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IN THIS ISSUE / DANS CE NUMÉRO

EDITORIAL

2020 has given us a lot to talk about, to think about, to reconsider, redo and relearn. Hopefully, we have come out better on the other side, but only the future will tell.

Year 2020 marked the 46th Volume of ***Crux***. Thanks to the generous support of our sponsors, ***Crux*** continues to flourish as an open access journal, gaining a wider audience that includes high school students, teachers and numerous other avid problem solvers of all ages and from all around the globe. The impact of the journal has grown dramatically since it became freely available online last year. Over the 12 month period from October 2019 to October 2020, the website has been visited by over 11,500 unique users accessing the website over 40,000 times. The number of submissions has also grown drastically to the point where we had to increase the size of the Editorial Board to moderate the incoming volume of solutions in a timely fashion. We now routinely receive around 200 submissions per issue and growing: what a great problem to have!

This year was marked by several losses to the mathematical community. In March 2020, we lost legendary Richard K. Guy whose life and work influenced so many of us. We dedicated issue 8 of this Volume to the memory of Richard Guy and received an unprecedented number of submissions. As a result, the memorial issue was the largest ***Crux*** issue to date with 103 pages to commemorate 103 years of Richard's life. The issue was used in the University of Calgary's events in honour of Guy held October 1–4, 2020: `https://science.ucalgary.ca/mathematics-statistics/about/richard-guy`

With all of our sections going strong, we are looking forward to 2021. The new year will start with a new cover for ***Crux*** and a new regular column *Exploring Indigenous Mathematics*.

Stay healthy.

Kseniya Garaschuk

MathemAttic

No. 20

The problems featured in this section are intended for students at the secondary school level.

Click here to submit solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2021**.

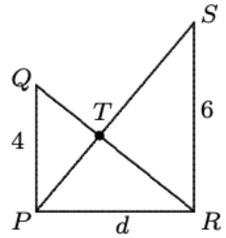
MA96.

a) A circle passes through points with coordinates $(0,1)$ and $(0,9)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle.

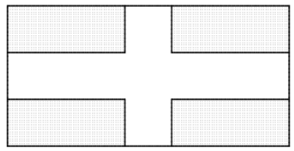
b) Let a and b be any real numbers of the same sign (either both positive or both negative). A circle passes through points with coordinates $(0,a)$ and $(0,b)$ and is tangent to the positive part of the x-axis. Find the radius and coordinates of the centre of the circle in terms of a and b.

MA97. In London there are two notorious burglars, A and B, who steal famous paintings. They hide their stolen paintings in secret warehouses at different ends of the city. Eventually all the art galleries are shut down, so they start stealing from each other's collection. Initially A has 16 more paintings than B. Every week, A steals a quarter of B's paintings, and B steals a quarter of A's paintings. After 3 weeks, Sherlock Holmes catches both thieves. Which thief has more paintings by this point, and by how much?

MA98. A pair of telephone poles d metres apart is supported by two cables which run from the top of each pole to the bottom of the other. The poles are 4 m and 6 m tall. Determine the height above the ground of the point T, where the two cables intersect. What happens to this height as d increases?



MA99. A flag consists of a white cross on a red field. The white stripes, both vertical and horizontal, are of the same width. The flag measures 48cm by 24cm. If the area of the white cross equals the area of the red field, what is the width of the cross?



MA100. Suppose the equation $x^3 + 3x^2 - x - 1 = 0$ has real roots a, b, c. Find the value of $a^2 + b^2 + c^2$.

. . . .

Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

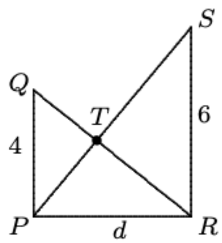
MA96.

a) Un certain cercle passe par $(0, 1)$ et $(0, 9)$ et est tangent à l'axe des x dans sa partie positive. Déterminer le rayon du cercle et les coordonnées de son centre.

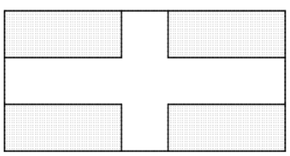
b) Soient a et b deux nombres réels de même signe, les deux étant positifs ou les deux étant négatifs. Un certain cercle passe par $(0, a)$ et $(0, b)$ et est tangent à l'axe des x dans sa partie positive. Déterminer le rayon du cercle et les coordonnées de son centre, en termes de a et b.

MA97. Deux cambrioleurs, A et B, ont la spécialité de voler des œuvres d'art et de les cacher, chacun dans son propre entrepôt. Éventuellement, toutes les galleries d'art ont été vidées et sont donc fermées. Les deux cabrioleurs commencent alors à voler l'un de l'autre. À ce moment, A possède 16 œuvres d'art de plus que B. Par la suite, chaque semaine, A vole le quart des œuvres d'art de B et B vole le quart de celles de A. Après 3 semaines, on attrappe les deux cambrioleurs. Lequel cambrioleur a alors le plus d'œuvres d'art, et par combien ?

MA98. Deux poteaux de téléphone, de hauteurs 4 mètres et 6 mètres et à d mètres de distance, sont stabilisés par deux cables, allant du haut de chaque poteau jusqu'à la base de l'autre. Déterminer la distance au sol du point T où les cables se rencontrent. Qu'arrive-t-il à cette hauteur lorsque d augmente ?



MA99. Un drapeau de taille 48 cm par 24 cm consiste d'une croix blanche sur un fond rouge, les rayures blanches, l'une horizontale et l'autre verticale, étant de même largeur. Si les surfaces rouge et blanche sont égales, déterminer la largeur des rayures de la croix.

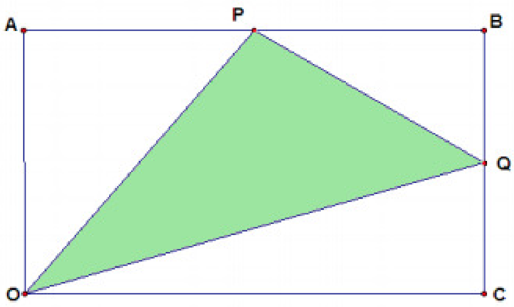


MA100. Supposer que l'équation $x^3 + 3x^2 - x - 1 = 0$ a les racine réelles a, b, c. Déterminer la valeur de $a^2 + b^2 + c^2$.

MATHEMATTIC SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(5), p. 199–210.

MA71. You are given a rectangle $OABC$ from which you remove three right-angled triangles, leaving a fourth triangle OPQ as shaded in the diagram below.



How must you position the points P and Q so that the area of each of the three removed triangles is the same? In other words, what are the ratios $PB : PA$ and $QB : QC$?

Originally Problem 2, Vermont State Mathematics Coalition Talent Search, 2009.

We received 12 submissions, all correct. We present the solution by T. Reji and B. Sneha.

Let $OA = a$ and $OC = b$ be the sides of the given rectangle $OABC$. Let P and Q be as given in the figure. Denote the length PB by x and the length BQ by y. Then length $AP = b - x$ and length $QC = a - y$.

We need to find the ratio $PB : PA$ and $QB : QC$ subject to the condition that the area of the three triangles except OPQ are the same. That is, we need the area of OAP to equal the area of PQB to equal the area of OCQ:

$$\frac{a}{2}(b - x) = \frac{xy}{2} = \frac{b}{2}(a - y)$$

$$ab - ax = xy = ab - by \tag{1}$$

The outer equalities of (1) give us $ab - ax = ab - by$ or

$$y = \frac{ax}{b}. \tag{2}$$

Substituting (2) for y into the left-most equalities of (1), $ab - ax = xy$, gives

$$b - x = \frac{x^2}{b} \implies x^2 + bx - b^2 = 0,$$

which has the solution

$$x = \frac{-b + \sqrt{b^2 + 4b^2}}{2} = \frac{b}{2}(\sqrt{5} - 1).$$

Resubstituting this value of x in (2) gives

$$y = \frac{a}{2}(\sqrt{5} - 1).$$

With these values for x and y, the ratios $PB : PA = \dfrac{x}{b - x}$ and $QB : QC = \dfrac{y}{a - y}$ are both

$$PB : PA = QB : QC = \frac{\sqrt{5} - 1}{3 - \sqrt{5}} = \frac{\sqrt{5} + 1}{2},$$

which one recognizes to be the golden section.

MA72. Consider four numbers x, y, z and w. The first three are in arithmetic progression and the last three are in geometric progression. If $x + w = 16$ and $y + z = 8$, find all possible solutions (x, y, z, w).

Originally (modified) Problem 8, Vermont State Mathematics Coalition Talent Search, 2009.

We received seven correct and complete and seven incomplete solutions. In each of the incomplete solutions the case $z = 0$ was overlooked. We present the solution by Joel Schlosberg, lightly edited.

Since x, y, z are in arithmetic progression and $z = 8 - y$,

$$x = 2y - z = 3y - 8.$$

Using that y, z, w are in geometric progression and the previous equation,

$$z^2 = yw = y(16 - x) = y(24 - 3y).$$

Therefore

$$0 = z^2 - z^2 = (8 - y)^2 - y(24 - 3y) = 4(y - 2)(y - 8),$$

so either

$$y = 2 \implies (x, y, z, w) = (-2, 2, 6, 18)$$

or

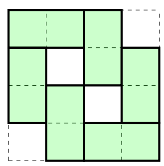
$$y = 8 \implies (x, y, z, w) = (16, 8, 0, 0).$$

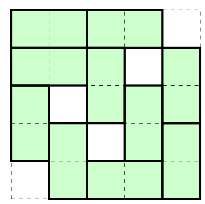
MA73. A checkerboard is "almost tileable" if there exists some way of placing non-overlapping dominoes on the board that leaves exactly one square in each row and column uncovered. (Note that dominoes are 2×1 tiles which may be placed in either orientation.) Prove that, for $n \geq 3$, an $n \times n$ checkerboard is almost tileable if and only if n is congruent to 0 or 1 modulo 4.

Originally Problem 6, Vermont State Mathematics Coalition Talent Search, 2015.

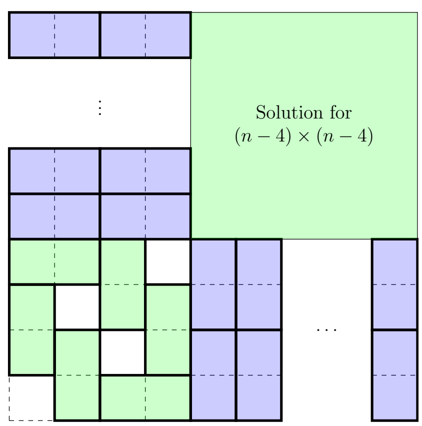
We received 3 solutions. We present the solution by Richard Hess, edited.

Possible solutions for $n = 4$ and $n = 5$ are shown below.





For any $n \equiv 0$ or $1 \mod 4$, we can inductively construct a solution from the $(n-4)\times(n-4)$ solution by appending the 4×4 solution to the bottom left corner and packing the rest of the board with dominoes:



We now show that it is impossible to find a solution for $n \equiv 2$ or $3 \mod 4$. Colour the squares of the board white and black alternatingly, with the top left corner white. Place n pawns on squares in any arrangement such that each column and row has a pawn; the squares with pawns on them will be our candidates for squares which are not covered by dominoes in an almost tiling. Let w and b be the number of pawns on white squares and black squares respectively. If all the pawns are on the main white diagonal then clearly $w - b = n$. Any arrangement of the pawns can be transformed into an arrangement with all the pawns on the diagonal by repeatedly taking the pawns at locations (a, b) and (c, a) and moving them to (a, a) and (c, b). Note that whenever we swap pawns from two different columns while keeping the pawns in their respective rows then $w - b$ will either change by 4 or remain the same. Hence, for any arrangement of pawns we must have $w - b \equiv n$

mod 4. Suppose now that the checkerboard is almost tileable; in any solution, each domino covers a white square and a black square. We place the pawns on the uncovered squares. If n is even there are as many white squares on the board as black squares, so we must have $w - b = 0$; if n is odd, there is one more white square than there are black squares, so we must have $w - b = 1$. This contradicts the earlier assertion that $w - b \equiv n \mod 4$.

MA74. A set of n distinct positive integers has sum 2015. If every integer in the set has the same sum of digits (in base 10), find the largest possible value of n.

Originally Problem 5, Vermont State Mathematics Coalition Talent Search, 2015.

We received three submissions, out of which one was correct and complete. We present the solution by Corneliu Mănescu-Avram, modified by the editor.

The numbers 8, 17, 26, 35, 44, 53, 62, 71, 80, 107, 116, 125, 134, 143, 152, 161, 170, 206, and 305 add up to 2015 and the sum of the digits of each number is 8. We deduce that $n \geq 19$. Since all n integers have to be congruent to each other modulo 9, their sum has to be at least

$$1 + 10 + 19 + \cdots + (9n - 8) = 9\frac{n(n-1)}{2} + n = \frac{n(9n-7)}{2}.$$

From this it follows $n < 22$.

