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# MathemAttic

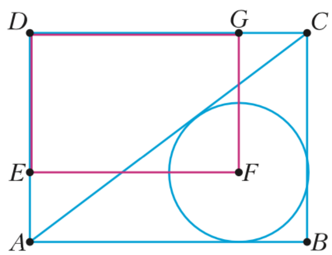
## No. 24

*The problems in this section are intended for students at the secondary school level.*

***Click here to submit solutions, comments and generalizations to any problem in this section.***

*To facilitate their consideration, solutions should be received by* **June 15, 2021**.

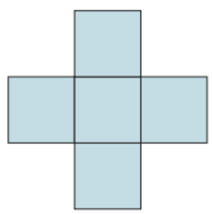
**MA116**. Let $ABCD$ and $DEFG$ be two rectangles so that the point $E$ lies on the side $AD$, the point $G$ lies on the side $CD$ and the point $F$ is the incenter of $\triangle ABC$. What is the ratio of the area of $ABCD$ and the area of $DEFG$?




**MA117**. For which natural numbers $n$ can the set $\{1, 2, \ldots, n\}$ be partitioned into two subsets so that the sum of numbers in one subset equals to the product of the numbers in the other subset?

**MA118**. Can you colour all natural numbers using exactly 7 colours so that the product of any two (not necessarily distinct) numbers of the same colour results in a number of that same colour? For example, if 3 and 4 and coloured red, then 9, 12 and 16 must also be coloured red.

**MA119**. Alice and Bob are playing tic-tac-toe on an infinite grid. The winner is declared when they place their sign over 5 squares in the shape of a plus. If Alice goes first, can Bob always prevent her from winning?



**MA120**. With grid paper and pencil, it is easy to draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths; for example, the so-called Egyptian triangle with side 3, 4 and 5 will do. Can you draw a right-angle triangle with vertices on intersections of grid lines and with integer side-lengths, but so that none of its sides follows grid lines?

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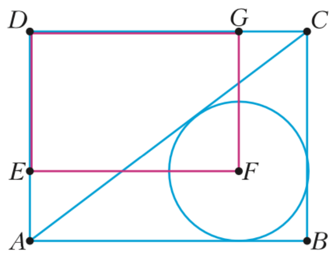
*Les problèmes proposés dans cette section sont appropriés aux étudiants de l'école secondaire.*

***Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le* **15 juin 2021**.

*La rédaction souhaite remercier Rolland Gaudet, professeur titulaire à la retraite à l'Université de Saint-Boniface, d'avoir traduit les problèmes.*

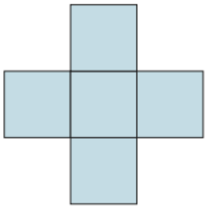
**MA116**. Soient $ABCD$ et $DEFG$ deux rectangles tels que le point $E$ se situe sur le côté $AD$, le point $G$ se situe sur le côté $CD$, et le point $F$ est le centre du cercle inscrit de $\triangle ABC$. Déterminer le ratio de la surface de $ABCD$ par rapport à celle de $DEFG$.




**MA117**. Déterminer pour lesquels nombres naturels $n$ l'ensemble $\{1, 2, \ldots, n\}$ peut être partitionné en deux de façon à ce que la somme des entiers dans un des deux sous ensembles égale le produit des nombres dans le deuxième.

**MA118**. Est-il possible de colorer tous les entiers naturels à l'aide d'exactement 7 couleurs, de façon à ce que le produit de deux nombres de même couleur donne toujours un nombre de la même couleur? (Par exemple, si 3 et 4 sont de la même couleur, alors 9, 12 et 16 sont aussi de cette même couleur.)

**MA119**. Alice et Bernard jouent à une variante de morpion (tic-tac-toe) sur une grille infinie. Dans ce jeu particulier, un gagnant est déclaré lorsque le joueur réussit à placer sa marque sur 5 cases en forme d'un plus. Si Alice joue première, Bernard peut-il l'empêcher de gagner ?



**MA120**. Sur du papier avec grille, il est aisé de placer aux points d'intersection de la grille les sommets d'un triangle rectangle quelconque à côtés entiers, par exemple le triangle 3-4-5 bien connu, deux des côtés étant aisément alignés avec la grille. Existe-t-il un triangle rectangle à côtés entiers dont les sommets peuvent être placés aux points d'intersection de la grille, mais dont aucun des côtés ne suit l'orientation du grillage ?

# MathemAttic SOLUTIONS

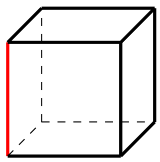
*Statements of the problems in this section originally appear in 2020: 46(9), p. 436–437.*

**MA91**. The points $(1, 2, 3)$ and $(3, 3, 2)$ are vertices of a cube. Compute the product of all possible distinct volumes of the cube.

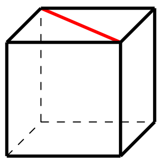
*Originally from American Regions Mathematics League (ARML) 2019 Contest, Team problems, problem 1.*

*We received 7 submissions of which 6 were correct and complete. We present the solution by Doddy Kastanya.*

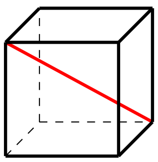
There are three types of cubes that could be constructed depending on the location of the vertices. Each case is depicted below.



(1) Vertices share an edge.



(2) Vertices share a face but not an edge.



(3) Vertices are in opposite corners.

The red lines in the above figures depict the distance between the two vertices.

Let $\lambda$ denote the distance between the two points. The volumes of the cubes in (1), (2), and (3) are given by

$$\lambda^3,\ \left(\frac{\lambda}{\sqrt{2}}\right)^3,\ \text{and}\ \left(\frac{\lambda}{\sqrt{3}}\right)^3,$$

respectively. The desired product is therefore

$$\lambda^3 \times \left(\frac{\lambda}{\sqrt{2}}\right)^3 \times \left(\frac{\lambda}{\sqrt{3}}\right)^3.$$

As

$$\lambda = \sqrt{(1-3)^2 + (2-3)^2 + (3-2)^2} = \sqrt{6},$$

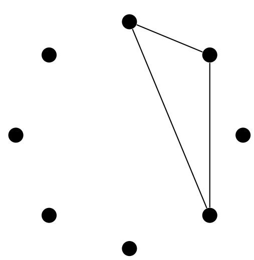
the product of all possible distinct volumes of the cube is 216.

**MA92**. Eight students attend a soccer practice. At the end of the practice, they decide to take selfies to celebrate the event. Each selfie will have either two or three students in the picture. Compute the minimum number of selfies so that each pair of the eight students appears in exactly one selfie.

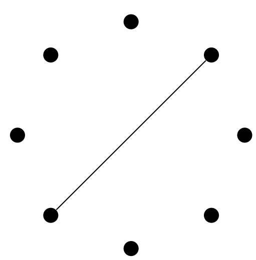
*Originally from American Regions Mathematics League (ARML) 2019 Contest, Team problems, problem 2.*

*We received 2 submissions of which 1 was correct and complete. We present the solution by Richard Hess, modified by the editor.*

Each student must appear in a selfie with each of the seven other students exactly once. The smallest number of selfies for each student is therefore four, three with two other students and one with one other student. The minimum number is then $3 \cdot 8/3 + 8/2 = 12$. The figure below shows how this is achieved. Consider the students in a circle as shown and, for each student, draw a triangle between them and the students one and three clockwise positions away. The triangles formed represent eight three person selfies. Once complete for each student, all pairings of students with the exception of four have occurred exactly once. The remaining pairings are achieved by matching opposite students. This represents two person selfies. The result is each pair of students is in exactly one of the twelve selfies.



Three person selfie



Two person selfie

**MA93**. Consider the system of equations

$$\begin{aligned}\log_4 x + \log_8(yz) &= 2,\\ \log_4 y + \log_8(xz) &= 4,\\ \log_4 z + \log_8(xy) &= 5.\end{aligned}$$

Given that $xyz$ can be expressed in the form $2^k$, compute $k$.

*Originally from American Regions Mathematics League (ARML) 2019 Contest, Team problems, problem 5.*

*We received 13 submissions of which 11 were correct and complete. We present the solution by William Digout.*

Since our goal is to compute $k$ and not $x$, $y$ and $z$, add the three given equations to get

$$\log_4 x + \log_4 y + \log_4 z + \log_8(yz) + \log_8(xz) + \log_8(xy) = 11.$$

Using the laws of logarithms, we can combine the six terms on the left hand side of the equation into two terms:

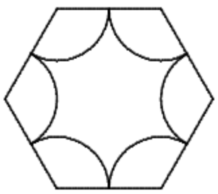
$$\log_4(xyz) + \log_8(x^2y^2z^2) = 11.$$

Substitute $xyz = 2^k$ to get $\log_4 2^k + \log_8 2^{2k} = 11$, which is equivalent to

$$k \log_4 2 + 2k \log_8 2 = 11.$$

Since $\log_4 2 = \frac{1}{2}$ and $\log_8 2 = \frac{1}{3}$ we find that $\frac{k}{2} + \frac{2k}{3} = 11$, which we solve to obtain $k = \frac{66}{7}$.

**MA94**. At each vertex of a regular hexagon, a sector of a circle of radius one-half of the side of the hexagon is removed. Find the fraction of the hexagon remaining.



*Originally from 2011 18th Blundon Mathematical Contest, Question 5.*

*We received* 6 *solutions, of which* 5 *were correct. The only incorrect solution was wrong due to a misunderstanding of what the problem was requesting as a final answer. We present the solution by Muhammad Robith.*

Let the side length of the hexagon be $s$. First, we compute the hexagon's area, It is well-known that one can dissect the hexagon into 6 equilateral triangles by connecting opposite vertices. Letting $H$ be the area of the hexagon, we get from the sine area formula for a triangle that

$$H = 6 \cdot \frac{1}{2} \cdot s^2 \cdot \sin 60^\circ = \frac{3\sqrt{3}}{2} \cdot s^2.$$

Next, we compute the total area of the six sectors. Since the hexagon is regular, all of its interior angles are $120^\circ$. From this, we know that each sector is one-third of a circle. Let $C$ be the total area of the six sectors. Then we have

$$C = 6 \cdot \frac{1}{3} \cdot \pi \left(\frac{1}{2}s\right)^2 = \frac{\pi}{2} \cdot s^2.$$

Let $F$ be the area of hexagon remaining after all of the sectors are removed. Then

$$F = H - C \quad \iff \quad \frac{F}{H} = 1 - \frac{C}{H} = 1 - \frac{\left(\frac{\pi}{2}\cdot s^2\right)}{\left(\frac{3\sqrt{3}}{2}\cdot s^2\right)} = 1 - \frac{\pi}{3\sqrt{3}}.$$

Therefore, the answer is $1 - \dfrac{\pi}{3\sqrt{3}}$.

