4}[CA'B'] + T, \end{aligned}$$

which gives us:

$$4H = 9T + (4T + [BC'A'] + [AB'C'] - [CA'B']) = 9T + T'.$$

Hence,

$$4H - T' = 9T. \tag{7}$$

By (6) and (7) it follows that $4H \leq 4T + 9T = 13 \cdot T$, so $H \leq \frac{13}{4}T$.

4642. *Proposed by Adam L. Bruce.*

Let $A \in \mathbb{R}^{n\times n}$ be a positive definite matrix and let $x \in \mathbb{R}^n$. Show that

$$(x^TA^2x)^3 \leq (x^TAx)(x^TA^2x)(x^TA^3x).$$

Six correct solutions were received from 5 respondents.

Solution 1, by Lucas Vantaggio, and the proposer (independently).

Define the inner product $\langle x, y\rangle = x^TAy$ with associated norm $\|x\| = \sqrt{\langle x, x\rangle}$. By the Cauchy-Schwarz inequality, $\langle x, y\rangle^2 \leq \|x\|^2\|y\|^2$. Setting $y = Ax$ and using $A^T = A$ yields

$$(x^TA^2x)^2 \leq (x^TAx)((Ax)^TA(Ax)) = (x^TAx)(x^TA^3x),$$

from which the desired result follows.

Solution 2, by Brian Bradie, and the UCLan Cyprus Problem Solving Group (independently).

A has n positive real eigenvectors λ_i with a corresponding orthonormal basis of eigenvectors v_i $(1 \leq i \leq n)$. We may suppose that $x \neq 0$ and that $x = \sum_{i=1}^n \mu_i v_i$. Then, for $k = 1, 2, 3$,

$$x^TA^kx = \sum_{i=1}^n \mu_i^2\lambda_i^k.$$

Then

$$(x^TA^2x)^2 = (\sum_{i=1}^n \mu_i^2\lambda_i^2)^2 \leq (\sum_{i=1}^n \mu_i^2\lambda_i)(\sum_{i=1}^n \mu_i^2\lambda_i^3) = (x^TAx)(x^TA^3x),$$

by an application of the Cauchy-Schwarz inequality to the vectors $(\mu_i \lambda_i^{1/2})$ and $(\mu_i \lambda_i^{3/2})$.

Editor's comment: Two solvers had an approach related to Solution 2 that began by representing A in the form PDP^T where D is a positive nonsingular diagonal matrix and P is an orthogonal transformation. Then it is sufficient to establish the result for D.

The UCLan Cyprus Problem Solving Group pointed out that some definitions of positive definite do not require the matrix to be symmetric. In this case, the 2×2 matrix $A = (1, 1; 0, 1)$ and vector $(1, 1)^T$ provide a counterexample, since $x^T Ax = 3$, $x^T A^2 x = 4$ and $x^T A^3 x = 5$. (Observe that $x^T Ax = x^2 + xy + y^2$.)

4643. *Proposed by Nguyen Viet Hung.*

Find all pairs (m, n) of positive integers such that $\gcd(m, n) = 1$ and integer $(2^m - 1)(2^n - 1)$ is a perfect square.

We received 17 *correct solutions. We present the one by the UCLan Cyprus Problem Solving Group.*

Suppose p is a prime such that $p|2^m - 1$ and $p|2^n - 1$.

Let k be the order of 2 modulo p. Then $k|m$ and $k|n$ and therefore $k = 1$, which is impossible since $2^1 \not\equiv 1 \pmod{p}$.

Thus $2^m - 1$ and $2^n - 1$ are relatively prime and therefore must both be perfect squares. For any natural number $r \geq 2$ we have that $2^r - 1$ cannot be a perfect square since $2^r - 1 \equiv 3 \pmod 4$ and 3 is not a quadratic residue modulo 4. Therefore we must have $m = n = 1$ for which $(2^m - 1)(2^n - 1) = 1$ is indeed a perfect square.

4644. *Proposed by Mihaela Berindeanu, modified by the Editorial Board.*

Let $z_1, z_2, z_3 \in \mathbb{C}$ be different numbers, with $|z_1| = |z_2| = |z_3| = 1$. Show that

$$|2z_1 - z_2 - z_3| + |z_3 - z_2| \geq \frac{1}{\sqrt{2}} \left(|z_2 - z_3||z_1 - z_3| + |z_2 - z_1||z_2 - z_3| \right).$$

There were 6 correct solutions received. There were two other submissions. One did not provide a solution but speculated as to whether the problem was known. The second provided what might have been a valid solution, but there were a number of complex steps involved that were not transparent.