Let s be the common digit sum. Then $ns \equiv 2015 \equiv 8 \pmod 9$. Thus n cannot be 21. If $n = 20$, then $20s \equiv 8 \pmod 9$, thus $s \equiv 4 \pmod 9$. The smallest 15 integers with digit sum 4 are 4, 13, 22, 31, 40, 103, 112, 121, 130, 202, 211, 220, 301, 310, and 400, which already sum to 2220. The smallest 15 integers with digit sum 13 are 49, 58, 67, 76, 85, 94, 139, 148, 157, 166, 175, 184, 193, 229, and 238, which already sum to 2058. If the digit sum is 22 or greater than the smallest number is at least 499 and the sum of the n numbers must be greater than 2015.

MA75. At the Mathville Tapas restaurant, the dishes come in three types: small, medium, and large. Each dish costs an integer number of dollars, with the small dishes being the cheapest and the large dishes being the most expensive. (Tax is already included, different sizes have different prices, and the prices have stayed constant for years.) This week, Jean, Evan, and Katie order 9 small dishes, 6 medium dishes, and 8 large dishes. When the bill arrives, the following conversation occurs:

Jean: "The bill is exactly twice as much as last week."

Evan: "The bill is exactly three times as much as last month."

Katie: "If we gave the waiter a 10% tip, the total would still be less than \$100."

Find the price of the group's meal next week: 2 small dishes, 9 medium dishes, and 11 large dishes.

Originally Problem 1, Vermont State Mathematics Coalition Talent Search, 2015.

We received 9 submissions of which 4 were correct and complete. We present the solution by Richard Hess.

Let the prices of dishes be $0 < S < M < L$. The language of the problem implies that S, M and L are distinct and it makes no sense that a small dish would be free. We are told $9S + 6M + 8L = T < 91$, where T is the total bill. We are further told that T is even, determining that S must be even. Also, T is divisible by 3, determining that L must be divisible by 3.

The only cases that satisfy the constraints are $(S, M, L, T) = (2, 3, 6, 84)$ and $(2, 4, 6, 90)$. The second is not possible since $90/2 = 45$ is odd and we cannot get an odd total when all prices are even. For the remaining case we can get a total of 42 with 6 large dishes and a total of 28 with 14 small dishes. Many other combinations achieve totals of 42 or 28. Therefore \$97 is the price for 2 small dishes, 9 medium dishes and 11 large dishes.

Folding Paper Geometry

Abbas Galehpour Aghdam

In [1], Ian VanderBurgh discussed a paper folding problem from the UK Intermediate Challenge 1999, giving the following problem as a challenge to the readers:

Problem *A rectangular sheet of paper $ABCD$ has $AB = 8$ and $BC = 6$. The paper is folded so that corner A coincides with the midpoint, M, of DC. What is the length of fold?*

We will solve this problem by examining four different approaches.

We first need to draw a diagram, but it is better to start with a practical thing. Get a rectangular sheet of paper with sides 8 and 6 (or with dimensions in the ratio 4:3), then fold it by following the method given in the problem:

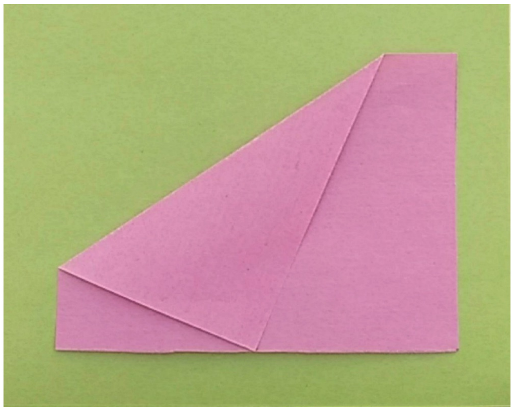


Figure 1: Paper folding

Now, we can draw a suitable diagram as shown in Figure 2.

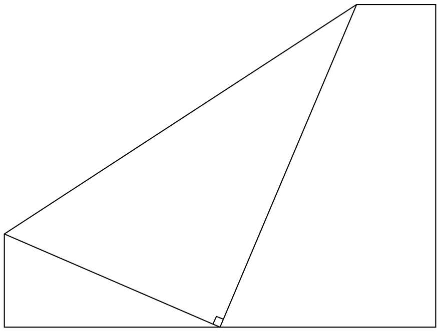


Figure 2: Basic diagram

Let's redraw the diagram by adding the edges of the full original rectangle (the dotted lines) and labelling the relevant points as shown in Figure 3.

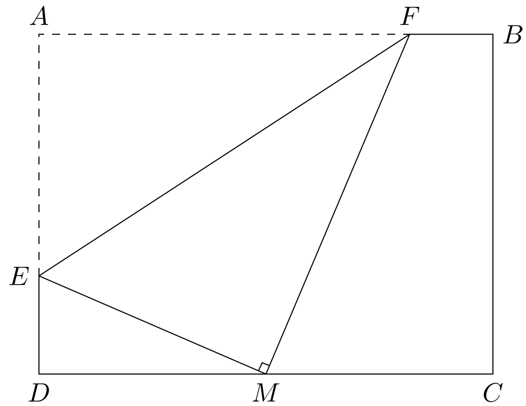


Figure 3: Complete diagram

We should calculate the length of EF. How should we start? We first focus on ΔDEM to determine the lengths of AE, DE and EM, then follow four different approaches.

Since EM is the folded image of AE, we have $EM = AE$. Since the paper has width 6, then $AD = 6$ and $DE = 6 - AE$. Since the paper has length 8 and ΔDEM is right-angled at D, we know one of the three side lengths, namely, $DM = 4$, and we know the other two side lengths in terms of AE, namely, $EM = AE$ and $DE = 6 - AE$. What should we do to determine the length of AE? Let us apply the Pythagorean Theorem in ΔDEM. We obtain

$$EM^2 = DE^2 + DM^2, \quad AE^2 = (6 - AE)^2 + 4^2, \quad AE = \frac{13}{3}.$$

Since $AE = \frac{13}{3}$, we have $EM = \frac{13}{3}$ and $DE = 6 - \frac{13}{3} = \frac{5}{3}$.

Method 1: Using area

Since AF becomes FM after folding, then $AF = FM$ and $\Delta AEF \cong \Delta EFM$ (because of side-side-side congruency), therefore ΔAEF is equal in area to ΔEFM. This implies that the area of $ABCD$ is equal to: $2\times$ (the area of the ΔAEF) + the area of the ΔDEM + the area of $MFBC$. Thus

$$\begin{aligned} 8 \times 6 &= 2\left(\frac{1}{2}AE \cdot AF\right) + \frac{1}{2}DE \cdot DM + \frac{1}{2}BC(FB + MC); \\ 48 &= \frac{13}{3}AF + \frac{10}{3} + 3\big[(8 - AF) + 4\big]; \\ AF &= \frac{13}{2}. \end{aligned}$$

We now have two of the three side lengths of ΔAEF, so we can use the Pythagorean Theorem to conclude that $EF^2 = AF^2 + AE^2 = \frac{169}{4} + \frac{169}{9} = \frac{2197}{36}$; since $EF > 0$, then $EF = \frac{\sqrt{2197}}{6} = \frac{13\sqrt{13}}{6}$.

Method 2: Using trigonometry

In Figure 3, let $\angle DEM = \alpha$ and $\angle EFM = \theta$. Since $\Delta AEF \cong \Delta MEF$, so $\angle AFE = \theta$. This yields

$$\angle AEF = \frac{\pi}{2} - \theta. \tag{1}$$

In right-angle triangle ΔMEF, $\angle MEF$ is complementary to $\angle EFM$, thus

$$\angle MEF = \frac{\pi}{2} - \theta. \tag{2}$$

We also have

$$\angle AEF + \angle MEF + \angle DEM = \pi. \tag{3}$$

With substitution (1) and (2) into (3), we get

$$\alpha = 2\theta. \tag{4}$$

Since ΔDEM, we have $\tan\alpha = \dfrac{DM}{DE} = \dfrac{12}{5}$, so $\tan 2\theta = \dfrac{12}{5}$ (because of (4)). This yields

$$\frac{2\tan\theta}{1-\tan^2\theta} = \frac{12}{5}, \quad \text{or} \quad 6\tan^2\theta + 5\tan\theta - 6 = 0.$$

If we solve this equation for $\tan\theta$, we get $\tan\theta = \dfrac{2}{3}$. Moreover, in right ΔAEF, we have $\tan\theta = \dfrac{AE}{AF}$, therefore $\dfrac{AE}{AF} = \dfrac{2}{3}$ or $AF = \dfrac{3}{2}AE = \dfrac{3}{2} \times \dfrac{13}{3} = \dfrac{13}{2}$. The length of EF is calculated similarly.

Method 3: Using trigonometry (again)

In Figure 3, connect AM as shown in the Figure 4.

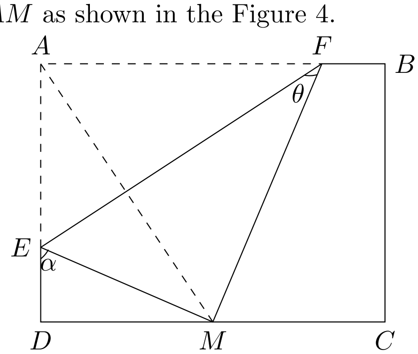


Figure 4: Diagram with angles

Notice that $AD = 6$ and $DM = 4$. Applying the Pythagorean Theorem to right ΔADM yields $AM = 2\sqrt{13}$. Now, let's do some angle-chasing in ΔAFM. We know that $\angle AFE = \angle EFM = \theta$, and this yields $\angle AFM = 2\theta = \alpha$. Moreover, since ΔAFM is isosceles, we have $\angle FAM = \angle FMA = \dfrac{\pi - \angle AFM}{2} = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$. Applying the Sine Law on ΔAFM yields:

$$\begin{aligned} \frac{AF}{\sin(\angle FMA)} &= \frac{AM}{\sin(\angle AFM)}; \\ \frac{AF}{\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)} &= \frac{AM}{\sin\alpha}; \\ AF &= \cos\frac{\alpha}{2} \cdot \frac{AM}{\sin\alpha}. \end{aligned} \tag{5}$$

In ΔDEM, we have $\sin\alpha = \dfrac{DM}{EM} = \dfrac{12}{13}$ and $\cos\alpha = \dfrac{DE}{EM} = \dfrac{5}{13}$. Using double-angle identity, we have $\cos^2\dfrac{\alpha}{2} = \dfrac{9}{13}$; since $\cos\dfrac{\alpha}{2} > 0$, then $\cos\dfrac{\alpha}{2} = \dfrac{3\sqrt{13}}{13}$. Substituting these values into (5), we get $AF = \dfrac{13}{2}$. The length of EF is determined similarly to method 1.

Method 4: Using similar triangles

We know that $\angle FAM = \dfrac{\pi}{2} - \dfrac{\alpha}{2}$ (Figure 4), so $\angle DAM = \dfrac{\alpha}{2}$. This implies that

$$\angle DAM = \frac{\alpha}{2} = \theta = \angle EFM,$$

therefore ΔEFM is similar to ΔADM, so $\dfrac{AM}{EF} = \dfrac{DM}{EM}$. Rearranging yields $EF = AM \times \dfrac{EM}{DM}$, so

$$EF = 2\sqrt{13} \times \frac{\frac{13}{3}}{4} = \frac{13\sqrt{13}}{6}.$$

References

[1] I. Vander Burgh, Problem of the month. *Crux Mathematicorum with Mathematical Mayhem*, Volume 36 (4), April, 2010.

OLYMPIAD CORNER

No. 388

The problems featured in this section have appeared in a regional or national mathematical Olympiad.

Click here to submit solutions, comments and generalizations to any problem in this section

To facilitate their consideration, solutions should be received by **February 15, 2021**.

OC506. A quadrilateral is called convex if the lines given by its diagonals intersect inside the quadrilateral. A convex quadrilateral has side lengths 3, 3, 4, 4 not necessarily in this order, and its area is a positive integer. Find the number of non-congruent convex quadrilaterals having these properties.

OC507. There are $2n$ consecutive integers written on a blackboard. In each move, they are divided into pairs and each pair is replaced with their sum and their difference, which may be taken to be positive or negative. Prove that no $2n$ consecutive integers can appear on the board again.

OC508. Let ABC be an isosceles triangle with $AC = BC$, whose incenter is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

OC509. Prove that for any odd prime p the number of positive integers n satisfying $p|n!+1$ is smaller than or equal to $cp^{2/3}$ where c is a constant independent of p.

OC510. 2019 points are chosen independently and uniformly in the unit disc $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Let C be the convex hull of the chosen points. Which probability is larger: that C is a polygon with three vertices, or a polygon with four vertices?

. . . .

Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.

Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

OC506. Un quadrilatère est dit convexe si les lignes associées aux diagonale intersectent à l'intérieur du quadrilatère. Or un certain quadrilatère convexe a des côtés de longueurs 3, 3, 4 et 4, pas nécessairement dans cet ordre ; de plus, sa surface est donnée par un entier positif. Déterminer le nombre de quadrilatères covexes non congrus ayant ces proriétés.

OC507. Sur un tableau à craie sont écrits $2n$ entiers consécutifs. Par la suite, on regroupe ces entiers en paires, puis chaque paire est effacée et remplacée par deux nouveaux entiers, la somme et la différence des deux entiers de la paire, où la différence peut être prise en positif ou en négatif. Démontrer que quel que soit le nombre de fois qu'on répète ce processus, on ne verra jamais réapparaître $2n$ entiers consécutifs.

OC508. Soit ABC un triangle isocèle tel que $AC = BC$ et soit I le centre de son cercle inscrit. Soit P un point sur le cercle circonscrit du triangle AIB, se situant à l'intérieur du triangle ABC. Les lignes passant par P et parallèles à CA et CB rencontrent AB en D et E, respectivement. La ligne passant par P et parallèle à AB rencontre CA et CB en F et G, respectivement. Démontrer que les lignes DF et EG intersectent en un point qui se trouve sur le cercle circonscrit du triangle ABC.

OC509. Démontrer que pour tout nombre premier impair p, le nombre d'entiers positifs n tels que $p|n! + 1$ est plus petit ou égal à $cp^{2/3}$, où c est une constante indépendante de p.

OC510. 2019 points sont choisis de façon aléatoire et indépendante les uns des autres, selon une distribution uniforme dans le disque unitaire $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Soit C l'enveloppe convexe de ces points. Laquelle probabilité est la plus élevée : C est un polygone à 3 sommets, ou C est un polygone à 4 sommets?

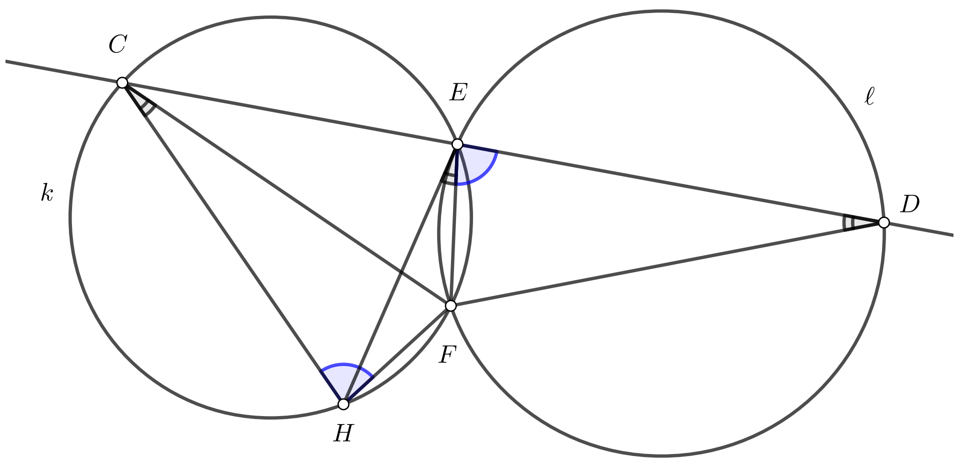
OLYMPIAD CORNER SOLUTIONS

Statements of the problems in this section originally appear in 2020: 46(5), p. 210–211.

OC481. In the plane, there are circles k and l intersecting at points E and F. The tangent to the circle l drawn from E intersects the circle k at point H ($H \neq E$). On the arc EH of the circle k, which does not contain the point F, choose a point C ($E \neq C \neq H$) and let D be the intersection of the line CE with the circle l ($D \neq E$). Prove that triangles DEF and CHF are similar.

Originally Czech-Slovakia Math Olympiad, 3rd Problem, Category B, Regional Round 2017.

We received 12 *submissions. We present the solution by the UCLan Cyprus Problem Solving Group.*



Since C, E, F, H are concyclic, with C on the opposite side of EF than H, then

$$\angle CHF = 180^\circ - \angle CEF = \angle FED\,.$$

We also have

$$\angle FCH = \angle FEH = \angle EDF\,.$$

Here, the first equality follows since C, E, F, H concyclic, and the second by the Chord-Tangent Theorem as HE is tangent to ℓ at E. So, triangles DEF and CHF have equal angles, which means that they are similar.

OC482. Let $a_1, a_2, \ldots, a_{2017}$ be real numbers such that

$$a_1 + a_2 + \cdots + a_{2017} = 2017.$$

Find the largest number of pairs (i, j) for which $1 \le i < j \le 2017$ and $a_i + a_j < 2$.

Originally Bulgaria Math Olympiad, 4th Problem, Grade 11, Second Round 2017.

We received 8 *submissions of which* 7 *were correct and complete. We present* 2 *solutions.*

Solution 1, by UCLan Cyprus Problem Solving Group.

We claim that we can have $\binom{2016}{2}$ pairs but no more.

This number of pairs is achieved by taking $a_1 = \cdots = a_{2016} = 0$ and $a_{2017} = 2017$. Then all $\binom{2016}{2}$ pairs not including a_{2017} has sum less than 2.

Let us now show that we cannot have more pairs. Equivalently, we will show that at least 2016 pairs do not have the property.

We may assume that $a_1 \leqslant \cdots \leqslant a_{2017}$. For $1 \leqslant i \leqslant 1008$ we define

$$b_i = a_i + a_{2018-i}.$$

We also define $b_{1009} = 2a_{1009}$. Then

$$b_1 + \cdots + b_{1008} + \frac{b_{1009}}{2} = a_1 + \cdots + a_{2017} = 2017 = 2 \cdot 1008 + 1\,.$$

So there is at least one $1 \leqslant i \leqslant 1009$ such that $b_i \geqslant 2$.

Note that if $b_i \geqslant 2$, then for each $j \geqslant i$ and $k \geqslant 2018 - i$ we have

$$a_j + a_k \geqslant a_i + a_{2018-i} = b_i \geqslant 2\,.$$

There are

$$(2018 - i)i - i - \binom{i}{2} = \frac{(4035 - 3i)i}{2}$$

such pairs (j, k) with $j < k$. (Since there are a total of $(2018 - i)i$ pairs (j, k) with $j \geqslant i$ and $k \geqslant 2018 - i$, out of which i of them satisfy $j = k$ and $\binom{i}{2}$ of them satisfy $j > k$.)

We have

$$\frac{(4035 - 3i)i}{2} - 2016 = \frac{(i - 1)(4032 - 3i)}{2} \geqslant 0$$

with equality if and only if $i = 1$. So at least 2016 pairs do not have this property as required.

Solution 2, by Roy Barbara.

More generally, let $n \geq 6$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\sum_{i=1}^n a_i = n$. Then, the largest number of pairs (i, j), $1 \leq i < j \leq n$ such that $a_i + a_j < 2$ is exactly

$$\binom{n}{2} - (n-1) = \binom{n-1}{2} \tag{1}$$

For $1 \leq i < j \leq n$, call (i, j) a green pair (a red pair) if $a_i + a_j < 2$ (if $a_i + a_j \geq 2$). Note that the number of green (red) pairs is invariant by any rearrangement of the a_i's, so we may assume, when needed, that the a_i's are ordered as $a_1 \leq a_2 \leq \ldots \leq a_n$.

The number $n-1$ of red pairs (i, j) is reached with $a_1 = a_2 = \ldots = a_{n-1} = 0$ and $a_n = n$.

Next we show that in any configuration $a_1, a_2, \ldots, a_n$ the number of red pairs is at least $n-1$ and (1) will follow.

Lemma. Let $n \geq 6$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = n$. Then, either there are at least $n-1$ red pairs or there is a red pair (i, j) with $a_i \leq 1$.

Proof. If $n = 2k$, $k \geq 3$, we take $a_1 \leq a_2 \leq \ldots \leq a_{2k}$ such that $\sum_{i=1}^{2k} a_i = 2k$ and if $n = 2k+1$, $k \geq 3$, we take $a_0 \leq a_1 \leq \ldots \leq a_{2k}$ such that $\sum_{i=0}^{2k} a_i = 2k+1$. We have two cases.

(i) $a_k > 1$. Then, $1 < a_k \leq a_{k+1} \leq \ldots \leq a_{2k}$ clearly shows that the number of red pairs is at least $\binom{k+1}{2}$. As $k \geq 3$, then $\dfrac{k+1}{2} \geq 2$, so $\dfrac{k(k+1)}{2} \geq 2k$, that is $\binom{k+1}{2} \geq 2k$, whence $2k \geq n-1$.

(ii) $a_k \leq 1$. We claim that $a_k + a_{2k} \geq 2$ (and we are done with the red pair $(k, 2k)$). Otherwise, since $a_1 \leq a_2 \leq \ldots \leq a_{2k}$, we would get $a_i + a_{k+i} < 2$ for $i = 1, 2, \ldots, k$. Adding, we get $\sum_{i=1}^k (a_i + a_{k+i}) < 2k$, that is $\sum_{i=1}^{2k} a_i < 2k$. If $n = 2k$, this is a contradiction. If $n = 2k+1$, adding $a_0 \leq 1$ (that obviously holds) to this inequality would imply $\sum_{i=0}^{2k} a_i < 2k+1$, contradiction.

We are now ready to prove by induction on n that in any configuration $a_1, a_2, \ldots, a_n$ the number of red pairs is at least $n-1$.

For $n = 6$, let $a_1 \leq a_2 \leq \ldots \leq a_6$ be real numbers with $\sum_{i=1}^6 a_i = 6$. If we had $a_1 + a_6 < 2$, $a_2 + a_5 < 2$ and $a_3 + a_4 < 2$, then $\sum_{i=1}^6 a_i < 6$, contradiction. Hence, we must have either $a_1 + a_6 \geq 2$ or $a_2 + a_5 \geq 2$ or $a_3 + a_4 \geq 2$.

If $a_1 + a_6 \geq 2$, the following pairs are clearly red: $(1, 6)$, $(2, 6)$, $(3, 6)$, $(4, 6)$, $(5, 6)$.

If $a_2 + a_5 \geq 2$, the following pairs are clearly red: $(2, 5)$, $(2, 6)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$, $(5, 6)$.

If $a_3 + a_4 \geq 2$, the following pairs are clearly red: $(3, 4)$, $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$, $(5, 6)$.

In any case, there are at least 5 red pairs, and the number 5 is reached with $a_1 = a_2 = a_3 = a_4 = a_5 = 0$, $a_6 = 6$.

Now, assume that the thesis holds for some $n \geq 6$. Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$ with

$$\sum_{i=0}^{n} a_i = n + 1.$$

By the lemma, either there are at least n red pairs (and we are done) or there is a red pair (i, j) with $a_i \leq 1$. In this case, without loss of generality, we may assume that this red pair is $(0, n)$. Thus, we have $a_0 \leq 1$ and $a_0 + a_n \geq 2$ for $i = 0, 1, \ldots, n-1$ and $b_n = a_n + a_0 - 1$ $(b_n \leq a_n)$. Then,

$$\sum_{i=1}^{n} b_i = n.$$

By the inductive hypothesis, there are at least $n - 1$ pairs (i, j), $1 \leq i < j \leq n$ with $b_i + b_j \geq 2$. Since $a_i \geq b_i$ for $i = 1, 2, \ldots, n$, then the same $n - 1$ pairs (i, j) satisfy $a_i + a_j \geq 2$. If we add the red pair $(0, n)$, we obtain (at least) n red pairs for the sequence $a_0, a_1, \ldots, a_n$.

OC483. Prove that for each prime number $p > 2$, there is exactly one positive integer n such that the number $n^2 + np$ is a perfect square.

Originally Poland Math Olympiad, 1st Problem, Second Round 2017.

We received 23 *submissions, of which* 22 *were correct and complete. We present* 3 *solutions.*

Solution 1, by Fernando Ballesta Yagüe.

Let n be a positive integer, and p a given prime, $p > 2$. Then, $n^2 + np$ is a perfect square if and only if $n^2 + np = k^2$ for some positive integer k, i.e. if and only if $4n^2 + 4np = 4k^2$ for some positive integer k. Then,

$$(2n + p)^2 - p^2 = 4k^2 \iff (2n + p)^2 - 4k^2 = p^2,$$

i.e.

$$(2n + p - 2k)(2n + p + 2k) = p^2.$$

Since p is a prime number, the only decompositions of p^2 as a product of two natural numbers are $p \cdot p$ and $1 \cdot p^2$. As $k > 0$,

$$2n + p + 2k > 2n + p - 2k,$$

so it cannot be

$$2n + p + 2k = 2n + p - 2k = p.$$

Therefore, it must be $2n + p + 2k = p^2$, $2n + p - 2k = 1$. Adding them up:

$$4n + 2p = p^2 + 1 \iff n = \frac{p^2 - 2p + 1}{4} = \left(\frac{p-1}{2}\right)^2$$

So $n = \left(\frac{p-1}{2}\right)^2$ is the only value that makes $n^2 + np$ a perfect square. Notice that $p > 2$ is prime, so it is an odd number, so $p - 1$ is even, which implies that n is a positive integer.