**MA95**. Find the smallest and the largest prime factors of $M$, where

$$M = 1 + 2 + 3 + \cdots + 2017 + 2018 + 2019 + 2018 + 2017 + \cdots + 3 + 2 + 1.$$

*Originally from 2019 36th Blundon Mathematical Contest, Question 3.*

*We received* 9 *solutions, of which* 7 *were correct. One incorrect solution was unclear and the other was incomplete. We present the solution by Konstantine Zelator.*

It is a well-known fact that, for all positive integers $n$,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Using this, we may determine that

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) + n + \cdots + 2 + 1 &= \frac{n(n+1)}{2} + (n+1) + \frac{n(n+1)}{2} \\ &= n(n+1) + (n+1) \\ &= (n+1)^2. \end{aligned}$$

In the special case of $n = 2018$, we obtain that

$$1 + 2 + \cdots + 2018 + 2019 + 2018 + \cdots + 2 + 1 = 2019^2.$$

Since the sum of the digits of 2019 is

$$2 + 0 + 1 + 9 = 12 = 3 \cdot 4,$$

we check for the quotient upon divisibility by 3 and find that

$$2019^2 = 3^2 \cdot 673^2.$$

If we can show that 673 is a prime, then the answer will be that 3 is the smallest prime factor and 673 is the largest prime factor. To this end, we note that

$$25^2 = 625 < 673 < 676 = 26^2 \implies 25 < \sqrt{673} < 26,$$

so it suffices to prove that no prime less than 25 divides 673. These primes are

$$2, 3, 5, 7, 11, 13, 17, 19, 23.$$

Since 673 does not have an even units digit, it is not divisible by 2. The sum of the digits of 673 is not divisible by 3, so 673 is not divisible by 3. The units digit of 673 is not 0 or 5, so 673 is not divisible by 5. Finally, the sum of the digits of 673 with alternating signs is not divisible by 11, so 673 is not divisible by 11. Now we use Euclidean division to find that

$$\begin{aligned}
673 &= 7 \cdot 96 + 1,\\
673 &= 13 \cdot 51 + 10,\\
673 &= 17 \cdot 39 + 10,\\
673 &= 19 \cdot 35 + 8,\\
673 &= 23 \cdot 29 + 6,
\end{aligned}$$

so 673 is divisible by none of these primes. Therefore, the answer is 3 and 673.

# OLYMPIAD CORNER

No. 392

*The problems featured in this section have appeared in a regional or national mathematical Olympiad.*

***Click here to submit solutions, comments and generalizations to any problem in this section***

*To facilitate their consideration, solutions should be received by* **June 15, 2021**.

**OC526**. Let $ABC$ be a triangle. The circle $\omega_A$ through $A$ is tangent to line $BC$ at $B$. The circle $\omega_C$ through $C$ is tangent to line $AB$ at $B$. Let $\omega_A$ and $\omega_C$ meet again at $D$. Let $M$ be the midpoint of line segment $BC$, and let $E$ be the intersection of lines $MD$ and $AC$. Show that $E$ lies on $\omega_A$.

**OC527**. Anna and Boris play a game with $n$ counters. Anna goes first, and turns alternate thereafter. In each move, a player takes either 1 counter or a number of counters equal to a prime divisor of the remaining number of counters. The player who takes the last counter wins. For which $n$ does Anna have a winning strategy?

**OC528**. Let $\{z_n\}_{n\geq 1}$ be a sequence of complex numbers, whose odd-indexed terms are real, even-indexed terms are purely imaginary, and for every positive integer $k$, $|z_k z_{k+1}| = 2^k$. Denote $f_n = |z_1 + z_2 + \cdots + z_n|$, for $n = 1, 2, \ldots$.

(1) Find the minimum of $f_{2020}$.

(2) Find the minimum of $f_{2020} \cdot f_{2021}$.

**OC529**. Let $ABCD$ be a cyclic quadrilateral with $E$, an interior point such that $AB = AD = AE = BC$. Let $DE$ meet the circumcircle of $BEC$ again at $F$. Suppose a common tangent to the circumcircle of $BEC$ and $DEC$ touches the circles at $F$ and $G$ respectively. Show that $GE$ is the external angle bisector of angle $BEF$.

**OC530**. Let $n > 3$ be a fixed integer and $x_1, x_2, \ldots, x_n$ positive real numbers. Find in terms of $n$, all possible values of

$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-2} + x_{n-1} + x_n} + \frac{x_n}{x_{n-1} + x_n + x_1}.$$

*Les problèmes présentés dans cette section ont déjà été présentés dans le cadre d'une olympiade mathématique régionale ou nationale.*

***Cliquez ici afin de soumettre vos solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

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**OC526**. Soit $ABC$ un triangle. Le cercle $\omega_A$ passant par $A$ est tangent à la ligne $BC$ en $B$, tandis que le cercle $\omega_C$ passant par $C$ est tangent à la ligne $AB$ en $B$. Supposons que $\omega_A$ et $\omega_C$ se recoupent de nouveau en $D$. Soit alors $M$ le point milieu du segment $BC$ et $E$ le point d'intersection des lignes $MD$ et $AC$. Démontrer que $E$ se trouve sur $\omega_A$.

**OC527**. Anne et Bonaventure pratiquent un jeu qui utilise une pile ayant $n$ pièces au départ. Anne joue première et par la suite les deux alternent, le jeu consistant à enlever soit 1 seule pièce soit un nombre de pièces égal à un diviseur premier du nombre de pièces restantes dans la pile. Le joueur qui enlève la dernière pièce gagne. Pour quels $n$ Anne a-t-elle une stratégie gagnante ?

**OC528**. Soit $\{z_n\}_{n\geq 1}$ une suite de nombres complexes, dont les termes d'indice impair sont réels et les termes d'indice paire sont purement imaginaires; aussi, pour tout entier $k$ on a $|z_k z_{k+1}| = 2^k$. Dénoter $f_n = |z_1 + z_2 + \cdots + z_n|$ pour $n = 1, 2, \ldots$.

(1) Déterminer le minimum possible pour $f_{2020}$.

(2) Déterminer le minimum possible de $f_{2020} \cdot f_{2021}$.

**OC529**. Soit $ABCD$ un quadrilatère cyclique et soit $E$ un point dans son intérieur tel que $AB = AD = AE = BC$, où le cercle circonscrit de $BEC$ rencontre $DE$ de nouveau en $F$. Supposer qu'une tangente commune aux cercles circonscrits de $BEC$ et $DEC$ touche ces cercles en $F$ et $G$ respectivement. Démontrer que $GE$ est la bissectrice extérieure de l'angle $BEF$.

**OC530**. Soit $n > 3$ un entier donné et $x_1, x_2, \ldots, x_n$ des nombres réels positifs arbitraires. En termes de $n$, déterminer toutes les valeurs possibles de

$$\frac{x_1}{x_n + x_1 + x_2} + \frac{x_2}{x_1 + x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-2} + x_{n-1} + x_n} + \frac{x_n}{x_{n-1} + x_n + x_1}.$$

# OLYMPIAD CORNER SOLUTIONS

*Statements of the problems in this section originally appear in 2020: 46(9), p. 445–447.*

**OC501**. Pavel alternately writes crosses and circles in the cells of a rectangular table (starting with a cross). When the table is completely filled, the resulting score is calculated as the difference $O - X$, where $O$ is the total number of rows and columns containing more circles than crosses and $X$ is the total number of rows and columns containing more crosses than circles.

**(a)** Prove that for a $2 \times n$ table the resulting score will always be 0.

**(b)** Determine the highest possible score achievable for the table $(2n+1)\times(2n+1)$ depending on $n$.

*Originally from 2018 Czech-Slovakia Math Olympiad, Problem 1, Category A.*

*We received 3 solutions, of which 2 were correct and complete. We present the solution by Sergey Sadov.*

**(a)** Let $O_r$ be the number of rows with more circles than crosses and let $O_c$, $X_r$ and $X_c$ have the similar meaning. Then $O - X = O_r + O_c - X_r - X_c$. The total number of crosses is the same as the total number of circles, so either $O_r = X_r = 1$ or $O_r = X_r = 0$. In any case, $O - X = O_c - X_c$.

Let $a$ be the number of columns of the form (in transposed notation) $(XX)$. Similarly, let $b$, $c$, $d$ be the numbers of columns of the form $(XO)$, $(OX)$, and $(OO)$, respectively. Then $X_c = a$, $O_c = d$.

The total number of crosses is $2a+b+c$ and the total number of circles is $b+c+2d$. It follows that $a = d$, hence $O - X = d - a = 0$.

**(b)** Answer: $\max(O - X) = 4n - 2$.

There are $2n(n + 1)$ circles in the filled table. Hence there exist a row and a column containing less than $n+1$ circles. So $X \geq 2$; correspondingly, $O \leq 4n$ and $O - X \leq 4n - 2$.

Let us show that a configuration with $O = 4n$, $X = 2$ does exist. The case $n = 0$ is trivial. The examples for $n = 1, 2, 3$ are pictured below.

| | | |
|---|---|---|
| o | o | x |
| o | o | x |
| x | x | x |

| | | | | |
|---|---|---|---|---|
| o | o | o | x | x |
| o | o | x | o | x |
| o | x | o | o | x |
| x | o | o | o | x |
| x | x | x | x | x |

| | | | | | | |
|---|---|---|---|---|---|---|
| o | o | o | o | x | x | x |
| o | o | o | x | o | x | x |
| o | o | o | x | x | o | x |
| o | x | x | o | o | o | x |
| x | o | x | o | o | o | x |
| x | x | o | o | o | o | x |
| x | x | x | x | x | x | x |

In general we can fill the squares $[1 : n, 1 : n]$ and $[n + 1 : 2n, n + 1 : 2n]$ with circles and add $2n$ circles in the positions $(i, n + i)$, $(n + i, i)$, $i = 1, \ldots, n$. Here $[a : b, c : d]$ denotes the rectangular submatrix with row indices running from $a$ to $b$ and column indices running from $c$ to $d$.

**OC502**. Find the largest possible number of elements of a set $M$ of integers having the following property: from each three different numbers from $M$ you can select two of them whose sum is a power of 2 with an integer exponent.

*Originally from 2018 Czech-Slovakia Math Olympiad, 6th Problem, Category A, First Round.*

*We received* 4 *submissions, of which* 3 *were correct and complete. We present* 2 *solutions.*

*Solution 1, by UCLan Cyprus Problem Solving Group.*

We can have 6 elements by taking $M = \{-5, -1, 3, 5, 7, 9\}$. This works since if we pick any three numbers from $M$, two of them will belong to $A = \{-1, 3, 5\}$ or $B = \{-5, 7, 9\}$. But any two numbers from $A$ or $B$ have a sum which is a power of two.