Solution 1, by Ben Ajiba Mohamed Amine.

Let A, B, C be the respective positions of z_1, z_2, z_3 in the complex plane and M, corresponding to $\frac{1}{2}(z_2 + z_3)$, be the midpoint of BC. Let a, b, c, m be the respective lengths of BC, CA, AB and the median AM. The desired inequality can be written as

$$\sqrt{2}(2m+a) \geq a(b+c).$$

Suppose that AM meets the circle again at D and that d is the length of AD. Since $AM \cdot MD = BM \cdot MC$ and $4m^2 = 2(b^2+c^2)-a^2$, it follows that $m(d-m) = a^2/4$, whence

$$md = m^2 + \frac{a^2}{4} = \frac{b^2+c^2}{2}.$$

Since $d \leq 2$, the diameter of the circle,

$$m \geq \frac{1}{4}(b^2+c^2) \geq \frac{1}{8}(b+c)^2.$$

Therefore

$$\begin{aligned}(2m+a) - \frac{1}{\sqrt{2}}a(b+c) &\geq \frac{(b+c)^2}{4} - \frac{1}{\sqrt{2}}a(b+c) + a \\ &= \left(\frac{b+c}{2} - \frac{a}{\sqrt{2}}\right)^2 - \frac{a^2}{2} + a \\ &= \left(\frac{b+c}{2} - \frac{a}{\sqrt{2}}\right)^2 + \frac{1}{2}a(2-a) \\ &\geq 0,\end{aligned}$$

as desired. Equality occurs if and only if $b = c$ and $a = 2$.

Solution 2, by C.R. Pranesachar.

We follow the notation and formulation of the previous solution. Let h be the length of the altitude from A to BC. Let E complete the parallelogram $AMCE$ and apply Apollonius' theorem, $2(AM^2 + MC^2) = AC^2 + ME^2$, to obtain

$$2\left(m^2 + \frac{a^2}{4}\right) = b^2 + c^2.$$

Using the fact that $a = 2\sin A$, we see that the difference between the squares of the two sides of the desired inequality is

$$\begin{aligned}2(2m+a)^2 - (b+c)^2a^2 &= 8m^2 + 8am + 2a^2 - 4(b+c)^2\sin^2 A \\ &= 4(b^2+c^2)(1-\sin^2 A) + 8am - 8bc\sin^2 A \\ &= 4(b^2+c^2)(1-\sin^2 A) + 8am - 16[ABC]\sin A \\ &= 4(b^2+c^2)(1-\sin^2 A) + 8am - 8ah\sin A \\ &= 4(b^2+c^2)(1-\sin^2 A) + 8a(m - h\sin A) \geq 0,\end{aligned}$$

which establishes the inequality. Equality occurs if and only if $\sin A = 1$ and $m = h$, *i.e.*, when triangle ABC is isosceles and right-angled at A.

4645. *Proposed by Leonard Giugiuc and Bogdan Suceava.*

Let a, b, c be positive real numbers such that $a+b+c+d = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$. Prove that

$$(a+b+c+d)^2 + 48abcd \geq 64.$$

We received 6 submissions, 5 of which are correct. We present the solution by Ioan Viorel Codreanu.

Let $S_1 = \frac{1}{4}\sum a$, $S_2 = \sqrt{\frac{1}{6}\sum ab}$, $S_3 = \sqrt[3]{\frac{1}{4}\sum abc}$, and $S_4 = \sqrt[4]{abcd}$ denote the elementary symmetric means of a, b, c, d.

Note first that $\frac{1}{4}\left(\sum a\right)\left(\prod a\right) = \frac{1}{4}\sum abc$ so

$$S_1 S_4^4 = S_3^3. \tag{1}$$

Hence the given inequality becomes $(4S_1)^2 + 48S_4^4 \geq 64$ or

$$S_1^2 + 3S_4^2 \geq 4. \tag{2}$$

By Newton's Inequality, we have $S_2^2 \geq S_1 S_3$ and

$$S_3^2 \geq S_2 S_4 \tag{3}$$

which implies

$$S_2 S_3 \geq S_1 S_4 \tag{4}$$

From (1) and (4), we then get $S_3^3 = S_1 {S_4}^4 \leq (S_2 S_3){S_4}^3$ or ${S_3}^2 \leq S_2 {S_4}^3$.