Solution 2, by Vincent Chan.

If $n \equiv 0 \pmod{p}$, say $n = pk$ for an integer k, then

$$n^2 + np = p^2k^2 + p^2k = p^2(k^2 + k).$$

But this cannot be a perfect square, since

$$k^2 < k^2 + k < k^2 + 2k + 1 = (k+1)^2.$$

Therefore, $\gcd(n, p) = 1$. In this case, $\gcd(n, p + n) = 1$, so $n^2 + np = n(n + p)$ is a perfect square if and only if both n and $n + p$ are perfect squares, say

$$\begin{aligned} n &= a^2, \\ n + p &= b^2. \end{aligned}$$

Subtracting, this gives $p = b^2 - a^2 = (b + a)(b - a)$. But p is prime, hence $b - a = 1$ and $b + a = p$, yielding

$$a = \frac{p-1}{2}.$$

Therefore, the unique value of n for which $n^2 + np$ is a perfect square is

$$n = \frac{(p-1)^2}{4}.$$

Solution 3, by UCLan Cyprus Problem Solving Group.

Assume that $n^2 + np = m^2$. Then $np = (m - n)(m + n)$. Since $m + n > n$, then $p > m - n$. Since p is prime, then $p|(m - n)$ or $p|(m + n)$. But $p > m - n$, so $p|(m + n)$. So we can write $m = kp - n$ for some positive integer k.

Then $np = (kp - 2n)kp$ which gives $k^2p = n(2k + 1)$. Since k^2 and $2k + 1$ are coprime (if a prime q divides k^2 and $2k + 1$, then it also divides k and therefore 1 as well, a contradiction) then $k^2|n$. So $n = k^2r$ for some positive integer r. But then $k^2p = k^2r(2k + 1)$, so $p = r(2k + 1)$. Since p is prime and $2k + 1 > 1$, then $r = 1$. This gives $p = 2k + 1$ and $n = k^2$.

So if $n^2 + np$ is a perfect square, then we must have $n = \left(\frac{p-1}{2}\right)^2$. Since p is odd then this is an integer and it is easy to check that in this case $n^2 + np = \left(\frac{p^2-1}{4}\right)^2$ is a perfect square.

OC484. Let x be a real number with $0 < x < 1$ and let $0.c_1c_2c_3\dots$ be the decimal expansion of x. Denote by $B(x)$ the set of all subsequences of $c_1, c_2, c_3\dots$ that consist of six consecutive digits. For instance, $B(1/22) = \{045454, 454545, 545454\}$.

Find the minimum number of elements of $B(x)$ as x varies among all irrational numbers with $0 < x < 1$.

Originally Italy Math Olympiad, 5th Problem, Final Round 2018.

We received 5 *submissions. We present the solution by UCLan Cyprus Problem Solving Group.*

Given $n \in \mathbb{N}$ and $x \in (0,1)$ we write $B_n(x)$ for the set of all subsequences of n consecutive digits in the decimal expansion of x. This is uniquely defined for an irrational x. We will show that $|B_n(x)| \geqslant n+1$. This is best possible as

$$x = 0.\underbrace{0\cdots0}_{n}1\underbrace{0\cdots0}_{n+1}1\underbrace{0\cdots0}_{n+2}1\cdots$$

is irrational with $|B_n(x)| = n+1$. To see the irrationality of x note that for any k, the decimal expansion of x cannot have period of k as there are infinitely many 1's in the decimal expansion which are followed by k or more 0's. To see the second claim about the size of $B_n(x)$ we just note that $B_n(x)$ consists of all possible sequences with at most one digit equal to 1 and all the other digits equal to 0).

We proceed to prove our claim by induction on n. The case $n = 1$ is trivial as $x \neq 0$. Assume that $|B_k(x)| \geqslant k+1$ for each irrational $x \in (0,1)$. For the inductive step, we want to show that $|B_{k+1}(x)| \geqslant k+2$ for each irrational $x \in (0,1)$. So assume for contradiction that there is an irrational $y \in (0,1)$ such that $|B_{k+1}(y)| \leqslant k+1$. Given $\mathbf{a} = a_1a_2\cdots a_k \in B_k(y)$ we write $B_{k+1}(y;\mathbf{a})$ for all elements of $B_{k+1}(y)$ which begin with $\mathbf{a}$. Then $|B_{k+1}(y;\mathbf{a})| \geqslant 1$ for each $\mathbf{a} \in B_k(y)$ and $B_{k+1}(y;\mathbf{a}) \cap B_{k+1}(y;\mathbf{b}) = \emptyset$ for each $\mathbf{a} \neq \mathbf{b}$. Since $|B_k(y)| \geqslant k+1$, we get that $|B_{k+1}(y)| \geqslant k+1$. Therefore $|B_{k+1}(y)| = k+1$ and the equality occurs if and only if $|B_{k+1}(y;\mathbf{a})| = 1$ for each $\mathbf{a} \in B_k(y)$. But in this case, every k consecutive digits in the decimal expansion of y completely determine the next digit. We will show that in this case y is rational. This contradiction completes the induction step and therefore our claim that $|B_n(x)| \geqslant n+1$ for each $n \in \mathbb{N}$ and each irrational $x \in (0,1)$.

We want to show that given the decimal expansion of $y \in (0,1)$, if every k consecutive digits of y completely determine the next digit, then y is rational. Let $y = 0.c_1c_2c_3\dots$. Since there are only finitely many sequences of k consecutive digits that can occur, then there are $r < s$ such that $c_{r+1} = c_{s+1}, \dots, c_{r+k} = c_{s+k}$. Then $c_{r+k+1} = c_{s+k+1}$ as every k consecutive digits completely determine the next one. Inductively, we can now easily get $c_{r+n} = c_{s+n}$ for every $n \in \mathbb{N}$ showing that y is rational as required.

Therefore the minimum possible number of elements of $B(x) = B_6(x)$ is 7.

OC485. Prove that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is increasing if and only if

$$(c-b)\int_a^b f(x)\ dx \leq (b-a)\int_b^c f(x)\ dx,$$

for all real numbers $a < b < c$.

Originally Romania Math Olympiad, 3rd Problem, Grade 12, District Round 2018.

We received 12 *submissions. We present the solution by Corneliu Avram Manescu.*

If f is increasing and $a < b < c$, then

$$(c-b)\int_a^b f(x)\ dx \leq (c-b)(b-a)f(b) = (b-a)(c-b)f(b) \leq (b-a)\int_b^c f(x)\ dx.$$

Conversely, if a and b are real numbers, $a < b$, let $F : \mathbb{R} \to \mathbb{R}$ be the primitive of f. If x and y are real numbers such that $a < x < y < b$, then the given relation implies

$$f(a) = F'(a) = \lim_{x \to a^+} \frac{F(x) - F(a)}{x - a} \leq \lim_{y \to b^-} \frac{F(b) - F(y)}{b - y} = F'(b) = f(b).$$

Remark from the solver. The condition that the function f is continuous is not necessary: for the direct implication the monotony of f implies the integrability of f on every compact interval and for the converse implication it suffices that f have primitives on $\mathbb{R}$.

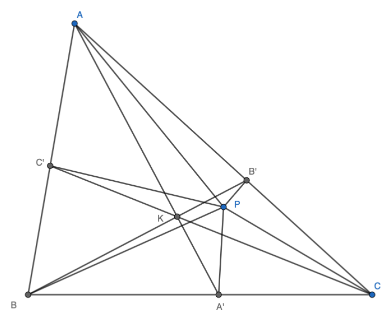
PROBLEMS

Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.

To facilitate their consideration, solutions should be received by **February 15, 2021**.

4591. *Proposed by Pericles Papadopoulos.*

Let point P be inside triangle ABC and let A', B' and C' be the points where the internal bisectors of $\angle BPC$, $\angle CPA$ and $\angle APB$ intersect sides BC, CA and AB, respectively.



Show that lines AA', BB' and CC' concur at a point K satisfying

$$\frac{AK}{KA'} = PA\left(\frac{1}{PB} + \frac{1}{PC}\right).$$

4592. *Proposed by Michel Bataille.*

Let ABC be a triangle with $\angle BAC \neq 90^\circ$ and let O be its circumcentre. Let γ be the circumcircle of ΔBOC. The perpendicular to OA at O intersects γ again at M and the line AM intersects γ again at N. Prove that

$$\frac{NO}{NA} = \frac{2OA^2}{AB \cdot AC}.$$

4593. *Proposed by Diaconu Radu.*

Solve the system of equations in real numbers:

$$\begin{cases} a^2 + bc = 7, \\ ab + bd = 3, \\ ac + dc = 2, \\ bc + d^2 = 6. \end{cases}$$

4594. *Proposed by Nguyen Viet Hung.*

Prove that for any point M on the incircle of triangle ABC,

$$\frac{MA^2}{h_a}+\frac{MB^2}{h_b}+\frac{MC^2}{h_c}=2R+r,$$

where h_a, h_b and h_c are the lengths of the altitudes from A, B and C respectively, while R and r denote circumradius and inradius, respectively.

4595. *Proposed by Nguyen Viet Hung.*

Let $n>2$ be an integer and let $S_n=\sum_{k=2}^{n}\sqrt{1+\frac{2}{k^2}}$. Determine $\lfloor S_n\rfloor$.

4596. *Proposed by Boris Čolaković.*

Let a, b, c be the lengths of the sides of triangle ABC with inradius r and circumradius R. Show that

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\le\frac{R}{r}-\frac{1}{2}$$

4597. *Proposed by George Apostolopoulos.*

Let a, b, c be positive real numbers with $a+b+c=1$. Prove that

$$a^2+b^2+c^2+\frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}\ge 2(ab+bc+ca).$$

4598. *Proposed by George Stoica.*

Let $P(z)$ be a polynomial of degree n with complex coefficients and with no zeroes z satisfying $|z|<1$. Prove that $|P(z)|\le 2^n|P(rz)|$ for all $|z|\le 1$ and $0<r<1$.

4599. *Proposed by Albert Natian.*

The sum of squares of the sides of a triangle ABC is 133. By enlarging two sides of ABC by a factor of 27, and a third side by a factor of 8, ABC is deformed into a larger but similar triangle RST whose area is 324 times that of ABC. Find the side lengths of ABC.

4600. *Proposed by Semen Slobodianiuk, modified by the Editorial Board.*

It is known (for example, by a formula of Euler, often attributed to Nicolas Fuss, giving the distance between the centers in terms of the two radii) that given a bicentric quadrilateral inscribed in one circle and circumscribed about a second, then every point A of the circumcircle is the vertex of a bicentric quadrilateral $ABCD$ that is inscribed in the first circle, and circumscribed about the second. Determine the locus of the centroid of the vertex set $\{A, B, C, D\}$ as the bicentric quadrilateral $ABCD$ travels around the first circle while its sides stay tangent to the second.

. . . .

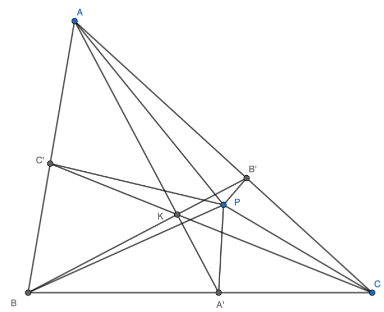
Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposś dans cette section.

Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le **15 février 2021**.

La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.

4591. *Proposée par Pericles Papadopoulos.*

Soit P un point à l'intérieur du triangle ABC et soient A', B' et C' les points où les bissectrices internes de $\angle BPC$, $\angle CPA$ et $\angle APT$ intersectent les côtés BC, CA et AB, respectivement.



Démontrer que les lignes AA', BB' et CC' sont concourantes en un point K tel que

$$\frac{AK}{KA'} = PA\left(\frac{1}{PB} + \frac{1}{PC}\right).$$

4592. *Proposée par Michel Bataille.*

Soit ABC un triangle tel que $\angle BAC \neq 90°$ et soit O le centre de son cercle circonscrit. Aussi, soit γ le cercle circonscrit de ΔBOC. Enfin, la perpendiculaire vers OA en O intersecte γ de nouveau en M et la ligne AM intersecte γ de nouveau en N. Démontrer que

$$\frac{NO}{NA} = \frac{2OA^2}{AB \cdot AC}.$$

4593. *Proposée par Diaconu Radu.*

Résoudre le système d'équations suivant pour les nombres réels:

$$\begin{cases} a^2 + bc = 7, \\ ab + bd = 3, \\ ac + dc = 2, \\ bc + d^2 = 6. \end{cases}$$

4594. *Proposée par Nguyen Viet Hung.*

Démontrer que pour tout point M sur le cercle inscrit de ABC, la suivante tient :

$$\frac{MA^2}{h_a} + \frac{MB^2}{h_b} + \frac{MC^2}{h_c} = 2R + r$$

où h_a, h_b et h_c sont les longueurs des altitudes émanant de A, B et C respectivement, et R et r sont les rayon du cercle circonscrit et du cercle inscrit, respectivement.