We proceed to show that if $|M| \geqslant 7$ then this is impossible. Note that $M$ can have at most two non-positive numbers otherwise we immediately have a contradiction. So letting $M_+$ be the set of positive elements of $M$ we have $|M_+| \geqslant 5$. Let $x$ be the largest element of $M_+$, say $2^r \leqslant x < 2^{r+1}$. Then, if $y \in M_+$, we have $2^r < x + y < 2^{r+2}$ so if $x + y$ is a power of 2 we must have $x + y = 2^{r+1}$ and therefore we must have $y = 2^{r+1} - x$.

It follows that there are three elements of $M_+$, call them $a, b, c$, such that none of $a+x, b+x, c+x$ is a power of 2. Looking at the sets $\{a, b, x\}$, $\{b, c, x\}$ and $\{c, a, x\}$ we deduce that $a + b, b + c, c + a$ are all powers of 2. Say $a + b = 2^k, b + c = 2^\ell$ and $c + a = 2^m$. We may assume that $k < \ell < m$. (We cannot e.g. have $k = \ell$ as this gives $a = c$.) Then

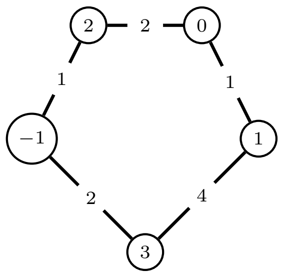
$$b = \frac{(a + b) + (b + c) - (c + a)}{2} = \frac{2^k + 2^\ell - 2^m}{2} < \frac{2^\ell + 2^\ell - 2^m}{2} = \frac{2^{\ell+1} - 2^m}{2} \leqslant 0\,,$$

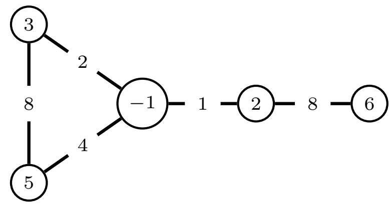
a contradiction.

*Solution 2, by Sergey Sadov.*

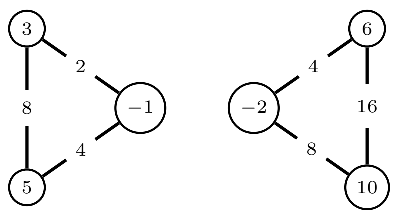
Examples of a 5-element set with the required property are: $M = \{-1, 0, 1, 2, 3\}$, $M = \{-1, 2, 3, 5, 6\}$, $M = \{-3, 0, 1, 4, 7\}$, $M = \{-1, 1, 3, 5, 7\}$, $M = \{-3, 1, 3, 5, 7\}$.

Let us introduce a convenient visual representation. Let $B = \{1, 2, 4, \ldots, \}$ be the set of powers of 2. Let us regard members of the set $M$ as vertices of a graph; the edges will connect those vertices whose values sum up to a member of the set $B$. The following picture is a graphical interpretation of two examples from the above list.




Note that if $M = \{x_1, \ldots, x_m\}$ is a suitable set, then for any $b \in B$ the set $bM = \{bx_1, \ldots, bx_m\}$ is also suitable. A 6-element example can be constructed so that its graph consists of two disjoint triangles corresponding to the sets $\{-1, 3, 5\}$ and $2^k\{-1, 3, 5\}$, $k \geq 1$. For instance, we can take $M = \{-2, -1, 3, 5, 6, 10\}$:




We will prove that a 7-element set $M$ does not exist.

Imagine the graph $\Gamma$ corresponding to a hypothetical $M = \{x_1, \ldots, x_7\}$.

**Lemma 1** *The graph $\Gamma$ does not contain a cycle of length 4.*

*Proof.* Suppose the vertices $x_1$, $x_2$, $x_3$, $x_4$ make a 4-cycle. Then

$$(x_1 + x_2) + (x_3 + x_4) = (x_4 + x_1) + (x_2 + x_3)$$

are two decompositions of the same integer $n = x_1 + x_2 + x_3 + x_4$ into a sum of powers of 2. By uniqueness of the binary representation of positive integers, we conclude that either $x_1 + x_2 = x_4 + x_1$ or $x_1 + x_2 = x_2 + x_3$. Any of these equations implies that the members of the set $M$ are not all distinct, a contradiction. □

*Remark.* The uniqueness of the decomposition in the above proof is the only place where we refer to a specific property of the set $B$.

**Lemma 2** *In the graph $\Gamma$, the degree of every vertex is at least 3.*

*Proof.* Suppose the vertex $x_1$, say, is not connected with $x_2, \ldots, x_5$. Then every pair of the latter vertices must be connected. In particular, there is a 4-cycle. Impossible by Lemma 1. □

**Lemma 3** *There exists a vertex of degree $\geq 4$ in $\Gamma$.*

*Proof.* If the degree of any vertex is exactly 3, then the number of "half-edges" (segments connecting a vertex with midpoint of an adjacent edge) is 21. But it is equal to twice the number of edges. Contradiction. □

*Proof.* [Obtaining a final contradiction] We may assume that degree of vertex $x_1$ is at least 4 and $x_1$ is connected with $x_2$, $x_3$, $x_4$ and $x_5$.

Since the degree of $x_6$ is at least 3, it must be connected with one of $x_2, \dots, x_5$. Without loss of generality we may assume that there is an edge $(x_2 x_6)$. Then $x_6$ is not connected with any of $x_3, x_4, x_5$ as otherwise we would obtain a 4-cycle of the form $(x_1, x_2, x_6, x_i)$. By necessity of making its degree, $x_6$ must be connected with $x_1$ and $x_7$.

Similarly we conclude that $x_7$ must be connected with with $x_1$ and $x_6$. So we have a 4-cycle $(x_1, x_2, x_6, x_7)$. Contradiction. □

The answer to the asked question is 6.

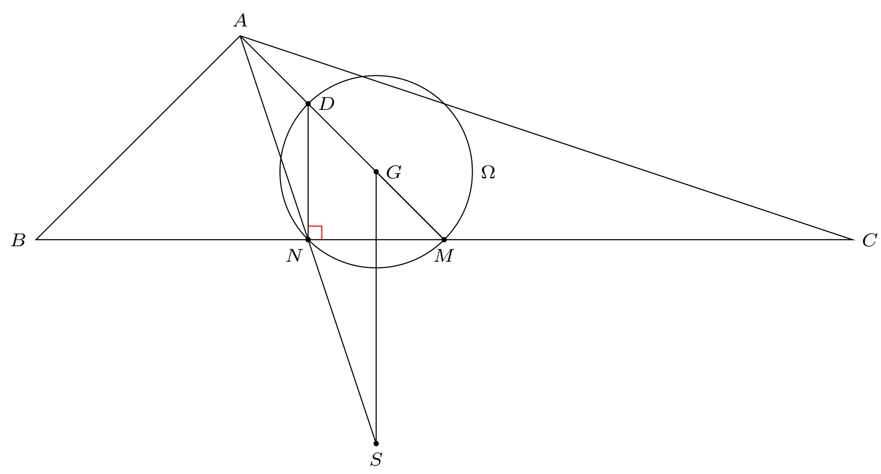
**OC503**. Let $ABC$ be a non-isosceles acute-angled triangle with centroid $G$. Let $M$ be the midpoint of $BC$, let $\Omega$ be the circle with center $G$ and radius $GM$, and let $N$ be the intersection point between $\Omega$ and $BC$ that is distinct from $M$. Let $S$ be the symmetric point of $A$ with respect to $N$, that is, the point on the line $AN$ such that $AN = NS$ $(A \neq S)$. Prove that $GS$ is perpendicular to $BC$.

*Originally from 2018 Italy Math Olympiad, 2nd Problem, Final Round.*

*We received* 14 *submissions. We present* 3 *solutions.*

*Solution 1, by Miguel Amengual Covas.*

Let $D$ be the point of $\Omega$ diametrically opposite to $M$. Then $DG = GM$.




Because $G$ is on the median $AM$, points $A$, $D$, $G$, $M$ are collinear. Since $G$ trisects $AM$, we have $AD = DG$. Therefore,

$$\frac{AD}{DG} = 1 = \frac{AN}{NS},$$

making $DN$ parallel to $GS$, so that $DN$ and $GS$ make equals angles with $BC$.

Since $\angle DNM$ is a right angle (since it is inscribed in a semicircle), the desired conclusion follows.

As shown in the proof, the condition '$\triangle ABC$ acute-angled' is not necessary.

*Solution 2, by UCLan Cyprus Problem Solving Group.*

We use vectors. Let $L$ be the midpoint of $MN$. We have

$$\begin{aligned}\overrightarrow{BC}\cdot\overrightarrow{GS} &= \overrightarrow{BC}\cdot\left(\overrightarrow{GA}+\overrightarrow{AS}\right) = \overrightarrow{BC}\cdot\left(\overrightarrow{GA}+2\overrightarrow{AN}\right) = \overrightarrow{BC}\cdot\left(\overrightarrow{GA}+2\overrightarrow{AG}+2\overrightarrow{GN}\right)\\ &= \overrightarrow{BC}\cdot\left(\overrightarrow{AG}+2\overrightarrow{GN}\right) = 2\overrightarrow{BC}\cdot\left(\overrightarrow{GM}+\overrightarrow{GN}\right) = 4\overrightarrow{BC}\cdot\overrightarrow{GL} = 0\,.\end{aligned}$$

In the penultimate line we have used the fact that $A, G, M$ are collinear with $AG/GM = 2$. In the last line we have used the fact that $GL$ is perpendicular to $MN$ and therefore to $BC$ as $MN$ is a chord of $\Omega$.

*Solution 3, by Corneliu Avram Manescu.*

Choose a Cartesian system of coordinates with $BC$ as $x$-axis, $M$ as origin and $MC$ as unit. Then we have $M(0,0)$, $B(-1,0)$, $C(1,0)$, $A(3a,3b)$, with $a,b$ nonzero real numbers and $G(a,b)$. Circle $\Omega$ has the equation

$$(x-a)^2+(y-b)^2=a^2+b^2,$$

therefore $N(2,0)$. From

$$x_S = 2x_N - x_A = 2\cdot 2a - 3a = a = x_G,$$

we deduce that $GS \perp BC$.

*Note.* The condition $a \neq 0$ means that the triangle $ABC$ is non-isosceles and the condition $b \neq 0$ means that the triangle is non-degenerate. The hypothesis that the triangle is acute-angled is not necessary.