Hence from (3) we have $S_2 S_4 \leq S_2 {S_4}^3$ so $S_4 \geq 1$. Then by the AM-GM Inequality we get $S_1 \geq S_4 \geq 1$. Thus, ${S_1}^2 + 3{S_4}^4 \geq 1 + 3 = 4$, establishing (2) and our proof is complete.

Editor's comments: Besides the proposers, two other solvers showed that $(a+b+c+d)^2 + 48abcd = 64$ if and only if $(a, b, c, d) = (1, 1, 1, 1)$ or $(\sqrt{10}, \frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{5})$ and its cyclic permutations.

4646. *Proposed by George Apostolopoulos.*

Let ABC be an acute triangle with inradius r and circumradius R. Prove that

$$\cot A + \cot B + \cot C \leq \sqrt{3}\left(\frac{R}{2r}\right)^2.$$

We received 42 solutions, all of which were correct. This included 19 solutions by Mehra Vivek. We present the solution by Sorin Rubinescu.

We will use the following well-known relations:

$$\cot A + \cot B + \cot C = \frac{a^2+b^2+c^2}{4F}, \text{ where } F \text{ is the area of } ABC; \tag{1}$$

$$a^2 + b^2 + c^2 \le 9R^2; \tag{2}$$

$$s \ge 3r\sqrt{3} \text{ (Mitrinovic)}. \tag{3}$$

We have

$$\cot A + \cot B + \cot C \overset{(1)}{=} \frac{a^2+b^2+c^2}{4F} \overset{(2)}{\le} \frac{9R^2}{4F} = \frac{9R^2}{4rs} \overset{(3)}{\le} \frac{9R^2}{4r \cdot 3r\sqrt{3}} = \sqrt{3}\left(\frac{R}{2r}\right)^2.$$

Editor's comment. As a number of solvers pointed out, the requirement that ABC be acute is unnecessary.

4647. *Proposed by Michel Bataille.*

In the plane, two circles Γ and γ intersect at A and B. Let M (resp. N) be a point of the arc of γ exterior (resp. interior) to Γ. If O is the centre of Γ, prove that

$$OM^2 - ON^2 = k(MA \cdot MB + NA \cdot NB)$$

for some real number k independent of the chosen points M and N.

All but one of the seven submissions were correct and complete. We feature a composite of the independent solutions from Marie-Nicole Gras, from Theo Koupelis, and from the UCLan Cyprus Problem Solving Group.

Let O_γ and r be the center and radius of γ, and denote by t the distance OO_γ between the centers of the two circles. We shall show that $k = \frac{t}{r}$, whence k is indeed independent of M and N.

In a Cartesian coordinate system we take $O_\gamma = (0,0)$, so that circle Γ has center $O = (-t, 0)$. Note that OO_γ is the perpendicular bisector of AB, so that for some $\alpha \in (0, \pi)$, $A = (r\cos\alpha, r\sin\alpha)$ and $B = (r\cos\alpha, -r\sin\alpha)$. Then we have $M = (r\cos\vartheta, r\sin\vartheta)$ for some $\vartheta \in (-\alpha, \alpha)$, and $N = (r\cos\varphi, r\sin\varphi)$ for some $\varphi \in (\alpha, 2\pi - \alpha)$.

By the Law of Cosines (which the French evidently call Alkashi's formula),

$$OM^2 = t^2 + r^2 + 2tr\cos\vartheta \quad \text{and} \quad ON^2 = t^2 + r^2 + 2tr\cos\varphi,$$

so that

$$OM^2 - ON^2 = 2tr(\cos\vartheta - \cos\varphi). \tag{1}$$

Since $0 < \vartheta < \alpha < \pi$, in the circle γ the chords MA and MB have lengths

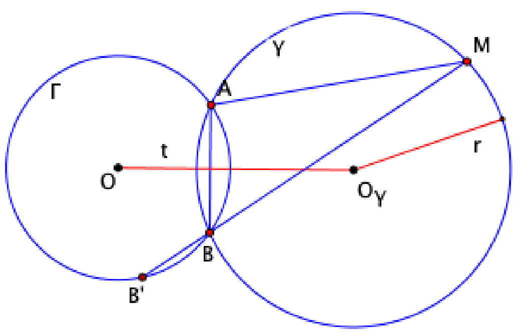
$$MA = 2r\sin\left(\frac{\alpha - \vartheta}{2}\right) \quad \text{and} \quad MB = 2r\sin\left(\frac{\alpha+\vartheta}{2}\right).$$