4595. *Proposée par Nguyen Viet Hung.*

Soit $n > 2$ un entier et soit $S_n = \sum_{k=2}^{n} \sqrt{1 + \frac{2}{k^2}}$. Déterminer $\lfloor S_n \rfloor$.

4596. *Proposée par Boris Čolaković.*

Soit a, b et c les longueurs des côtés d'un triangle ABC dont le cercle inscrit et le cercle circonscrit ont les rayons r et R respectivement. Démontrer que

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \leq \frac{R}{r} - \frac{1}{2}$$

4597. *Proposée par George Apostolopoulos.*

Soient a, b, c des nombres réels positifs tels que $a + b + c = 1$. Démontrer que

$$a^2 + b^2 + c^2 + \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq 2(ab + bc + ca).$$

4598. *Proposée par George Stoica.*

Soit $P(z)$ un polynôme de degré n à coefficients complexes avec aucune racine z telle que $|z| < 1$. Démontrer que $|P(z)| \leq 2^n |P(rz)|$ pour tout $|z| \leq 1$ et $0 < r < 1$.

4599. *Proposée par Albert Natian.*

La somme des carrés des côtés d'un triangle ABC est 133. En allongeant deux côtés de ABC par un facteur de 27 et le troisième par un facteur de 8, ABC est transformé en un triangle similaire RST dont la surface est 324 celle de ABC. Déterminer les longueurs des côtés de ABC.

4600. *Proposée par Semen Slobodianiuk, modifé par le comité éditorial.*

C'est un fait connu (par exemple, par une formule d'Euler, souvent attribuée à Nicolas Fuss, donnant la distance entre centres en termes de rayons) : pour tout quadrilatère bicentrique inscrit dans un cercle et circonscrit autours d'un second, chaque point A sur le cercle circonscrit est le sommet d'un quadrilatère bicentrique $ABCD$ inscrit dans le premier cercle et circonscrit autours du second. Déterminer le lieu géométrique de l'ensemble de sommets $\{A, B, C, D\}$ lorsque le quadrilatère bicentrique $ABCD$ se déplace autours du premier cercle, tout en faisant en sorte que ses côtés restent tangents au second.

BONUS PROBLEMS

These problems appear as a bonus. Their solutions will **not** *be considered for publication.*

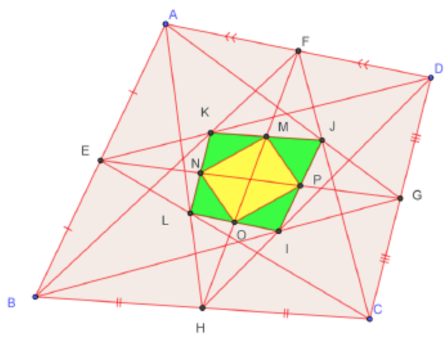
B51. *Proposed by Leonard Giugiuc.*

Let $a \in (1, e)$ be a fixed real number. Find

$$\sup_{x>0}\left(a^{-x}+a^{-\frac{1}{x}}\right).$$

B52. *Proposed by Dao Thanh Oai and Leonard Giugiuc.*

Let $ABCD$ be a convex quadrilateral and let E, H, G, F be the midpoints of AB, BC, CD, DA. Show that $MNOP$ is a parallelogram and area($ABCD$)/area($KJIL$)=9.



B53. *Proposed by Leonard Giugiuc.*

Let S be a point in the interior of triangle ABC. Prove that for the existence and uniqueness of a triangle XYZ for which (i) its sides YZ, ZX, XY contain the points A, B, C, respectively, and (ii) the lines XA, YB, and ZC are concurrent in S, it is sufficient that the largest of the three areas $[SAB], [SBC], [SCA]$ is less than the sum of the other two.

B54. *Proposed by Leonard Giugiuc.*

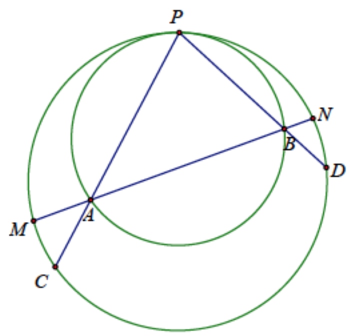
Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$ and $s = \frac{a+b+c+d}{2}$. Prove that if $ab + cd = 2\sqrt{(s-a)(s-b)(s-c)(s-d)}$, then there exist positive numbers x, y, z such that $a = \sqrt{x^2+y^2}$, $b = z\sqrt{x^2+y^2}$, $c = x\sqrt{1+z^2}$ and $d = y\sqrt{1+z^2}$.

B55. *Proposed by Mihaela Berindeanu.*

In $\triangle\ ABC$, $\measuredangle(BAC) = 60^\circ$, $O =$ circumcenter, $I =$ incenter and $H =$ orthocenter. If $BO \parallel HI$, find $\measuredangle(HAO)$.

B56. *Proposed by Arsalan Wares*

In figure below, two circles touch each other internally at point P. Line segments PC and PD are chords to the larger circle that intersect the smaller circle at points A and B, respectively. Point M and N are on the larger circle so that points M, A, B and N are collinear. Suppose $BD : AC = 3 : 4$, $BN = 10$, and $AM = 20$. Find the exact length of AB.



B57. *Proposed by Mihaela Berindeanu.*

Given the acute triangle ABC and its circumcircle Γ, define E to be the point where the bisector of the angle at C intersects the tangent to Γ at B. Furthermore, for the midpoint F of AB define $X = EF \cap AC$, and Y to be the second point where BX intersects Γ. Prove that $YB = YA$.

B58. *Proposed by Mihaela Berindeanu.*

Let A' and B' be the points where the tangents to a given circle at B and A, respectively, meet the tangent that touches the circle at C. If O is the center of the circle, define

$$X = AB \cap OB', \quad Y = AB \cap OA', \quad \text{and } Z = A'X \cap B'Y.$$

Prove that the circle determined by X, Y, and the midpoint of $A'B'$ also contains the midpoint of OZ.

B59. *Proposed by Leonard Giugiuc.*

Let a_i with $1 \leq i \leq 6$ be real numbers such that

$$\sum_{i=1}^{6} a_i = \frac{15}{2} \quad \text{and} \quad \sum_{i=1}^{6} a_i^2 = \frac{45}{4}.$$

Find the extrema for the expression

$$\sum_{1 \leq i < j < k < l \leq 6} a_i a_j a_k a_l.$$

B60. *Proposed by Leonard Giugiuc.*

Let k be a real number with $k \geq \sqrt{3}$. Consider non-negative real numbers a, b, c such that

$$ab + bc + ca + (k^2 - 3)abc = k^2.$$

Prove that

$$a + b + c + (2k - 3)abc \geq 2k.$$

B61. *Proposed by Leonard Giugiuc.*

Let a, b and c be positive real numbers. Prove that

$$\sqrt{a^2c^2 - abc + b^2} + \sqrt{b^2c^2 - abc + a^2} \geq \sqrt{a^2 + ab + b^2}.$$

When does the equality hold?

B62. *Proposed by Leonard Giugiuc and Daniel Dan.*

Find all real numbers k such that for all triangles ABC we have

$$\cos^2 A + \cos^2 B + \cos^2 C + k(\cos A \cos B + \cos A \cos C + \cos B \cos C) \geq \frac{3(1+k)}{4}.$$

B63. *Proposed by Leonard Giugiuc and George Apostolopoulos.*

Prove that in any triangle ABC with the customary notations, we have

$$\frac{m_a^2}{b^2 + c^2} + \frac{m_b^2}{c^2 + a^2} + \frac{m_c^2}{a^2 + b^2} \geq \frac{9r}{4R}.$$

B64. *Proposed by Leonard Giugiuc.*

Let $0 \leq a < b < c \leq \frac{1}{\sqrt{3}}$ and $a + b + c = 1$. Find the best upper bound for

$$\frac{\arctan a - \arctan b}{a - b} + \frac{\arctan b - \arctan c}{b - c} + \frac{\arctan c - \arctan a}{c - a}.$$

B65*. *Proposed by Leonard Giugiuc.*

Let $n \geq 4$ be an integer and let a_i for $i = 1, 2, \ldots, n$ be positive real numbers. Prove or disprove the following:

$$\sqrt{(n-1)\left(\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}\right)} + \frac{n(\sqrt{n} - \sqrt{n-1})}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{\sqrt{n}}.$$

B66. *Proposed by Leonard Giugiuc and Nguyen Viet Hung.*

Let ABC be a non-obtuse angles triangle, none of whose angles are less that $\pi/4$. Prove that

$$\frac{\cos 2A}{\sin B \sin C} + \frac{\cos 2B}{\sin C \sin A} + \frac{\cos 2C}{\sin A \sin B} \leq -2.$$

B67. *Proposed by Lorian Saceanu.*

a) For any triangle ABC, prove that

$$\frac{aA + bB + cC}{a + b + c} \geq \operatorname{arccot} \frac{3r}{s} \geq \frac{\pi}{3},$$

where s is the semiperimeter of ABC and r is the inradius.

b) For an acute angled triangle ABC, prove that

$$\frac{aA + bB + cC}{a + b + c} \leq \arccos \frac{r}{R},$$

where r is the inradius and R is the circumradius.

B68. *Proposed by Daniel Sitaru.*

Prove that for $\frac{\sqrt{3}}{3} \leq a, b, c \leq 1$, we have

$$\sqrt[3]{abc} \cdot \tan^{-1}\left(\sqrt{\frac{ab + bc + ca}{3}}\right) \leq \sqrt{\frac{ab + bc + ca}{3}} \cdot \tan^{-1}(\sqrt[3]{abc}).$$

When does equality occur?

B69. *Proposed by George Apostolopoulos.*

Let ABC be a triangle with circumradius R and inradius r. Prove that

$$\csc A + \csc B + \csc C - \cot A - \cot B - \cot C \leq \sqrt{3} \left(\frac{R}{2r}\right)^2.$$

B70. *Proposed by Leonard Giugiuc, Diana Trailescu and Dan Stefan Marinescu.*

Let $f : [0, \infty) \to \mathbb{R}$ be a differentiable function, whose derivative is concave for all positive real numbers and $f(0) = 0$. Prove that for any integer $n \geq 3$ and any non-negative numbers x_k, $k = 1, 2, \ldots, n$, we have

$$f(x_1 + \cdots + x_n) + (n - 2) \sum_{k=1}^{n} f(x_k) \leq \sum_{1 \leq i < j \leq n} f(x_i + x_j).$$

B71. *Proposed by Hoang Le Nhat Tung.*

Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\frac{a}{\sqrt{2(b^4+c^4)}+7bc}+\frac{b}{\sqrt{2(a^4+c^4)}+7ac}+\frac{c}{\sqrt{2(a^4+b^4)}+7ab}\geq\frac{1}{3}$$

B72. *Proposed by Daniel Sitaru.*

Prove that in triangle ABC, the following relationship holds:

$$\frac{\sin A}{\sin\frac{B}{2}\sin\frac{C}{2}}+\frac{\sin B}{\sin\frac{C}{2}\sin\frac{A}{2}}+\frac{\sin C}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq\frac{2s}{r}.$$

B73. *Proposed by Leonard Giugiuc and Diana Trailescu.*

Let a, b and c be real numbers that satisfy $a\geq 1\geq b\geq c$, $a+b+c<0$ and $ab+bc+ca=3$. Determine the maximum value of $a+b+c$ and the values of a, b and c for which this maximum is attained

B74. *Proposed by Mihaela Berindeanu.*

Let $a, b, c\geq 0$ be real numbers with the property $a^2+b^2+c^2=1$. Show that:

$$\frac{a^2\ln(1+bc)}{bc}+\frac{b^2\ln(1+ac)}{ac}+\frac{c^2\ln(1+ab)}{ab}\geq\ln\frac{64}{27}.$$

B75. *Proposed by George Apostolopoulos.*

If a, b, c are positive real numbers such that $a+b+c=1$, prove that

$$4(ab+bc+ca)^2\leq ab+bc+ca+3abc.$$

SOLUTIONS

No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.

Statements of the problems in this section originally appear in 2020: 46(5), p. 226–230.

4541. *Proposed by Michel Bataille.*

Let a, b, c be positive real numbers such that $abc = 1$ and let $S_k = a^k + b^k + c^k$. Prove that

$$\frac{S_2 + S_4}{2} \geq 1 + \sqrt{1 + S_3}.$$

We received 24 submissions, of which 22 were correct and complete. We present the solution by Corneliu Manescu-Avram.

From the AM-GM inequality we have $S_3 \geq 3$. On the other hand,

$$\frac{S_2 + S_4}{2} - S_3 = \frac{(a^2 - a)^2 + (b^2 - b)^2 + (c^3 - c)^2}{2} \geq 0,$$

therefore it suffices to prove that $S_3 \geq 1 + \sqrt{1 + S_3}$. Separating the radical, squaring and reducing similar terms, we get $S_3(S_3 - 3) \geq 0$, which is true.