**OC504**. Let $\mathcal{F}$ be the set of continuous functions $f:[0,1]\to\mathbb{R}$ such that $\max_{0\le x\le 1}|f(x)| = 1$ and let $I:\mathcal{F}\to\mathbb{R}$,

$$I(f) = \int_0^1 f(x)\ dx - f(0) + f(1).$$

**(a)** Prove that $I(f) < 3$ for all $f \in \mathcal{F}$.

**(b)** Determine $\sup\{I(f) \mid f\in\mathcal{F}\}$.

*Originally from 2018 Romania Math Olympiad, 1st Problem, Grade 12.*

*We received* 8 *submissions. We present the solution by Oliver Geupel.*

Since $\max\limits_{0\le x\le 1}|f(x)| = 1$, we have

$$\int_0^1 f(x)\,\mathrm{d}x \le \int_0^1 1\,\mathrm{d}x = 1, \qquad -1\le f(0), \qquad \text{and} \qquad f(1)\le 1. \qquad (1)$$

Thus, $I(f) \leq 3$. We show to the contrary that in fact $I(f) < 3$. Assume that $I(f) = 3$ for some appropriate $f$. Then, equality holds in each of the three relations in (1), specifically, $f(0) = -1$. Since $f$ is continuous, there is a positive number $\varepsilon$ such that $f(x) < 0$ for every $x \in [0, \varepsilon]$. Therefore,

$$\int_0^1 f(x)\,\mathrm{d}x = \int_0^\varepsilon f(x)\,\mathrm{d}x + \int_\varepsilon^1 f(x)\,\mathrm{d}x < 0 + (1-\varepsilon) < 1,$$

a contradiction. Hence the result (a).

We show that the supremum in (b) is 3. With a parameter $p \in (0, 1/2)$, let

$$f_p(x) = \begin{cases} \dfrac{x-p}{p} & \text{if } 0 \leq x \leq 2p \\ 1 & \text{if } 2p \leq x \leq 1. \end{cases}$$

Then $f_p$ is continuous,

$$\int_0^1 f_p(x)\,\mathrm{d}x = 1 - 2p, \qquad f(0) = -1, \qquad \text{and} \qquad f(1) = 1,$$

so that $I(f_p) = 3 - 2p$ and $\lim\limits_{p \searrow 0} I(f_p) = 3$.

**OC505**. Let $n$ be a positive integer. We will say that a set of positive integers is *complete of order* $n$ if the set of all remainders obtained by dividing an element in $A$ by an element in $A$ is $\{0, 1, 2, \ldots, n\}$. For example, the set $\{3, 4, 5\}$ is a complete set of order 4. Determine the minimum number of elements of a complete set of order 100.

*Originally from 2018 Romania Math Olympiad, 4th Problem, Grade 6, Final Round.*

*We received only one correct submission. We present the solution by UCLan Cyprus Problem Solving Group.*

Assume first that $A = \{x_1, \ldots, x_n\}$ is a complete set of order 100. Without loss of generality $x_1 < x_2 < \cdots < x_n$. Note that for $i \leqslant j$, the remainder obtained from dividing $x_i$ by $x_j$ is either 0 (if $i = j$) or $x_i$ (if $i < j$). So the total number of distinct remainders is at most

$$m = 1 + n + \binom{n}{2}.$$

We must therefore have $n \geqslant 14$, as otherwise,

$$m \leqslant 1 + 13 + \binom{13}{2} = 92 < 101\,,$$

a contradiction.

We now proceed to show that a complete set of order 100 and size 14 exists. We will show that the set

$$A = \{56, 57, 61, 68, 81, 83, 91, 97, 100, x_1, \ldots, x_5\}$$

for appropriate choices of $x_1, \ldots, x_5$ works.

Let $B = \{56, 57, 61, 68, 81, 83, 91, 97, 100\}$ and observe that all pairwise differences between elements of $B$ are distinct and take the values 1–17, 19-20, 22-27, 29-30, 32, 34-36, 39-41, 43-44. Because each element of $B$ is larger than 44 all of these occur as remainders of divisions of elements of $A$. Of course 0 and each element of $A$ also occurs as remainder. (100 will occur because each of $x_1, \ldots, x_5$ will be larger than 100, so 100 can occur as remainder when dividing 100 by $x_1$ say.)

We will demand that $x_i \bmod b_j$ are as in the following table:

| | 100 | 97 | 91 | 83 | 81 | 68 | 61 | 57 | 56 |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 89 | 65 | 82 | 75 | 76 | 21 | 18 | 28 | 33 |
| $x_2$ | 77 | 66 | 87 | 60 | 74 | 49 | 42 | 47 | 45 |
| $x_3$ | 99 | 69 | 86 | 62 | 70 | 63 | 48 | 46 | 51 |
| $x_4$ | 59 | 72 | 80 | 38 | 71 | 55 | 54 | 50 | 31 |
| $x_5$ | 73 | 93 | 79 | 64 | 67 | 53 | 58 | 52 | 37 |

The residues were carefully chosen such that by the Chinese Remainder Theorem these congruences have solutions. For example, for $x_1$ they are equivalent to

$$x_1 \equiv 1 \bmod 2^3,\ x_1 \equiv 76 \bmod 3^4,\ x_1 \equiv 14 \bmod 5^2,\ x_1 \equiv 5 \bmod 7,\ x_1 \equiv 4 \bmod 13,$$
$$x_1 \equiv 4 \bmod 17,\ x_1 \equiv 9 \bmod 19,\ x_1 \equiv 18 \bmod 61,\ x_1 \equiv 75 \bmod 83,\ x_1 \equiv 65 \bmod 97$$

So we get a unique solution for each $x_i$ modulo $N = 2^3 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 61 \cdot 83 \cdot 97$. Furthermore, by the choice of the residues, each $x_i$ is coprime to $N$ and so by Dirichlet's theorem on arithmetic progressions we can choose each $x_i$ to be a large enough prime number.

We now see that the only remainders remaining to be achieved are

$$78, 84, 85, 88, 90, 92, 94, 95, 96, 98.$$

We pick $x_1 < x_2 < x_3 < x_4 < x_5$ such that they are prime numbers achieving the congruences given in the above table. Furthermore, once we chose $x_1$ we further demand that $x_2 \equiv 78 \bmod x_1$, once we choose $x_2$ we further demand that $x_3 \equiv 84 \bmod x_1$ and $x_3 \equiv 85 \bmod x_2$, and so on. These additional congrunces can also be satisfied since each time we are taking $x_1, x_2, \ldots$ to be prime numbers and so they do not contradict any of our previous congruences. Since we have 5 numbers to choose and $10 = \binom{5}{2}$ additional remainders, we can guarantee that all these remainders are achieved.

Therefore $A$ is a complete set of order 100.

# Equations involving positive divisors of a given integer (Part I)

Salem Malikić

## 1 Introduction

In this article we present several distinct strategies for solving equations or systems of equations which involve divisors of a positive integer. We illustrate this through eleven solved examples of various difficulty from math competitions around the world, including one problem shortlisted for the International Mathematical Olympiad. In addition, we also provide a list of related problems for self-study. The article is published in two related parts and Part II is going to be published in one of the future issues of this journal. When classifying problems, we aimed to have lower average difficulty of problems included in Part I compared to the problems in Part II.

## 2 Some Theoretical Background

**Definition 2.1** *Let $1 = d_1 < d_2 < \cdots < d_k = n$ denote all positive divisors of a positive integer $n$. We define the sum of positive divisors of $n$, commonly denoted as $\sigma(n)$, as*

$$\sigma(n) = d_1 + d_2 + \cdots + d_k.$$

*Similarly, the number of distinct positive divisors of $n$ is commonly denoted as $\tau(n)$ and in this case it equals $k$.*

The following theorem is well known.

**Theorem 2.2** *Functions $\sigma$ and $\tau$ are multiplicative, that is, if $m$ and $n$ are relatively prime positive integers then $\sigma(mn) = \sigma(m)\sigma(n)$ and $\tau(mn) = \tau(m)\tau(n)$.*

Using the above theorem, and assuming that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$ is the canonical representation of positive integer $n$, we can easily prove the following two theorems.

**Theorem 2.3 (Sum of divisors)** *The sum of all distinct positive divisors of $n$ is given by the following formula:*

$$\sigma(n) = (1 + p_1 + p_1^2 + \cdots + p_1^{\alpha_1})(1 + p_2 + p_2^2 + \cdots + p_2^{\alpha_2}) \ldots (1 + p_s + p_s^2 + \cdots + p_s^{\alpha_s}).$$

*Note that the last formula is also equivalent to*

$$\sigma(n) = \frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \frac{p_2^{\alpha_2+1} - 1}{p_2 - 1} \ldots \frac{p_s^{\alpha_s+1} - 1}{p_s - 1}.$$

**Theorem 2.4 (Number of divisors)** *The number of all distinct positive divisors of $n$ is given by the following formula:*

$$\tau(n) = (\alpha_1 + 1)(\alpha_2 + 1) \dots (\alpha_s + 1).$$

We also present the following observations that are useful in solving some of the problems.

**Observation 2.5** *If $1 = d_1 < d_2 < \cdots < d_k = n$ are all positive divisors of $n$ sorted in the ascending order then*

$$\frac{n}{d_1} > \frac{n}{d_2} > \cdots > \frac{n}{d_k}$$

*are all positive divisors of $n$ sorted in the descending order.*

**Observation 2.6** *We have $n = d_i d_{k+1-i}$ for each $i \in \{1, 2, \dots, k\}$.*

**Observation 2.7** *If $1 = d_1 < d_2 < \cdots < d_k = n$ are all positive divisors of integer $n > 1$ then $d_2$ is a prime number.*

## 3 Solved Problems

**Problem 3.1** *Let $1 = d_1 < d_2 < \cdots < d_k = n$ denote all positive divisors of integer $n$.*

*i) Find all positive integers $n$ such that $d_1 d_2 \dots d_k = 15n$.*

*ii) Prove that there does not exist a positive integer $n$ so that $d_1 d_2 \dots d_k = 15n^2$.*

(Netherlands, 2014)

**Solution.** Define $P(n) = d_1 d_2 \dots d_k$.

i) From $P(n) = 15n$ it follows that $n$ is divisible by 3 and 5 implying that $n \geq 15$. Clearly $n = 15$ is a solution. If $n > 15$ then we would have that $3, 5, 15$ and $n$ are distinct divisors of $n$, which implies

$$P(n) \geq 3 \cdot 5 \cdot 15 \cdot n > 15n.$$

In summary, $n = 15$ is the only solution.

ii) Suppose on the contrary that such $n$ exists. Similarly as above, we first conclude that $n$ is divisible by 3 and 5. Direct verification shows that $n \neq 15$, hence $n \geq 30$. But then $\frac{n}{5}$ and $\frac{n}{3}$ are two divisors of $n$ that are distinct from 3 and 5 (because each of them is greater than 5). This implies that

$$P(n) \geq 3 \cdot 5 \cdot \frac{n}{5} \cdot \frac{n}{3} \cdot n = n^3 > 15n^2,$$

a contradiction.