Thus

$$(MA)(MB) = 4r^2 \sin\left(\frac{\alpha-\vartheta}{2}\right)\sin\left(\frac{\alpha+\vartheta}{2}\right) = 2r^2(\cos\vartheta - \cos\alpha).$$

Similarly we obtain
$$(NA)(NB) = 2r^2(\cos\alpha - \cos\varphi).$$
Thus
$$(MA)(MB) + (NA)(NB) = 2r^2(\cos\vartheta - \cos\varphi). \tag{2}$$
By dividing both sides of the equation in (1) by the equation in (2) we conclude that
$$k = \frac{OM^2 - ON^2}{(MA)(MB) + (NA)(NB)} = \frac{t}{r},$$
as claimed.

Editor's comments. Continuing with the above notation, let us define B' to be the point where the line MB again meets Γ. Did you know that the ratio $\frac{MB'}{MA}$ is independent of the position of M on the circle γ? Indeed, much to this editor's surprise,
$$\frac{MB'}{MA} = \frac{t}{r}.$$
The person who submitted the incomplete treatment of the solution provided a neat proof that that the desired quantity k (from the statement of our problem) equals the quotient $\frac{MB}{MA}$, but they failed to prove that the ratio is constant.



4648. *Proposed by Corneliu Manescu-Avram.*

Let a be a positive integer and let $p > 3$ be a prime number such that $a^2+a+1 \equiv 0 \pmod{p}$. Prove that
$$(a+1)^p \equiv a^p + 1 \pmod{p^3}.$$

We received 10 submissions, all of which are correct. We present the solution by Michel Bataille.

We first show that $p \equiv 1 \pmod 3$. Since
$$a^2 + a + 1 \equiv 0 \pmod{p},$$
we have
$$a^3 - 1 = (a-1)(a^2+a+1) \equiv 0 \pmod{p},$$

so $a^3 \equiv 1 \pmod{p}$. It follows that the order of a modulo p is 1 or 3.

Since $a^2 + a + 1 \equiv 0 \pmod{p}$ and $p > 3$, we have $a \not\equiv 1$ so it must be 3.

Since $a^{p-1} \equiv 1 \pmod{p}$ (by Fermat's Little Theorem), $p - 1$ is a multiple of 3. Thus $p \equiv 1 \pmod{3}$.

Then what we have now is exactly the ***Crux*** problem proposal #3704 in the January 2012 issue (p. 24) and a solution to which appeared in the January 2013 issue (p. 42). (*Ed*: one of the 2 solvers there was the current solver here.)

4649. *Proposed by Mihaela Berindeanu.*

For $x, y, z \in \mathbb{R}$, show that

$$\frac{2^{10x}}{2^y + 2^z} + \frac{2^{10y}}{2^x + 2^z} + \frac{2^{10z}}{2^x + 2^y} \geq 2^{6x+2y+z-1} + 2^{6y+2z+x-1} + 2^{6z+2x+y-1}.$$

We received 12 submissions of which 11 were correct and complete. We present the solution by Oliver Geupel.

Put $a = 2^x$, $b = 2^y$, and $c = 2^z$. Then, a, b, and c are positive numbers. We have to show that

$$\frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b} \geq \frac{1}{2}\left(a^6b^2c + b^6c^2a + c^6a^2b\right). \tag{1}$$

By symmetry of the left hand side of (1) , there is no loss of generality in assuming that $a \geq b \geq c$. As a consequence,

$$\frac{a}{b+c} \geq \frac{b}{c+a} \geq \frac{c}{a+b}; \qquad a^9 \geq b^9 \geq c^9.$$

Applying Chebyshev's inequality and Nesbitt's inequality in succession, we obtain

$$\begin{aligned} \frac{a^{10}}{b+c} + \frac{b^{10}}{c+a} + \frac{c^{10}}{a+b} &\geq \frac{1}{3}\left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right)\left(a^9 + b^9 + c^9\right) \\ &\geq \frac{1}{3} \cdot \frac{3}{2}\left(a^9 + b^9 + c^9\right) \\ &= \frac{1}{2}\left(a^9 + b^9 + c^9\right). \end{aligned}$$

By the arithmetic vs. geometric mean inequality, we conclude

$$\begin{aligned} a^9 + b^9 + c^9 &= \frac{6a^9 + 2b^9 + c^9}{9} + \frac{6b^9 + 2c^9 + a^9}{9} + \frac{6c^9 + 2a^9 + b^9}{9} \\ &\geq a^6b^2c + b^6c^2a + c^6a^2b. \end{aligned}$$

Hence the result (1).