Editor's comment. Roy Barbara, Walther Janous, and Borche Joshevski submitted generalizations.

4542. *Proposed by Leonard Giugiuc and Alexander Bogomolny.*

Let ABC be a triangle with centroid G. Denote by D, E and F the midpoints of the sides BC, CA and AB respectively. Find the points M on the plane of ABC such that

$$MA + MB + MC + 3MG = 2(MD + ME + MF).$$

We received 12 solutions, 5 of which were incomplete. All of the complete solutions used Hlawka's inequality. As several solvers (Walther Janous, Ioannis Sfikas, UCLan Cyprus Problem Solving Group) pointed out, closely related problems have occurred both in this journal (problems 3052 and 2482) and The American Mathematical Monthly (problem 12015). The problems in this journal prove that inequality holds, while the notes to problem 2482 provide a reference to the case of equality. For completeness, we include a solution by Theo Koupelis here.

By considering the coordinates of the points in the complex plane we have

$$2D = B + C,\ 2E = C + A,\ 2F = A + B,\ \text{and } 3G = A + B + C.$$

Letting

$$x = M - A,\ y = M - B,\ \text{and } z = M - C,$$

we get

$$2(M-D) = y+z,\ 2(M-E) = z+x,\ 2(M-F) = x+y,\ \text{and}\ 3(M-G) = x+y+z.$$

Therefore, the given equality is equivalent to

$$|x| + |y| + |z| + |x + y + z| = |x + y| + |y + z| + |z + x|. \tag{1}$$

However, Hlawka's inequality for complex x, y, z is

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|. \tag{2}$$

We can prove this inequality by multiplying both sides by $|x|+|y|+|z|+|x+y+z|$ and by rearranging the terms to find the equivalent inequality

$$(|x|+|y|-|x+y|)(|z|+|x+y+z|-|x+y|)+(|y|+|z|-|y+z|)(|x|+|x+y+z|-|y+z|)$$
$$+(|z| + |x| - |z + x|)(|y| + |x + y + z| - |x + y|) \geq 0. \tag{3}$$

This inequality is obvious by the triangle inequality; in each of the three terms, both factors are non-negative. (In rearranging the terms to prove this inequality we used the identity

$$|x|^2 + |y|^2 + |z|^2 + |x + y + z|^2 = |x + y|^2 + |y + z|^2 + |z + x|^2.)$$

From (3) we see that equality is reached (and therefore we obtain (1)) when each of the three terms in (3) vanishes. Considering the first term of (3) as an example, we see that it vanishes when M is on the line AB (excluding the interior points of segment AB) or on the segment CG. Similarly for the other two terms in (2). Therefore, equality occurs when M is a vertex of the triangle or its centroid.

4543. *Proposed by Cherng-tiao Perng.*

Let $n = 4k + 2$ ($k \geq 1$) be an integer and $A_1A_2 \cdots A_n$ be a polygon with parallel opposite sides, i.e.

$$A_iA_{i+1} \parallel A_{n/2+i}A_{n/2+i+1}, \qquad i = 1, 2, \cdots , n/2,$$

where one sets $A_{n+1} = A_1$. Starting with a point B_1 and a circle C through B_1, define $B_2, B_3, \cdots , B_{n+1}$ inductively by requiring that the circle $(A_iA_{i+1}B_i)$ intersects C again at B_{i+1}, for $i = 1, 2, \cdots , n$. Prove that $B_{n+1} = B_1$.

We received 2 solutions; we present them both.

Solution 1, by the UCLan Cyprus Problem Solving Group, annotated and slightly edited.

Let $A_{n+1} = A_1$. For $1 \leq i \leq n$, denote by Γ_i the circumcircle of $\triangle A_iB_iA_{i+1}$.

We show in Figure 1 an example of the construction with $n = 6$. *We will use this specific example to illustrate the main steps of the proof.*

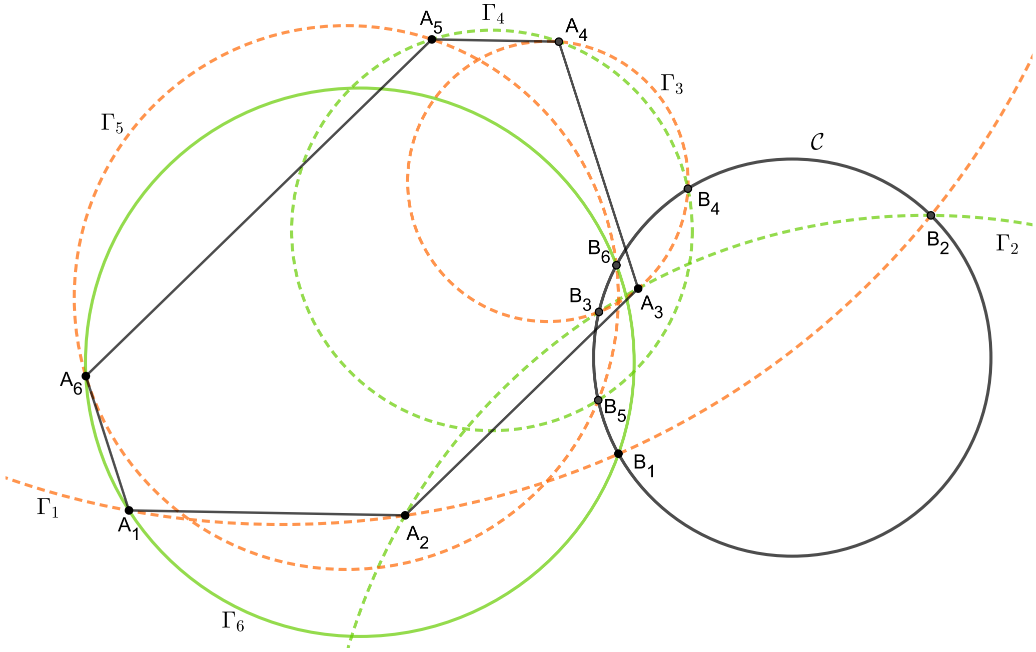


Figure 1

Note that, in order to show that B_{n+1} and B_1 coincide, it is sufficient to show that B_1 is on Γ_n or equivalently, that B_1, A_n, A_1 and B_n are concyclic. We use directed angles, for which we use the symbol $\measuredangle$.

Claim 1: $0 = \sum_{r=0}^{2k} \measuredangle A_{2r+1}A_{2r+2}A_{2r+3}$. *In Figure 2a, these are the angles drawn in green.*

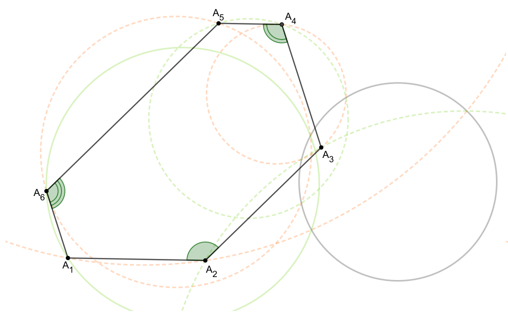


Figure 2a

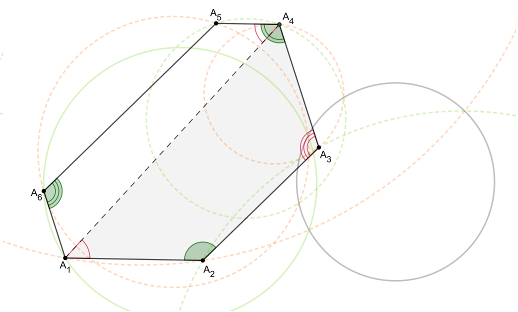


Figure 2b

Consider the closed polygon $A_1A_2 \cdots A_{2k+1}A_{2k+2}A_1$ *(shaded in Figure 2b)*. The sum of the directed angles in this polygon is 0; that is,

$$\measuredangle A_1A_2A_3 + \measuredangle A_2A_3A_4 + \cdots + \measuredangle A_{2k+1}A_{2k+2}A_1 + \measuredangle A_{2k+2}A_1A_2 = 0\,. \qquad (1)$$

$A_2A_1 \parallel A_{2k+2}A_{2k+3}$ gives us that $\measuredangle A_{2k+2}A_1A_2 = \measuredangle A_1A_{2k+2}A_{2k+3}$, so

$$\begin{aligned}\measuredangle A_{2k+1}A_{2k+2}A_1 + \measuredangle A_{2k+2}A_1A_2 &= \measuredangle A_{2k+1}A_{2k+2}A_1 + \measuredangle A_1A_{2k+2}A_{2k+3} \\ &= \measuredangle A_{2k+1}A_{2k+2}A_{2k+3}.\end{aligned}$$

Substitute in (1) to get

$$\sum_{j=1}^{2k+1} A_jA_{j+1}A_{j+2} = 0. \tag{2}$$

Finally, for each $1 \leqslant i \leqslant k$, we have

$$A_{2i}A_{2i+1} \parallel A_{2i+2k+1}A_{2i+2k+2} \quad \text{and} \quad A_{2i+1}A_{2i+2} \parallel A_{2i+2k+2}A_{2i+2k+3},$$

so

$$\measuredangle A_{2i}A_{2i+1}A_{2i+2} = \measuredangle A_{2i+2k+1}A_{2i+2k+2}A_{2i+2k+3}.$$

In our example, the equal angles are those decorated with three lines in Figure 2b; that is, $\measuredangle A_2A_3A_4 = \measuredangle A_5A_6A_1$. Substitute in (2) to obtain equality in Claim 1.

Claim 2: A_1, B_1, B_n, B_{n+1} are concyclic.

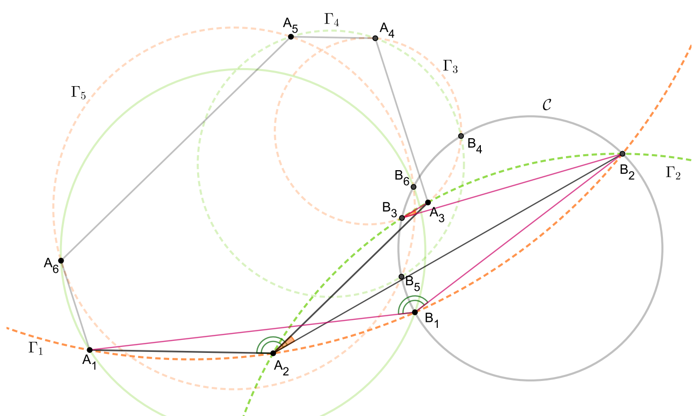


Figure 3

For $0 \leq r \leq 2k$ we have

$$\begin{aligned}\measuredangle A_{2r+1}A_{2r+2}A_{2r+3} &= \measuredangle A_{2r+1}A_{2r+2}B_{2r+2} + \measuredangle B_{2r+2}A_{2r+2}A_{2r+3} \\ &= \measuredangle A_{2r+1}B_{2r+1}B_{2r+2} + \measuredangle B_{2r+2}B_{2r+3}A_{2r+3},\end{aligned}$$

where in the last line we used the fact that by construction A_{2r+1}, A_{2r+2}, B_{2r+2} and B_{2r+1} are concyclic (they are all on Γ_{2r+1}), as are A_{2r+2}, B_{2r+2}, A_{2r+3} and B_{2r+3} (which are all on Γ_{2r+2}). *In Figure 3, we show the relevant angles in our example for the case* $r = 0$.

Substituting into the equality proven in Claim 1 we get

$$\begin{aligned}
0 &= \sum_{r=0}^{2k} \measuredangle A_{2r+1}A_{2r+2}A_{2r+3}\\
&= \sum_{r=0}^{2k} (\measuredangle A_{2r+1}B_{2r+1}B_{2r+2} + \measuredangle B_{2r+2}B_{2r+3}A_{2r+3})\\
&= \measuredangle A_1B_1B_2 + \measuredangle B_nB_{n+1}A_1 + \sum_{r=1}^{2k} (\measuredangle B_{2r}B_{2r+1}A_{2r+1} + \measuredangle A_{2r+1}B_{2r+1}B_{2r+2})\\
&= \measuredangle A_1B_1B_2 + \measuredangle B_{4k+2}B_{4k+3}A_1 + \sum_{r=1}^{2k} \measuredangle B_{2r}B_{2r+1}B_{2r+2}, \qquad (3)
\end{aligned}$$

where the third line in the above follows from regrouping the terms. By construction, all the B points are concyclic (they are all on the circle $\mathcal{C}$), so

$$\measuredangle B_{2r}B_{2r+1}B_{2r+2} = \measuredangle B_{2r}B_1B_{2r+2}.$$

Angle addition gives us that

$$\sum_{r=1}^{2k} \measuredangle B_{2r}B_1B_{2r+2} = B_2B_1B_{4k+2},$$

so from (3) and $n = 4k+2$ we conclude that

$$\begin{aligned}
0 &= \measuredangle A_1B_1B_2 + \measuredangle B_nB_{n+1}A_1 + \measuredangle B_2B_1B_n\\
&= \measuredangle A_1B_1B_n + \measuredangle B_nB_{n+1}A_1.
\end{aligned}$$

From this we deduce that A_1, B_1, B_n and B_{n+1} are concyclic, as claimed.