**Problem 3.2** *Prove that there does not exist a positive integer $n$ which is*

*(i) smaller than* 1995*,*

*(ii) can be represented as $abcd$, where $a$, $b$, $c$ and $d$ are distinct primes, and*

*(iii) $d_9 - d_8 = 22$, where $1 = d_1 < \cdots < d_{16} = n$ are all positive divisors of $n$.*

(Ireland, 1995 - slightly modified)

**Solution.** It is sufficient to prove that, if for some positive integer $n$ conditions (i) and (ii) are satisfied, then condition (iii) can not be satisfied. Assume, without loss of generality, that $a < b < c < d$. Note that, since $n$ has 16 divisors, we must have $n = d_8 d_9$. We discuss two cases:

- $a = 2$

  If $a = 2$, then $n = d_8 d_9$ is divisible by 2 but not by 4 (since $n = abcd$), hence in this case one of the numbers $d_8$ and $d_9$ is even and the other is odd. But then $d_9 - d_8$ is odd so it can not be equal to 22.

- $a \neq 2$

  In this case we must have $a = 3$, as otherwise
  $$n = abcd \geq 5 \cdot 7 \cdot 11 \cdot 13 > 1995.$$
  Similarly, if $b > 5$, then
  $$abcd \geq 3 \cdot 7 \cdot 11 \cdot 13 > 1995.$$
  Therefore $b = 5$. Analogously we conclude that $c = 7$. Then $n < 1995$ implies $d < 19$. Direct inspection for $d \in \{11, 13, 17\}$ shows that in each case $d_9 - d_8 \neq 22$.

**Problem 3.3** *For a positive integer $n$ write down all its positive integer divisors in increasing order: $1 = d_1 < d_2 < \cdots < d_k = n$. Find all positive integers $n$ divisible by* 2019 *such that $n = d_{19} \cdot d_{20}$.*

(Belarus, 2019)

**Solution.** First note that $2019 = 3 \cdot 673$ is a canonical representation of 2019. Since $d_i d_{k+1-i} = n$ for each $i \in \{1, 2, \ldots, k\}$ we must have $k = 38$. Let
$$n = 3^x 673^y p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_s^{\alpha_s}$$
be a canonical representation of $n$. Then
$$(x+1)(y+1)(\alpha_1+1)(\alpha_2+1)\ldots(\alpha_s+1) = 38.$$
Since $n$ is divisible by 2019 we must have $x \geq 1$ and $y \geq 1$. Observe now that the only factorizations of 38 that have two or more factors are $2 \cdot 19$ and $19 \cdot 2$. Combining this with all of the above we conclude that $s = 0$ and either $(x, y) = (18, 1)$ or $(x, y) = (1, 18)$. Each of the two cases gives a number $n$ satisfying all conditions of the problem. In other words, $n = 3^{18} \cdot 673$ and $n = 3 \cdot 673^{18}$ are the only solutions.

**Problem 3.4** *Find all positive integers $n$ with exactly 16 positive divisors $1 = d_1 < d_2 < \cdots < d_{16} = n$ such that $d_6 = 18$ and $d_9 - d_8 = 17$.*

(Ireland, 1998)

**Solution.** Given that $n$ is divisible by 18, it must also be divisible by 1, 2, 3, 6 and 9. Since $d_6 = 18$ we must have $d_1 = 1$, $d_2 = 2$, $d_3 = 3$, $d_4 = 6$, $d_5 = 9$. From here we also conclude that $n$ is divisible by 2, but not by 4.

Since the number of divisors of $n$ is 16 and the power of 2 in its canonical representation is 1, the power of 3 increased by 1 can be either 2, 4 or 8. It can not be 2 because $n$ is divisible by $3^2$. If it is 8 then $n$ can not have any other prime factor so we must then have $n = 2 \cdot 3^7$. Direct verification shows that this is not a solution. If the power of 3 in canonical representation of $n$ increased by 1 is equal to 4, then $n$ has exactly one prime factor other than 2 and 3. Furthermore, the power of that prime factor in canonical representation of $n$ is 1 and it must be greater than 18. In other words, $n = 2 \cdot 3^3 \cdot p$, where $p$ is a prime number greater than 18. Note that $d_8d_9 = 54p$ so exactly one of $d_8$ and $d_9$ is divisible by $p$. We know that 27 and 54 are divisors of $n$ and motivated by this we discuss three possible cases for the range of values of $p$.

*Case* $p < 27$. Here we can do manual inspection as $p$ can be either 19 or 23. Alternatively, we can first observe that $d_7 = p$. Next, $d_8$ is either equal to $2p$ or 27. As $2p > 2 \cdot 18 > 27$ we must have $d_8 = 27$. From $d_8d_9 = 54p$ we get $d_9 = 2p$. Then $d_9 - d_8 = 2p - 27 = 17$ implying that $p = 22$, which is not a prime number and therefore we do not have any solution in this case.

*Case* $27 \leq p \leq 54$.

In this case $d_7 = 27$, $d_8 = p$ and $d_9 = 54$. Equation $d_9 - d_8 = 17$ implies $p = 37$ and we have solution $2 \cdot 2^3 \cdot 37 = 1998$.

*Case* $p > 54$.

In this case, $d_7 = 27$, $d_8 = 54$. From $d_8d_9 = 54p$ it follows that $d_9 = p$ and from $d_9 - d_8 = p - 54 = 17$ we get $p = 71$. This leads to another solution, namely $2 \cdot 2^3 \cdot 71 = 3834$.

In summary, the only two numbers satisfying all conditions of the problem are 1998 and 3834.

**Problem 3.5** *A natural number $n$ has exactly 12 positive divisors $d_1, d_2, \ldots, d_{12}$, where $1 = d_1 < d_2 < \cdots < d_{12} = n$. Find all such $n$ for which*

$$d_{d_4-1} = (d_1 + d_2 + d_4)d_8.$$

(USSR Olympiad, 1989)

**Solution.** First, observe that $d_5d_8 = n$. Second, the number $d_1 + d_2 + d_4$ divides the divisor of $n$ (namely $d_{d_4-1}$) so it is also a divisor of $n$. Then obviously $d_1 + d_2 + d_4 \geq d_5$. As

$$d_{12} \geq d_{d_4-1} = (d_1 + d_2 + d_4)d_8 \geq d_5d_8 = n = d_{12},$$

we have that the equality must be achieved in both of the above inequalities. This implies that $d_4 - 1 = 12$ and $d_1 + d_2 + d_4 = d_5$ or, equivalently, $d_4 = 13$ and $d_5 = 14 + d_2$. Using the fact that $d_2$ is prime and $d_2 < d_4 = 13$ we discuss several possible cases:

- $d_2 = 2$

  In this case, $d_5 = 16$ so $n$ is divisible by 1, 2, 4 and 8 so $d_4 \leq 8$, which contradicts $d_4 = 13$ and therefore we do not have any solution in this case.

- $d_2 = 3$

  In this case $d_5 = 17$ so $n$ already has 3 distinct prime divisors: 3, 13 and 17. It can not have a fourth prime divisor, otherwise it would have at least 16 divisors in total. Consider $d_3$. We know that $3 < d_3 < 13$ and that all primes that can divide $d_3$ are those that divide $n$, i.e., 3, 13 and 17. Clearly the only possible value for $d_3$ is $3^2$. We can now conclude that $n = 3^2 \cdot 13 \cdot 17 \cdot A$, where $A$ is some positive integer. As number $3^2 \cdot 13 \cdot 17$ has $(2+1)(1+1)(1+1) = 12$ divisors, same as $n$, we must have $A = 1$. It is easy to verify that $n = 3^2 \cdot 13 \cdot 17 = 1989$ is a solution.

- $d_2 = 5$

  Similarly as above, $d_5 = 19$ so three primes dividing $n$ are 5, 13 and 19. But then $d_3$ must not be divisible by any prime other than 5, 13 and 19 and $5 < d_3 < 13$. Obviously such $d_3$ does not exist so we do not have a solution in this case.

- $d_2 = 7$

  In this case $d_5 = 21$ so $n$ is divisible by 3, which is impossible as $d_2 = 7$ is the smallest prime divisor of $n$. Therefore there is no solution in this case either.

- $d_2 = 11$

  Similarly as in the previous case, we have that $d_5 = 25$ so $n$ is divisible by 5, which is impossible as 11 is the smallest prime dividing $n$. In other words, this is another case in which we have no solution.

In summary, $n = 1989$ is the only solution.

**Problem 3.6** *Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer $n$. Find all $n$ such that $k \geq 4$ and*

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2.$$

(Balkan Mathematical Olympiad 1989)

**Solution.** We first observe that $d_2 = 2$. Namely, if $d_2 \neq 2$, then $n$ must be an odd number, which would imply that all of its divisors are odd. But in that case the number $d_1^2 + d_2^2 + d_3^2 + d_4^2$, which is by the condition of the problem equal to $n$, is the sum of four odd numbers, hence it is even. In other words, if $d_2 \neq 2$, then $n$ is both odd and even, which is obviously impossible. Therefore $d_2 = 2$.

As $n$ is even, $n = 1^2 + 2^2 + d_3^2 + d_4^2$ implies that one of the numbers $d_3$ and $d_4$ must be odd and the other must be even. Therefore

$$n = 1^2 + 2^2 + d_3^2 + d_4^2 \equiv 2 \pmod 4$$

implying that $n$ is not divisible by 4. Here we conclude that $d_3$ is an odd prime. Let $d_3 = p$. Then $p$ is obviously the smallest odd prime dividing $n$.

Now we analyze the possible values of $d_4$. We know that $d_4$ must be an even number greater than 2, but not divisible by 4. Therefore $d_4 = 2(2t+1)$ for some positive integer $t$. As $2p$ divides $n$ and $d_3$ is equal to $p$ we know that $d_4 \leq 2p$, that is $2t+1 \leq p$. We claim that $2t+1 = p$. To prove this, we use a proof by contradiction assuming that it is possible that $2t+1 < p$. As $t \geq 1$, there exists a prime number $q$ dividing $2t+1$. Then $q \leq 2t+1 < p$. As $q \mid 2t+1$ and $2t+1 \mid n$ this implies that $q \mid n$ so we have found an odd prime divisor of $n$ smaller than $p$, which is a contradiction with the choice of $p$. Therefore $2t+1 = p$, which is equivalent to $d_4 = 2p$. We are now left with the equation:

$$n = 1^2 + 2^2 + p^2 + (2p)^2 = 5(1+p^2).$$

This implies that $n$ is divisible by 5 so $p \in \{3, 5\}$. For $p = 3$ and $p = 5$ we get $n = 50$ and $n = 130$, respectively. Direct verification shows that $n = 130$ is the only solution.