4650. *Proposed by Roberto F. Stöckli.*

Let $I_n = ((n-1)^2, n^2]$. Define $f(n) = 1$ if I_n contains exactly one triangular number (recall that the nth triangular number is $t_n = n(n+1)/2$) and $f(n) = 0$ otherwise. Find the value of

$$\lim_{n\to\infty} \frac{f(1)+f(2)+\cdots+f(n)}{n}$$

We received 8 submissions and 7 of them were complete and correct. We present the solution by the majority of solvers.

Let c_n be the number of triangular numbers in $(0, n^2]$; it is easy to show that

$$\lim_{n\to\infty} c_n/n = \sqrt{2}.$$

Note that the k-th triangular number $t_k = k(k+1)/2$ belongs to $I_n = \big((n-1)^2, n^2\big]$ if and only if

$$\begin{aligned} & (n-1)^2 < \frac{k(k+1)}{2} \le n^2 \\ \iff & 8(n-1)^2 + 1 < (2k+1)^2 \le 8n^2 + 1 \\ \iff & 2k+1 \in J_n := \left(\sqrt{8(n-1)^2+1}, \sqrt{8n^2+1}\,\right]. \end{aligned}$$

Let

$$L_n = \sqrt{8n^2+1} - \sqrt{8(n-1)^2+1}$$

be the length of J_n. It is easy to verify that $2 \le L_n < 4$ directly; alternatively, one can establish so by showing that L_n is increasing in n, and observing that as $n \to \infty$,

$$L_n = \frac{16n-8}{\sqrt{8n^2+1} + \sqrt{8(n-1)^2+1}} \to \frac{16}{4\sqrt{2}} = 2\sqrt{2}.$$

It follows that, there exists exactly one or two odd integers in J_n, and consequently one or two triangular numbers in I_n.

Let a_n be the number of I_k containing exactly one triangular number with $1 \le k \le n$ and b_n the number of I_k containing two triangular numbers with $1 \le k \le n$. Then we have

$$\begin{aligned} a_n + b_n &= n \\ a_n + 2b_n &= c_n \end{aligned}$$

Solving this system gives $a_n = 2n - c_n$. Therefore,

$$\lim_{n\to\infty} \frac{f(1)+f(2)+\cdots+f(n)}{n} = \lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \frac{2n - c_n}{n} = 2 - \sqrt{2}.$$

4409. *Proposed by Cristian Chiser. Correction.*

Let A and B be two matrices in $M_2(\mathbb{R})$ such that $A^2 = O_2$ and B is invertible. Prove that the polynomial $P = \det(xB^2 - AB + BA)$ has all real roots.

*This problem was originally published in **Crux** 45(1) with a typo: the conclusion asked for integer roots instead of real roots, as was stated in the original proposal. We apologize for the typo and publish the proposer's intended original solution here, slightly modified by the editor.*

Let $\mathrm{Tr}(X)$ denote the trace of the matrix X. Then the polynomial $p(x)$ above can be written as

$$p(x) = x^2 \det(B^2) + mx + \det(BA - AB),$$

where

$$m = \mathrm{Tr}(BA - AB)\mathrm{Tr}(B^2) - \mathrm{Tr}[(BA - AB)B^2].$$

Using basic properties of the trace function, we have

$$m = -\mathrm{Tr}(BAB^2) + \mathrm{Tr}(AB^3) = -\mathrm{Tr}(AB^3) + \mathrm{Tr}(AB^3) = 0.$$

Furthermore, if we set $t(x) = \det(BA + xAB)$, then

$$t(x) = x^2 \det(AB) + [\mathrm{Tr}(AB)\mathrm{Tr}(BA) - \mathrm{Tr}(BAAB)]x + \det(BA).$$

Since $A^2 = 0$, A is singular, and it follows that $t(x) = x(\mathrm{Tr}(AB))^2$. Thus setting $x = -1$, we deduce that $\det(BA - AB) = -(\mathrm{Tr}(AB))^2$

Hence $p(x) = x^2 \det(B^2) - (\mathrm{Tr}(AB))^2$ and the roots of $p(x)$ are $\pm\mathrm{Tr}(AB)/\det(B)$.