However, by the definition of the point B_{n+1}, the circle through $A_1 = A_{n+1}$, B_n, B_{n+1} is Γ_n, so B_1 is on Γ_n. Hence Γ_n and the circle C intersect at points B_1, B_n and B_{n+1}, so we must have that $B_1 = B_{n+1}$ as required.

Solution 2, by the proposer, slightly edited.

For $j = 1, \ldots, n$, associate the points A_j and B_j with complex numbers z_j and w_j. It suffices to show that the four points A_n, B_n, B_1, A_1 lie on the same circle; that is, that the cross ratio

$$R := (z_1, w_n; w_1, z_n) = \frac{(z_n - w_n)(w_1 - z_1)}{(z_n - z_1)(w_1 - w_n)}$$

is a real number.

Note that by construction, for $j = 1, \ldots, n-1$,

$$R_j := (z_{j+1}, w_j; w_{j+1}, z_j) = \frac{(z_j - w_j)(w_{j+1} - z_{j+1})}{(z_j - z_{j+1})(w_{j+1} - w_j)}$$

is a real number since A_j, B_j, A_{j+1} and B_{j+1} are concyclic. Consider the product

$$P = R_1 R_2^{-1} R_3 \cdots R_{n-2}^{-1} R_{n-1},$$

which must also be a real number. Since in the given polygon opposite sides are parallel, we have for $j = 1, \ldots, 2k+1$ that

$$\frac{z_{j+1} - z_j}{z_{2k+2+j} - z_{2k+1+j}}$$

is real. Thus up to sign or a real factor, by rearranging terms and cancellation, we get that P is equivalent to

$$\frac{(z_1 - w_1)(z_n - w_n)}{(z_{2k+1} - z_{2k+2})(w_1 - w_n)} \cdot \frac{(w_2 - w_3)(w_4 - w_5) \cdots (w_n - w_1)}{(w_1 - w_2)(w_3 - w_4) \cdots (w_{n-1} - w_n)}.$$

The first factor in the above formula is a real multiple of R since $A_{2k+1}A_{2k+2} \parallel A_n A_1$. To prove that R is real, it thus remains to prove that

$$\frac{(w_2 - w_3)(w_4 - w_5) \cdots (w_n - w_1)}{(w_1 - w_2)(w_3 - w_4) \cdots (w_{n-1} - w_n)}$$

is real. We note that this expression is equal to

$$-\prod_{j=1}^{2k} (w_{2j}, w_{2j+2}; w_{2j+1}, w_1).$$

Each of the cross ratios $(w_{2j}, w_{2j+2}; w_{2j+1}, w_1)$ is real since the points B_1, B_{2j}, B_{2j+1} and B_{2j+2} are on the same circle $\mathcal{C}$, concluding the proof.

Editor's comments. In the original question it was not addressed how to handle the case when A_i, A_{i+1} and B_i are collinear for some i. However, the proof should work the same way if we consider the line through A_i, A_{i+1} (and B_i) as a generalized circle, and define B_{i+1} as the second intersection of the line with $\mathcal{C}$.

4544. *Proposed by Burghelea Zaharia.*

Calculate

$$\int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right) \frac{dx}{x}.$$

We received 17 correct, 1 incomplete, and 1 incorrect solutions. The majority of the solutions used transformations using the substitution method for definite integrals and virtuoso elimination of non-elementary integrals. There were also some solutions which used power series and their definite integrals where Euler's answer for the Basel Problem was helpful. Some of the mistakes in the solutions *could have been avoided if the solvers checked their answer using, for example, software integral calculators. W. Janous and S. Jason generalized the problem by proving that*

$$\int_1^a \ln\left(\frac{x^4+a^2}{x^2+a^2}\right)\frac{dx}{x} = \frac{1}{2}(\ln a)^2.$$

Here we present the solution by the Missouri State University Problem Solving Group.

In the integral, let $x = 2y$, then

$$\begin{aligned} I &= \int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right)\frac{dx}{x} \qquad (1)\\ &= \int_{\frac{1}{2}}^1 \ln\left(\frac{4y^4+1}{y^2+1}\right)\frac{dy}{y}\\ &= \int_{\frac{1}{2}}^1 \ln(1+4y^4)\frac{1}{y}dy - \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy. \end{aligned}$$

In the first integral, let $z = 2y^2$, or $y = \sqrt{z}/\sqrt{2}$, then

$$\begin{aligned} I &= \int_{\frac{1}{2}}^2 \ln(1+z^2)\frac{1}{2z}dz - \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy\\ &= \frac{1}{2}\int_1^2 \ln(1+z^2)\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy\\ &= \frac{1}{2}\int_1^2 (\ln z^2 + \ln(1+z^{-2}))\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz\\ &= \int_1^2 \frac{\ln z}{z}dz + \frac{1}{2}\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz\\ &= \frac{1}{2}(\ln 2)^2 + \frac{1}{2}\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz. \qquad (2) \end{aligned}$$

However, let $y = 1/z$, we will have

$$\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz = \int_1^{\frac{1}{2}} \ln(1+y^2)y(-y^{-2})dy = \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy.$$

Hence, the second and the third terms in Eq. (1) add to zero.

As a result, we have

$$I = \int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2.$$

4545. *Proposed by Mihaela Berindeanu.*

Solve the following equation over $\mathbb{N}$:

$$6^n - 19 = \left[5\sqrt{n^2+4n}\right].$$

We received 36 submissions of which 33 were correct and complete. We present the solution of Jiahao Chen and Madhav Modak (independently), slightly modified.

The only solution to the equation is $n = 2$.

It is easy to check that $n = 1$ is not a solution.

When $n = 2$, we have $6^2 - 19 = 17 = \left[5\sqrt{12}\right]$, so $n = 2$ is a solution. Finally, if $n \geq 3$, by binomial theorem, the left-hand side of the equation is

$$6^n - 19 = (5+1)^n - 19 \geq 5^n + \binom{n}{1} \cdot 5 + 1 - 19 \geq 5^n + 5n - 18 \geq 5n + 107,$$

while the right-hand side of the equation is

$$\left[5\sqrt{n^2+4n}\right] < \left[5\sqrt{n^2+4n+4}\right] = 5n + 10,$$

thus the equation has no solution when $n \geq 3$.

4546. *Proposed by Thanos Kalogerakis, Leonard Giugiuc and Kadir Altintas.*

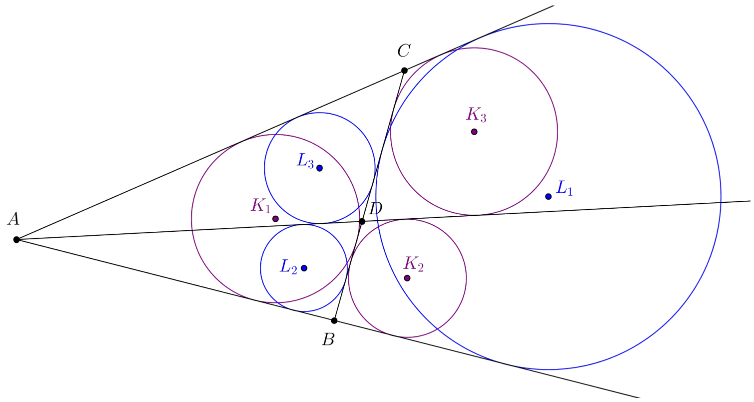
Let D be a point on the side BC of triangle ABC and consider the following tri-tangent circles:

(K_1, k_1) is the incircle and (L_1, l_1) is the A-excircle of ABC

(L_2, l_2) is the incircle and (K_2, k_2) is the A-excircle of ABD

(L_3, l_3) is the incircle and (K_3, k_3) is the A-excircle of ACD

Prove that Area$(\triangle K_1K_2K_3)$ = Area$(\triangle L_1L_2L_3)$.



Two versions of this problem appeared, due to a typo on our part. Of the 12 solutions received, all of which were correct, half addressed each version of the problem. We feature one of each.

Solution by Prithwijit De.

This version of the problem asked for a proof that $k_1k_2k_3 = l_1l_2l_3$.

Let $\angle DAB = 2\alpha$, $\angle DAC = 2\beta$ and $\angle ADB = 2\theta$. Then

$$\frac{l_1}{k_1} = \frac{4R_{ABC}\sin(A/2)\cos(B/2)\cos(C/2)}{4R_{ABC}\sin(A/2)\sin(B/2)\sin(C/2)} = \cot(B/2)\cot(C/2), \tag{1}$$

$$\frac{l_2}{k_2} = \frac{4R_{ABD}\sin\alpha\sin(B/2)\sin\theta}{4R_{ABD}\sin\alpha\cos(B/2)\cos\theta} = \tan(B/2)\tan\theta, \tag{2}$$

$$\frac{l_3}{k_3} = \frac{4R_{ACD}\sin\beta\sin(C/2)\cos\theta}{4R_{ACD}\sin\beta\cos(C/2)\sin\theta} = \tan(C/2)\cot\theta. \tag{3}$$

Multiplying the three equations we get

$$k_1k_2k_3 = l_1l_2l_3.$$

Solution by Andrea Fanchini.

This version of the problem asked for a proof that the area of triangle $K_1K_2K_3$ equals the area of triangle $L_1L_2L_3$.

We use barycentric coordinates with reference to the triangle ABC. We have

$$K_1 = (a : b : c), \quad L_1 = (-a : b : c), \quad D = (0 : t : 1-t),$$

for some t, $0 \le t \le 1$. This implies that $CD = ta$, $BC = (1-t)a$, and, by the Law of Cosines, $AD = \sqrt{a^2t^2 + b^2 - t\,(a^2 + b^2 - c^2)}$. The barycentric coordinates of the other points of interest are then

$$\begin{aligned} K_2 &= (a(t-1) : p + ct : c(1-t)) \\ K_3 &= (-at : bt : p + b(1-t)) \\ L_2 &= (a(1-t) : p + ct : c(1-t)) \\ L_3 &= (at : bt : p + b(1-t))\,, \end{aligned}$$

where

$$p = AD = \sqrt{a^2t^2 + b^2 - t\,(a^2 + b^2 - c^2)}.$$

We thus have

$$\begin{aligned} [\Delta K_1K_2K_3] &= [\Delta L_1L_2L_3] \\ &= \frac{[\Delta ABC]}{4s(s-a)(b-(b-c)t+p)}\begin{vmatrix} -a & b & c \\ a(1-t) & p+ct & c(1-t) \\ at & bt & p+b(1-t) \end{vmatrix}. \end{aligned}$$

4547. *Proposed by George Stoica, modified by the Editorial Board.*

Consider the complex numbers a, b, c such that $|a| = |b| = |c| = 1$. Prove that if

$$|a + b - c|^2 + |b + c - a|^2 + |c + a - b|^2 = 12,$$

then a, b, c represent the vertices of an equilateral triangle inscribed in the unit circle.

We received 28 submissions, one of which was incorrect. We feature two of them.

Solution 1. A composite of similar solutions by Jiahao Chen, Madhav Modak, and Ioannis Sfikas.

Because $|z| = 1$ implies that $\bar{z} = \frac{1}{z}$, we see that the assumption $|a + b - c|^2 + |b + c - a|^2 + |c + a - b|^2 = 12$ has the following sequence of implications:

$$\begin{aligned}
\sum_{cyc}(a + b - c)\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) &= 12 \\
\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a} - \frac{b}{c} - \frac{c}{b} - \frac{c}{a} - \frac{a}{c}\right) &= 3 \\
\sum_{cyc}\left(\frac{a}{b} + \frac{b}{a}\right) &= -3 \\
(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}) &= 0 \\
|a + b + c|^2 &= 0.
\end{aligned}$$

It follows that $a + b + c = 0$. This implies that if the vertices of a triangle are represented by a, b, c, then its centroid (namely $\frac{a+b+c}{3}$) coincides at the origin with its circumcenter, which can happen if and only if the triangle is equilateral.

Solution 2. A generalization by Corneliu Manescu-Avram.

Theorem. Let $t \neq \frac{1}{2}$ be a real number and a, b, c be complex numbers with modulus 1. If

$$\sum_{cyc}|b + c + ta|^2 = 3(t - 1)^2,$$

then the points represented by a,b,c are the vertices of an equilateral triangle.

Indeed, for $h = a + b + c$, we have

$$\begin{aligned}
3(t - 1)^2 = \sum_{cyc}|b + c + ta|^2 &= \sum_{cyc}|h + (t - 1)a|^2 \\
&= \sum_{cyc}(h + (t - 1)a)(\bar{h} + (t - 1)\bar{a}) \\
&= (2t + 1)|h|^2 + 3(t - 1)^2,
\end{aligned}$$

whence $h = 0$ and the conclusion follows [as in solution 1]. Note that in the original problem, we are given $t = -1$.