## 4 Problems for Self-study

**Problem 4.1** *How many distinct integers $n$ greater than 1 exist such that $d_{k-1} = 11d_2$, where $1 = d_1 < d_2 < \cdots < d_k = n$ are all the positive divisors of $n$?*

(Switzerland, 2021)

**Problem 4.2** *For a given positive integer $n$, let $1 = d_1 < d_2 < \cdots < d_k = n$ denote all of its divisors. Find all integers $n \geq 2$ such that:*

*(a) there exists $i \in \{1, 2, \ldots, k\}$ such that*

$$n = d_2^2 + d_i^2$$

(Austria, 2017)

*(b) there exists $i \in \{1, 2, \ldots, k\}$ such that*

$$n = d_2^3 + d_i^3$$

(North Macedonia, 2018)

**Problem 4.3** *Determine all composite positive integers $n$ with the following property: if $1 = d_1 < d_2 < \cdots < d_k = n$ are all the positive divisors of $n$, then*

$$(d_2 - d_1) : (d_3 - d_2) : \ldots : (d_k - d_{k-1}) = 1 : 2 : \ldots : (k-1).$$

(Austria, 2016)

**Problem 4.4** *Find all positive integers $n$ such that*

$$n^3 = d_1 d_2 \ldots d_k$$

*where $1 = d_1 < d_2 < \cdots < d_k = n$ are all the positive divisors of $n$.*

(Switzerland, 2011)

**Problem 4.5** *Find all positive integers $n$ which are not powers of 2 and which satisfy the equation $n = 3D + 5d$, where $D$ and $d$ respectively denote the largest and the smallest odd divisor of $n$ greater than 1 (i.e., $d > 1$).*

(Czech-Slovak, 2014)

**Problem 4.6** *Find all positive integers $n$ such that $1 = d_1 < d_2 < \cdots < d_{16} = n$ are all the positive divisors of $n$ and*

$$d_{d_5} = (d_2 + d_4) d_6.$$

(Junior Balkan Mathematical Olympiad, 2002)

**Problem 4.7** *Find all positive integers $n$ such that*

$$5(n+1) = d_1^2 + d_2^2 + \cdots + d_{k-1}^2$$

*where $1 = d_1 < d_2 < \cdots < d_k = n$ are all the positive divisors of $n$.*

(Switzerland, 2016)

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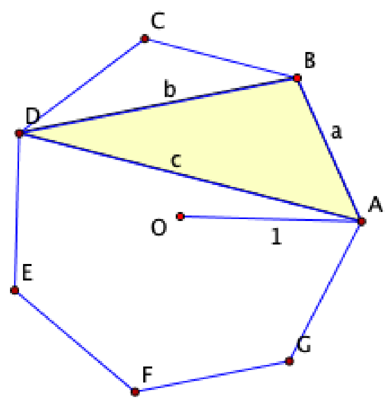
# Further Properties of the Regular Heptagon

J. Chris Fisher

A few years ago, an article in this journal [2] by Michel Bataille investigated results that describe relationships among the parts of a regular heptagon. Two recent problems, one appearing in *Mathematics Magazine*, the other in *Crux*, suggest that much remains to be discovered. When the regular heptagon is inscribed in the unit circle, its side $a$, short diagonal $b$, and long diagonal $c$ satisfy

$$a = 2\sin\frac{\pi}{7}, \quad b = 2\sin\frac{2\pi}{7}, \quad \text{and} \quad c = 2\sin\frac{4\pi}{7}.$$




Our goal here is to prove theorems about the sequences

$$S_n = b^n+c^n+(-a)^n,\ n \in \mathbb{Z} \quad \text{and} \quad T_n = (-a)^n c^{n-2}+b^n(-a)^{n-2}+c^n b^{n-2},\ n \in \mathbb{Z};$$

specifically,

**Theorem 1** (See [6], Proposition 4, page 30). $S_{n+1} = \sqrt{7}(S_n - S_{n-2}),\ n \in \mathbb{Z}$,

and

**Theorem 2**. $T_{n+2} = 7(T_n - T_{n-1}),\ n \in \mathbb{Z}$.

### Review of basic formulas

Let's begin with a list of the results from [2] that will be needed here.

From $b = 2\sin\frac{2\pi}{7} = 4\sin\frac{\pi}{7}\cos\frac{\pi}{7} = 2a\cos\frac{\pi}{7}$ and analogous calculations for $c = 2\sin\frac{4\pi}{7}$ and $-a = 2\sin\frac{8\pi}{7}$, we deduce that

$$\cos\frac{\pi}{7} = \frac{b}{2a}, \quad \cos\frac{2\pi}{7} = \frac{c}{2b}, \quad \cos\frac{4\pi}{7} = -\frac{a}{2c}. \tag{1}$$

It follows immediately, of course, that

$$\tan\frac{\pi}{7}=\frac{a^2}{b},\quad \tan\frac{2\pi}{7}=\frac{b^2}{c},\quad \tan\frac{4\pi}{7}=-\frac{c^2}{a}. \tag{2}$$

Applying Ptolemy's theorem to the quadrangles $ABCE$, $ABDE$, $ABCD$, and $ABDF$ in the accompanying figure yields, respectively,

$$bc=ab+ac,\quad c^2=a^2+bc,\quad b^2=a^2+ac,\quad \text{and}\quad c^2=b^2+ab. \tag{3}$$

These relations appear on page 56 of [2], and come with three alternative proofs. Finally, by starting with the expansion of $\sin 7\theta$, namely,

$$\sin 7\theta = 64\sin^6\theta - 112\sin^4\theta + 56\sin^2\theta - 7\sin\theta,$$

we see that for $x=\sin\theta$ when $\theta=\pm\frac{\pi}{7},\pm\frac{2\pi}{7}$, and $\pm\frac{4\pi}{7}$, the equation becomes

$$0=x^6-\frac{7}{4}x^4+\frac{7}{8}x^2-\frac{7}{64},$$

whose roots are $\pm\sin\theta$. The coefficients equal sums of $\sin^2\frac{\pi}{7},\sin^2\frac{2\pi}{7},\sin^2\frac{4\pi}{7}$ taken one, two, and three at a time; in terms of the heptagon we have

$$a^2+b^2+c^2=7,\quad a^2b^2+b^2c^2+c^2a^2=14,\quad \text{and}\quad a^2b^2c^2=7. \tag{4}$$

I took the proof of (4) from p. 13 of [1]. According to the authors in [5], Kepler found these relations using a similar argument some 400 years ago. Bataille [2] exhibited them as exercises (with suggestions for alternative proofs) on p. 57–58.

I end the preliminary material with a formula not in [2], namely

$$b+c-a=\sqrt{7}. \tag{5}$$

Since $b+c-a>0$ by the triangle inequality, it suffices to prove that $(b+c-a)^2=7$, as follows:

$$(b+c-a)^2=a^2+b^2+c^2+2(bc-ab-ca)=7-0,$$

by the formulas in (3) and (4).

## Proof of the two Theorems

*Proof of Theorem* 1.

In the second line, $\sqrt{7}$ is replaced first by $b+c-a$ (equation (5)), then by $abc$ (equation (4)):

$$\begin{aligned}
&\sqrt{7}(S_n-S_{n-2})\\
&=(b+c-a)\cdot(b^n+c^n+(-a)^n)+(-a)bc\cdot(b^{n-2}+c^{n-2}+(-a)^{n-2})\\
&=\left(b^{n+1}+c^{n+1}+(-a)^{n+1}\right)+b^n(c-a)+c^n(b-a)\\
&\quad+(-a)^n(b+c)-acb^{n-1}-abc^{n-1}+bc(-a)^{n-1}\\
&=\left(b^{n+1}+c^{n+1}+(-a)^{n+1}\right)+b^{n-1}(bc-ba-ac)\\
&\quad+c^{n-1}(bc-ca-ab)+(-a)^{n-1}(-ab-ac+bc)\\
&=b^{n+1}+c^{n+1}+(-a)^{n+1}\;=\;S_{n+1},
\end{aligned}$$

where (3) was used to obtain the last line.

We already know four values of $S_n$: $S_{-1} = \frac{1}{b} + \frac{1}{c} - \frac{1}{a} = 0$ (clear of fractions to obtain the first item in (3)), $S_0 = 1 + 1 + 1 = 3$, $S_1 = \sqrt{7}$ (equation (5)), $S_2 = 7$ (the first item in (4)). Others can be obtained using Theorem 1.

| $n$ | $\dots$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ | $7$ | $8$ | $\dots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_n$ | | $2$ | $0$ | $3$ | $\sqrt{7}$ | $7$ | $4\sqrt{7}$ | $21$ | $14\sqrt{7}$ | $70$ | $49\sqrt{7}$ | $5 \cdot 7^2$ | |

*Comment.* N. J. A. Sloane's *On-Line Encyclopedia of Integer Sequences* deals with the quantity $S_n$ by way of two sequences of integers, A215493 and A215494, which he refers to as "Berndt-type sequences": $S_n = 7S_{n-2} - 14S_{n-4} + 7S_{n-6}$. This result gives the values of $S_{2k}$ as integers, and of $S_{2k+1}$ as integer multiples of $\sqrt{7}$.

The proof of the second theorem depends on the following lemma:

**Lemma**. $b^2(a^2 + c^2 + (-a)c) = c^2(b^2 + a^2 + b(-a)) = a^2(c^2 + b^2 + cb) = 7$.

The equality on the left becomes

$$\begin{aligned} a^2b^2 + b^2c^2 - ab^2c &= b^2c^2 + a^2c^2 - bac^2 \\ (-a)b^2 + b^2c &= (-a)c^2 + bc^2 \\ a(b^2 - c^2) &= bc(b - c) \\ a(b + c) &= bc, \end{aligned}$$

which is the left entry in (3). A similar calculation holds for the equality on the right. To see that the three equal expressions all equal 7, observe that they sum to 21:

$$\begin{aligned} &b^2(a^2 + c^2 + (-a)c) + c^2(b^2 + a^2 + b(-a)) + a^2(c^2 + b^2 + cb) \\ &\quad = 2(a^2b^2 + b^2c^2 + c^2a^2) - abc(b + c - a) \\ &\quad = 2 \cdot 14 - 7 \; = \; 21, \end{aligned}$$

from (4) and (5).