Editor's comments. Borche Joshevski generalized the problem to a given set of $n > 1$ complex numbers $z_1, \ldots, z_n$. He showed that

$$\sum_{k=1}^{n} \left| -2z_k + \sum_{m=1}^{n} z_m \right|^2 = 4\sum_{k=1}^{n} |z_k|^2 + (n-4)\left|\sum_{k=1}^{n} z_k\right|^2.$$

From this he concluded that for $n \neq 4$,

$$\sum_{k=1}^{n} \left| -2z_k + \sum_{m=1}^{n} z_m \right|^2 = 4\sum_{k=1}^{n} |z_k|^2$$

is equivalent to $\sum_{k=1}^{n} z_k = 0$, which occurs if and only if the centroid of the polygon whose vertices correspond to the given complex numbers coincides with the origin.

Mihaela Berindeanu informed us that she proposed a problem that was identical to our 4547 for use on a Romanian mathematics contest in March of 2019. The problem was subsequently published in a supplement to the *Gazeta Matematică* (dealing with the National Olympiad that was written in Deva, Romania, April 22-26, 2019). By coincidence, the journal appeared in 2019 at almost the same time as Stoica's problem appeared in *Crux*.

4548. *Proposed by Lazea Darius.*

Find the maximum k for which

$$ab + bc + ca + k(a-b)^2(b-c)^2(c-a)^2 \leq 3$$

for all non-negative real numbers a,b,c such that $a + b + c = 3$.

We received 12 submissions of which 5 were correct and complete. We present the solution by the UCLan Cyprus Problem Solving Group.

Since

$$\begin{aligned}(a-b)^2 + (b-c)^2 + (c-a)^2 &= 2(a^2 + b^2 + c^2) - 2(ab + bc + ca)\\ &= 2(a + b + c)^2 - 6(ab + bc + ca)\\ &= 18 - 6(ab + bc + ca)\end{aligned}$$

the problem becomes to maximize k such that

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geqslant 6k(a-b)^2(b-c)^2(c-a)^2$$

under the stated conditions. Without loss of generality we assume that $a \geqslant b \geqslant c$. Let $x = a - b$ and $y = b - c$. Then $x, y \geqslant 0$, $x + 2y = 3 - 2c$, and we are to maximize k so that

$$x^2 + y^2 + (x+y)^2 \geqslant 6k(xy(x+y))^2$$

holds for all $x, y \geqslant 0$ for which $x + 2y \leqslant 3$. (Given any such x, y it is easy to determine a, b, c and check that they satisfy the given conditions.)

We may assume that $x, y > 0$ since otherwise the inequality is satisfied for any k. We will maximize

$$\frac{(xy(x+y))^2}{x^2+y^2+(x+y)^2}$$

under the conditions $x, y > 0$ and $x + 2y \leqslant 3$.

If $x + 2y < 3$, letting $x' = rx$ and $y' = ry$ where $r = \frac{3}{x+2y}$, we have $x', y' > 0$, $r > 1$, and $x' + 2y' = 3$. Thus,

$$\frac{(x'y'(x'+y'))^2}{(x')^2+(y')^2+(x'+y')^2} = r^4 \frac{(xy(x+y))^2}{x^2+y^2+(x+y)^2} > \frac{(xy(x+y))^2}{x^2+y^2+(x+y)^2}$$

and we may assume that $x + 2y = 3$. Letting $y = t$ and $x = 3 - 2t$, we will maximize

$$f(t) = \frac{((3-2t)t(3-t))^2}{(3-2t)^2+t^2+(3-t)^2}$$

for $t \in (0, 3/2)$. Motivated by the symmetry of f about $t = 3/2$, we make the substitution $t = s + 3/2$ and for $s \in (-3/2, 0)$ let

$$g(s) = f\left(s + \frac{3}{2}\right) = \frac{(-2s(s+\frac{3}{2})(\frac{3}{2}-s))^2}{(-2s)^2+(s+\frac{3}{2})^2+(\frac{3}{2}-s)^2} = \frac{4s^2\left(s^2-\frac{9}{4}\right)^2}{6s^2+\frac{9}{2}} = \frac{s^2(4s^2-9)^2}{6(4s^2+3)}.$$

Next, we let $u = s^2$ and for $u \in (0, 9/4)$ we will maximize

$$h(u) = 6g(\sqrt{u}) = \frac{u(4u-9)^2}{4u+3}.$$

We have

$$\begin{aligned} h'(u) &= \frac{(4u+3)((4u-9)^2+8u(4u-9)) - 4u(4u-9)^2}{(4u+3)^2} \\ &= \frac{(4u-9)((4u+3)(12u-9) - 4u(4u-9))}{(4u+3)^2} \\ &= \frac{(4u-9)(32u^2+36u-27)}{(4u+3)^2}. \end{aligned}$$

The zeroes of $32u^2 + 36u - 27$ are $u_+ = -\frac{3}{16}(3 - \sqrt{33})$ and $u_- = -\frac{3}{16}(3 + \sqrt{33})$ with

$$u_- < 0 < u_+ < \frac{3(6-3)}{16} < \frac{9}{4}.$$

Furthermore,

$$h'(u) = \frac{32(4u-9)(u-u_+)(u-u_-)}{(4u+3)^2}$$

and so, $h'(u) > 0$ in $(0, u_+)$ and $h'(u) < 0$ in $(u_+, 9/4)$. So, h is maximized at u_+ with maximum

$$\begin{aligned} h(u_+) &= \frac{16u_+(16u_+ - 36)^2}{64(16u_+ + 12)} \\ &= \frac{3(\sqrt{33} - 3)(3\sqrt{33} - 45)^2}{64(3\sqrt{33} + 3)} \\ &= \frac{27(69 - 11\sqrt{33})}{32}. \end{aligned}$$

The maximum of g is therefore

$$\frac{9(69 - 11\sqrt{33})}{64}$$

and is achieved at $s = -\frac{1}{4}\sqrt{3\sqrt{33} - 9}$. The maximum of f is the same as that of g and is achieved at $t = \frac{3}{2} - \frac{1}{4}\sqrt{3\sqrt{33} - 9}$. Hence,

$$k \leqslant \frac{64}{6 \cdot 9(69 - 11\sqrt{33})} = \frac{32(69 + 11\sqrt{33})}{27 \cdot 768} = \frac{69 + 11\sqrt{33}}{648}.$$

Equality is achieved when

$$c = 0, \ b = y = \frac{3}{2} - \frac{1}{4}\sqrt{3\sqrt{33} - 9} \text{ and } a = 3 - b = \frac{3}{2} + \frac{1}{4}\sqrt{3\sqrt{33} - 9}.$$

4549. *Proposed by Lorian Saceanu, Leonard Giugiuc and Kadir Altintas.*

Let a_k and b_k be real numbers for $k = 1, 2, \ldots, n$. Prove that

$$\sqrt{\sum_{k=1}^{n}(2a_k - b_k)^2} + \sqrt{\sum_{k=1}^{n}(2b_k - a_k)^2} \geq \sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}.$$

There were 13 correct solutions. We present three of them here.

Solution 1, by Anil Kumar, Prithwijit De, C.R. Pranesachar, and Zoltan Retkes, all done independently.

Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$. Then

$$2\|a\| = \|2a - b + b\| \leq \|2a - b\| + \|b\|$$

and

$$2\|b\| = \|2b - a + a\| \leq \|2b - a\| + \|a\|,$$

where $\|x\|$ is the Euclidean norm of the $n-$vector x. Adding these inequalities and rearranging terms yields the required

$$\|2a - b\| + \|2b - a\| \geq \|a\| + \|b\|.$$

Equality occurs when $2a - b = \lambda b$ and $2b - a = \mu a$ where λ and μ are nonnegative and satisfy $4 = (1+\lambda)(1+\mu)$. Thus, both must lie in the interval $[0, 3]$. Hence, equality holds if and only if $a = \nu b$ where

$$\frac{1}{2} \le \nu = \frac{1+\lambda}{2} = \frac{2}{1+\mu} \le 2.$$

Solution 2, by Oliver Geupel.

Let a and b be the vectors defined in Solution 1. Suppose that ABC is the triangle whose respective vertices are at $2a$, $2b$ and 0 in $n-$space. Let V be the midpoint of AC, U be the midpoint of BC and G be the intersection of the medians AU and BV. Then

$$\begin{aligned} \|2a-b\| + \|2b-a\| &= |AU| + |BV| \\ &= |AG| + |GU| + |BG| + |GV| \\ &= |AG| + |GV| + |BG| + |GU| \\ &\ge |AV| + |BU| \\ &= |VC| + |CU| \\ &= \|a\| + \|b\|. \end{aligned}$$

Solution 3, by Marie-Nicole Gras.

Let $x = \left(\sum a_i^2\right)^{1/2}$, $y = \left(\sum b_i^2\right)^{1/2}$, $z = \sum a_i b_i$.

By the Cauchy-Schwarz Inequality, $x^2y^2 \ge z^2$. Now

$$\sum (2a_i - b_i)^2 = 4x^2 + y^2 - 4z \ge 4x^2 + y^2 - 4xy = (2x-y)^2$$

and

$$\sum (2b_i - a_i)^2 \ge (2y - x)^2.$$

Hence the left side of the desired inequality is greater than or equal to

$$|2x - y| + |2y - x| \ge |(2x-y) + (2y-x)| = x + y.$$

Comment by the editor. Walther Janous generalized the inequality to

$$\sum_{k=1}^{m} \|\lambda x_k - x_{k+1} - \cdots - x_{k+m-1}\| \ge (\lambda - m + 1) \sum_{k=1}^{m} \|x_k\|$$

and Borche Joshevski to

$$\|sa - tb\| + \|sb - ta\| \ge |s - t|(\|a\| + \|b\|),$$

where s and t are nonnegative.

4550. *Proposed by Leonard Giugiuc and Kunihiko Chikaya.*

Let α be a real number greater than 2 and let x, y and z be positive real numbers such that $x \geq y \geq z$. Prove that

$$\frac{(x^\alpha - y^\alpha)(y^\alpha - z^\alpha)(z^\alpha - x^\alpha)}{(x^{\alpha-1} + y^{\alpha-1})(y^{\alpha-1} + z^{\alpha-1})(z^{\alpha-1} + x^{\alpha-1})} \geq \frac{\alpha^3}{24}((x-y)^3 + (y-z)^3 + (z-x)^3).$$

When does equality hold?

We received 11 correct solutions and 1 incomplete submission. We present the solution by Michel Bataille.

Since $(x-y) + (y-z) + (z-x) = 0$, the identity

$$X^3 + Y^3 + Z^3 - 3XYZ = (X+Y+Z)(X^2 + Y^2 + Z^2 - XY - YZ - ZX)$$

yields

$$(x-y)^3 + (y-z)^3 + (z-x)^3 = 3(x-y)(y-z)(z-x).$$

We deduce that equality holds if at least two of the numbers x, y, z are equal (then both sides are 0). Otherwise, that is, if $x > y > z$, we show that the strict inequality holds, or equivalently, that

$$\begin{aligned}&(x^\alpha - y^\alpha)(y^\alpha - z^\alpha)(x^\alpha - z^\alpha)\\ &< \frac{\alpha^3}{8}(x-y)(y-z)(x-z)(x^{\alpha-1} + y^{\alpha-1})(y^{\alpha-1} + z^{\alpha-1})(z^{\alpha-1} + x^{\alpha-1})\end{aligned}$$

or

$$f\left(\frac{y}{x}\right) \cdot f\left(\frac{z}{y}\right) \cdot f\left(\frac{z}{x}\right) < 1,$$

where

$$f(t) = \frac{2(1-t^\alpha)}{\alpha(1-t)(1+t^{\alpha-1})}.$$

Since $\frac{y}{x}, \frac{z}{y}, \frac{z}{x}$ are in the interval $(0,1)$ and $f(t) > 0$ if $t \in (0,1)$, it suffices to prove that $f(t) < 1$ or equivalently that $g(t) = \alpha - 2 - \alpha t + \alpha t^{\alpha-1} - (\alpha-2)t^\alpha$ is positive whenever $t \in (0,1)$.

Now $g'(t) = -\alpha h(t)$ with $h(t) = 1 - (\alpha-1)t^{\alpha-2} + (\alpha-2)t^{\alpha-1}$.

For $t \in (0,1)$ we have $h'(t) = -(\alpha-1)(\alpha-2)(1-t)t^{\alpha-3} < 0$, hence $h(t) > h(1) = 0$ and so $g'(t) < 0$ and therefore $g(t) > g(1) = 0$. This completes the proof.

Editor's comments: All of the solvers used the methods of Differential and Integral Calculus. For the majority of the solvers, the convexity of some functions was the main tool for the proof of the inequality.