*Proof of Theorem* 2. We use cyclic sums whose three summands are obtained by a cyclic permutation of the numbers $(-a), b$, and $c$.

$$\begin{aligned} &7(T_n + T_{n-1}) \\ &= 14T_n - 7T_n + 7T_{n-1} \\ &= (a^2b^2 + b^2c^2 + c^2a^2) \sum_{cyc} (-a)^n c^{n-2} - 7T_n + ((-a)bc)^2 \sum_{cyc} (-a)^{n-1} c^{n-3} \\ &= \sum_{cyc} (-a)^{n+2} c^n + \sum_{cyc} \big((-a)^n c^{n-2} b^2((-a)^2 + c^2)\big) - 7T_n + \sum_{cyc} \big((-a)^n c^{n-2} b^2 \cdot (-a)c\big) \\ &= T_{n+2} + \sum_{cyc} \big((-a)^n c^{n-2} b^2((-a)^2 + c^2 + (-a)c)\big) - 7T_n \\ &= T_{n+2} + 7 \sum_{cyc} (-a)^n c^{n-2} - 7T_n = T_{n+2}. \end{aligned}$$

To start the recursion we can use

$T_0 = \dfrac{1}{c^2} + \dfrac{1}{(-a)^2} + \dfrac{1}{b^2} = \dfrac{b^2c^2 + a^2b^2 + c^2a^2}{((-a)bc)^2} = \dfrac{14}{7} = 2$ (from (3)),

$T_1 = \dfrac{-a}{c} + \dfrac{b}{(-a)} + \dfrac{c}{b} = -1$ (which appears at the top of page 58 in [2]), and

$T_2 = (-a)^2 + b^2 + c^2 = 7.$

Here is the resulting sequence.

| $n$ | $\ldots$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\ldots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_n$ | | 2 | $-1$ | 7 | 7 | 42 | 98 | 343 | 980 | $7^3 \cdot 9$ | $7^3 \cdot 27$ | $7^3 \cdot 83$ | |

**The motivating problems**

*Crux* **Problem 4587** [4]. Find an elementary proof of $\tan\frac{2\pi}{7} = -\sqrt{7} + 4\sin\frac{4\pi}{7}$.
*Solution.* By (1) and (2) we have $\tan\frac{2\pi}{7} = \frac{b^2}{c}$ and $\sin\frac{4\pi}{7} = \frac{c}{2}$, so the problem is equivalent to showing that

$$\sqrt{7} = 2c - \frac{b^2}{c} = \frac{c^2 + c^2 - b^2}{c} = \frac{c^2 + ab}{c} = \frac{c^2 + bc - ac}{c} = c + b - a,$$

which is formula (5).

*Mathematics Magazine* **Problem 2102** [3].
Let $\alpha = \frac{\pi}{7}, \beta = \frac{2\pi}{7}$, and $\gamma = \frac{4\pi}{7}$. Prove the following trigonometric identities.

$$\frac{\cos^2\alpha}{\cos^2\beta} + \frac{\cos^2\beta}{\cos^2\gamma} + \frac{\cos^2\gamma}{\cos^2\alpha} = 10,$$

$$\frac{\sin^2\alpha}{\sin^2\beta} + \frac{\sin^2\beta}{\sin^2\gamma} + \frac{\sin^2\gamma}{\sin^2\alpha} = 6,$$

$$\frac{\tan^2\alpha}{\tan^2\beta} + \frac{\tan^2\beta}{\tan^2\gamma} + \frac{\tan^2\gamma}{\tan^2\alpha} = 83.$$

We leave this problem as an exercise, and end with three further problems.

**Exercise 1**. Prove that

$$4\sin\frac{2k\pi}{7} - \tan\frac{k\pi}{7} = \begin{cases} \sqrt{7} & k = 1, 2, 4 \\ -\sqrt{7} & k = 3, 5, 6 \end{cases}$$

This result is Theorem 2 of [6].

**Exercise 2**. Prove that $\frac{\sqrt{2}}{2}$ equals the distance from the midpoint of the side $AB$ of a regular convex heptagon $ABCDEFG$, inscribed in a unit circle, to the midpoint of the radius perpendicular to $BC$ and cutting this side. [Problem E1154 proposed by Victor Thébault in the *American Mathematical Monthly*, **62**:8 (Oct. 1955) 584-585.]

**Exercise 3**. A nonisosceles triangle that shares its vertices with three vertices of a regular heptagon is called a *heptagonal triangle*. When inscribed in a unit circle, its sides have lengths $a, b$, and $c$. Prove that the sum of the squares of its altitudes equals $\frac{7}{2}$. [Additional property 6 on page 14 of [1]].

**References**.

[1] Leon Bankoff and Jack Garfunkel, The Heptagonal Triangle, *Math. Magazine*, **46**:1 (Jan., 1973), p. 7–19.

[2] Michel Bataille, About the Side and Diagonals of the Regular Heptagon, *Crux Mathematicorum*, **43**:2 (Feb., 2017) , p. 55–60.

[3] Problem 2102, Proposed by Donald Jay Moore, *Math. Magazine*, **93**:4 (Oct., 2020) , p. 309.

[4] Problem 4587, Proposed by Kai Wang, *Crux Mathematicorum*, **46**:9 (Nov., 2020), p. 462 and this issue p. 212.

[5] D.Y. Savio and E.R. Suryanarayan, Chebychev Polynomials and Regular Polygons, *Amer. Math. Monthly*, **100**:7 (Aug.-Sep. 1993), p. 657–661.

[6] Kai Wang, Heptagonal Triangle and Trigonometric Identities, *Forum Geometricorum*, **19** (2019), p. 29–38.

# PROBLEMS

***Click here to submit problems proposals as well as solutions, comments and generalizations to any problem in this section.***

*To facilitate their consideration, solutions should be received by* **June 15, 2021**.

**4631**. *Proposed by Nguyen Viet Hung.*

Let $P$ be any point on a triangular face of a tetrahedron with centroid $O$ and let $L, M, N$ be respectively projections of $P$ onto the other three triangular faces. Prove that $PO$ passes through the centroid $G$ of triangle $LMN$ and determine ratio $\frac{\overline{PO}}{\overline{PG}}$.

**4632**. *Proposed by Michel Bataille.*

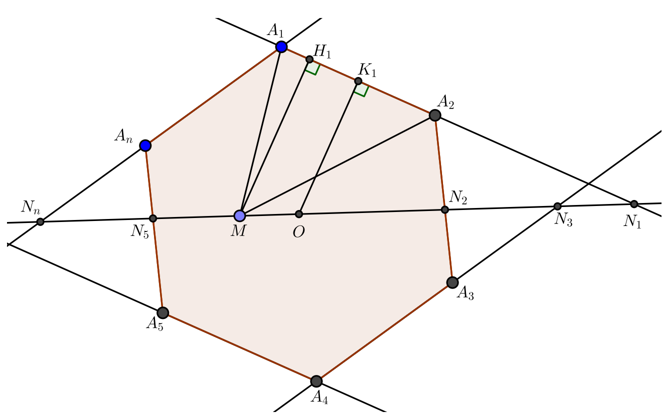
Let $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ be the $n$th harmonic number. Prove that for $n \geq 1$,

$$\sum_{k=1}^{n} \binom{2n+1}{2k-1} \frac{H_{2k-1}}{2k} = \sum_{k=1}^{n} \binom{2n+1}{2k} \frac{H_{2k}}{2k+1}.$$

**4633**. *Proposed by Nguyen Viet Hung.*

Let $A_1A_2\dots A_n$ be a a regular $n-$sided polygon with center $O$ and let $M$ be any point inside the polygon. Suppose that the line $OM$ intersects the lines $A_iA_{i+1}$ at $N_i$ ($i = 1, 2, \dots, n$ and $A_{n+1} \equiv A_1$), respectively. Find

$$\sum_{i=1}^{n} \frac{MN_i}{ON_i}.$$



**4634**. *Proposed by George Stoica.*

Let $\sum_{n=1}^{\infty} a_n < \infty$ for $a_n > 0$, $n = 1, 2, \ldots$ Find $\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n}$.

**4635***. *Proposed by S. Chandrasekhar.*

For a prime $p$ dividing $n!$, let $e_p(n!)$ denote the highest power of $p$ in $n!$, that is if $e_p(n!) = k$, it means $p^k \mid n!$ whereas $p^{k+1} \nmid n!$. Prove or disprove that $e_3(n!) \mid e_2(n!)$ for infinitely many $n$.

**4636**. *Proposed by Mihaela Berindeanu.*

Solve the following equation over the set of real numbers:

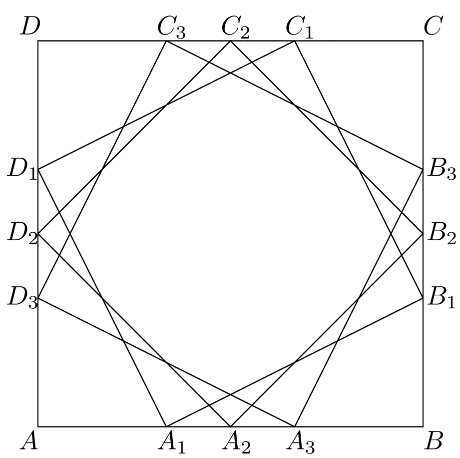
$$(3^x + 7)^{\log_4 3} - (4^x - 7)^{\log_3 4} = 4^x - 3^x - 14.$$

**4637**. *Proposed by Titu Zvonaru.*

Let the incircle of triangle $ABC$ touch the sides $BC, CA$, and $AB$ at points $D, E$, and $F$, respectively. The internal bisector of the angle $\angle BCA$ intersects the line $EF$ at $M$. Let $P$ be the reflection of the point $E$ with respect to $M$. Prove that the triangle $BPF$ is isosceles.

**4638**. *Proposed by Marie-Nicole Gras.*

We consider a square $ABCD$ of side $AB = 6a$, $a \in \mathbb{R}$ ; we put on the side $AB$ points $A_1, A_2, A_3$ such that $AA_1 = 2a$, $A_1A_2 = A_2A_3 = a$, then we draw the squares $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ and $A_3B_3C_3D_3$ as shown on the figure.




The region common to the interiors of the three squares is a dodecagon. Find the relationship between the areas of the dodecagon and the largest square.

*This problem was inspired by 4449.*

**4639**. *Proposed by Seán Stewart.*

If $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$, show that

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k(k+m)!(n-k)!} = \frac{1}{m!n!} \sum_{k=1}^{n} \frac{1}{m+k}.$$

**4640**. *Proposed by Mihaela Berindeanu.*

Let $ABC$ be an acute triangle and $A_1, B_1, C_1$ be the feet of the medians from $A, B, C$ respectively. Denote by $I_1$ and $I_2$ the centers of the inscribed circles of $\triangle\ ABA_1$ and $\triangle\ AA_1C$ respectively. The circumcircle of $\triangle\ ABI_1$ cuts the circumcircle of $\triangle\ AI_2C$ for the second time in $A'$. Define $B'$ and $C'$ analogously. Show that if $\overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} = \overrightarrow{0}$, then $\triangle\ ABC$ is an equilateral triangle.

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

***Cliquez ici afin de proposer de nouveaux problèmes, de même que pour offrir des solutions, commentaires ou généralisations aux problèmes proposés dans cette section.***

*Pour faciliter l'examen des solutions, nous demandons aux lecteurs de les faire parvenir au plus tard le* **15 juin 2021**.

*La rédaction souhaite remercier Frédéric Morneau-Guérin, professeur à l'Université TÉLUQ, d'avoir traduit les problèmes.*

**4631**. *Soumis par Nguyen Viet Hung.*

Soit $P$ un point quelconque de la face triangulaire d'un tétraèdre de centre de masse $O$. Soient $L$, $M$ et $N$ les projections respectives de $P$ sur les trois autres faces triangulaires. Montrez que $PO$ passe par le centre de masse $G$ du triangle $LMN$ et déterminez le rapport $\frac{\overline{PO}}{\overline{PG}}$.

**4632**. *Soumis par Michel Bataille.*

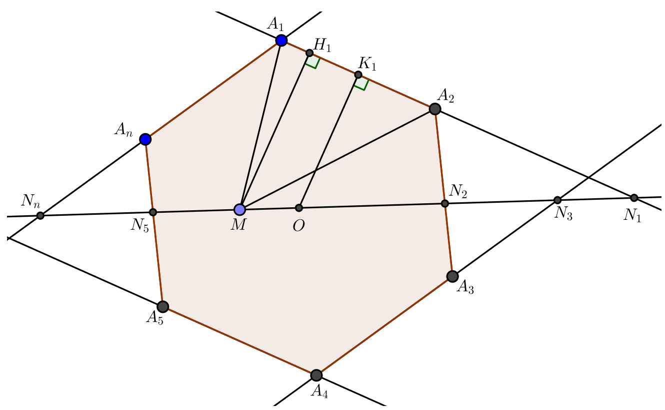
Soit $H_n = \sum_{k=1}^{n} \frac{1}{k}$ le $n$-ième nombre harmonique. Montrez que pour $n \geq 1$,

$$\sum_{k=1}^{n} \binom{2n+1}{2k-1} \frac{H_{2k-1}}{2k} = \sum_{k=1}^{n} \binom{2n+1}{2k} \frac{H_{2k}}{2k+1}.$$

**4633**. *Soumis par Nguyen Viet Hung.*

Soit $A_1A_2\ldots A_n$ un polygone à $n$ côtés et de centre $O$. Soit $M$ un point situé à l'intérieur de ce polygone. Supposons que la droite $OM$ rencontre chacune des droites $A_iA_{i+1}$ ($i = 1, 2, \ldots, n$ et $A_{n+1} \equiv A_1$) et notons par $N_i$ ces points d'intersection respectifs. Trouvez

$$\sum_{i=1}^{n} \frac{MN_i}{ON_i}.$$




**4634**. *Soumis par George Stoica.*

Soit $\displaystyle\sum_{n=1}^{\infty} a_n < \infty$, où $a_n > 0$ et $n = 1, 2, \ldots$. Trouvez $\displaystyle\lim_{n\to\infty} n \cdot \sqrt[n]{a_1 \cdots a_n}$.

**4635***. *Soumis par S. Chandrasekhar.*

Étant donné un nombre premier $p$ divisant $n!$, notons par $e_p(n!)$ la plus grande puissance de $p$ divisant $n!$. Autrement dit, si $e_p(n!) = k$ alors $p^k \mid n!$ mais $p^{k+1} \nmid n!$. Prouvez ou infirmez l'affirmation suivante : $e_3(n!) \mid e_2(n!)$ pour une infinité de valeurs de $n$.

**4636**. *Soumis par Mihaela Berindeanu.*

Trouvez tous les nombres réels vérifiant l'équation suivante :

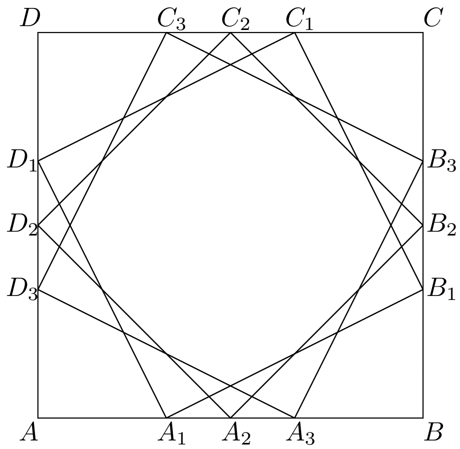
$$(3^x + 7)^{\log_4 3} - (4^x - 7)^{\log_3 4} = 4^x - 3^x - 14.$$

**4637**. *Soumis par Titu Zvonaru.*

Le cercle inscrit du triangle $ABC$ est tangent aux droites $BC, CA$ et $AB$ en $D, E$, et $F$, respectivement. La bissectrice intérieure de l'angle $\angle BCA$ rencontre la droite $EF$ en $M$. Soit $P$ la réflexion du point $E$ par rapport à $M$. Montrez que le triangle $BPF$ est isocèle.

**4638**. *Soumis par Marie-Nicole Gras.*

Considérons un carré $ABCD$ de côté $AB = 6a$, $a \in \mathbb{R}$ ; on ajoute sur les côté $AB$ des points $A_1, A_2, A_3$ tels que $AA_1 = 2a$, $A_1A_2 = A_2A_3 = a$. On trace ensuite les carrés $A_1B_1C_1D_1$, $A_2B_2C_2D_2$ et $A_3B_3C_3D_3$ tel qu'illustré dans la figure suivante.




La région commune à l'intérieur de ces trois carrés est un dodécagone. Trouvez la relation entre l'aire du dodécagone à celle du plus grand carré.

*Ce problème s'inspire de 4449.*

**4639**. *Soumis par Seán Stewart.*

Étant donné $m \in \mathbb{N} \cup \{0\}$ et $n \in \mathbb{N}$, montrez que

$$\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k(k+m)!(n-k)!} = \frac{1}{m!n!} \sum_{k=1}^{n} \frac{1}{m+k}.$$

**4640**. *Soumis par Mihaela Berindeanu.*

Soit $ABC$ un triangle acutangle. Soient $A_1$, $B_1$ et $C_1$ les pieds des médianes relatives à $A$, $B$ et $C$ respectivement. Désignons par $I_1$ et $I_2$ respectivement les centres des cercles inscrits de $\triangle\, ABA_1$ et $\triangle\, AA_1C$. Le cercle circonscrit de $\triangle\, ABI_1$ rencontre le cercle circonscrit de $\triangle\, AI_2C$ une première fois en $A$ et une seconde fois en $A'$. De façon analogue, ils définissent respectivement $B'$ et $C'$. Montrez que si $\overrightarrow{A'A_1} + \overrightarrow{B'B_1} + \overrightarrow{C'C_1} = \overrightarrow{0}$, alors $\triangle\, ABC$ est un triangle équilatéral.

# SOLUTIONS

*No problem is ever permanently closed. The editor is always pleased to consider for publication new solutions or new insights on past problems.*

*Statements of the problems in this section originally appear in 2020: 46(9), p. 461–464.*

**4581**. *Proposed by Mihaela Berindeanu.*

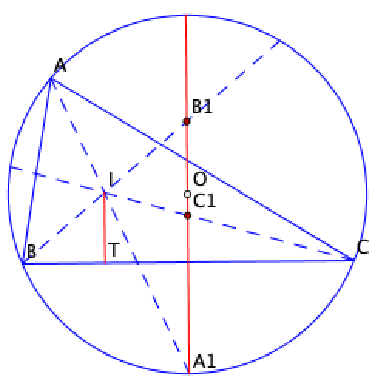
Let $ABC$ be a triangle, with $AB < AC$ and with circumcircle $\Gamma$, circumcenter $O$ and incenter $I$. Denote $AI \cap \Gamma = A_1$, $BI \cap A_1O = B_1$, $CI \cap A_1O = C_1$. Prove that

$$\frac{BC_1 - C_1I}{B_1I + B_1C} = \frac{BI \cdot A_1C_1}{CI \cdot A_1B_1}.$$

*We received 13 submissions, all correct, and feature a composite of the solutions by Marie-Nicole Gras and by Sergey Sadov.*

Let $a = BC$, $b = AC$, $c = AB$ be the sides of $\triangle ABC$, and $s$ be its semi-perimeter; we use the labels $A, B, C$ to denote the vertices as well as the (interior) angles at those vertices. Because $AA_1$ bisects angle $A$, $A_1$ is the midpoint of the arc $BC$ opposite $A$; consequently $A_1O$ is the perpendicular bisector of $BC$; since $B_1$ and $C_1$ belong to $A_1O$, we deduce

$$B_1B = B_1C \quad \text{and} \quad C_1B = C_1C.$$




Moreover, since we take $AB < AC$, the line $A_1O$ must separate $C$ from the points $A, B$, and $I$; consequently, $C_1$ is between $I$ and $C$ and $I$ is between $B_1$ and $B$, whence,

$$C_1I = CI - CC_1 \quad \text{and} \quad B_1I = BB_1 - BI.$$

It follows that

$$\begin{aligned} BC_1 - C_1I &= CC_1 - C_1I = CC_1 - (CI - CC_1) = 2CC_1 - CI, \\ B_1C + B_1I &= BB_1 + B_1I = BB_1 + (BB_1 - BI) = 2BB_1 - BI. \end{aligned}$$

The relation to be established thus becomes

$$\frac{2CC_1 - CI}{A_1C_1} \cdot \frac{A_1B_1}{2BB_1 - BI} = \frac{BI}{CI}. \tag{1}$$

If $T$ is the point where the incircle touches $BC$, we have $BT = s-b$ and $TC = s-c$, so that $BI = \frac{s-b}{\cos\frac{B}{2}}$, $CI = \frac{s-c}{\cos\frac{C}{2}}$, $BB_1 = \frac{a}{2\cos\frac{B}{2}}$, and $CC_1 = \frac{a}{2\cos\frac{C}{2}}$, whence

$$2CC_1 - CI = \frac{a-s+c}{\cos\frac{C}{2}} = \frac{s-b}{\cos\frac{C}{2}}\ . \tag{2}$$

For the value of $A_1C_1$ we first observe that $\angle A_1BC = \angle A_1AC = \frac{A}{2}$ (since the angles are inscribed in the circumcircle of $\Delta ABC$). The Law of Sines applied to $\Delta A_1C_1C$ gives us

$$\frac{A_1C_1}{\sin(\angle C_1CA_1)} = \frac{CC_1}{\sin(\angle C_1A_1C)}.$$

It follows that

$$A_1C_1 = CC_1\frac{\sin\left(\frac{A}{2}+\frac{C}{2}\right)}{\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)} = CC_1\frac{\cos\frac{B}{2}}{\cos\frac{A}{2}} = \frac{a}{2\cos\frac{C}{2}}\cdot\frac{\cos\frac{B}{2}}{\cos\frac{A}{2}}. \tag{3}$$

By interchanging in (2) and (3) the roles of $C, C_1$, and $c$ with those of $B, B_1$, and $b$, respectively, we get

$$2BB_1 - BI = \frac{a-s+b}{\cos\frac{B}{2}} = \frac{s-c}{\cos\frac{B}{2}} \quad \text{and} \quad A_1B_1 = \frac{a}{2\cos\frac{B}{2}}\cdot\frac{\cos\frac{C}{2}}{\cos\frac{A}{2}}. \tag{4}$$

Plugging the expressions obtained in (2), (3), and (4) into the left side of (1) yields

$$\frac{2CC_1 - CI}{A_1C_1} \cdot \frac{A_1B_1}{2BB_1 - BI} = \frac{2(s-b)\cos\frac{A}{2}}{a\cos\frac{B}{2}}\cdot\frac{a\cos\frac{C}{2}}{2(s-c)\cos\frac{A}{2}} = \frac{(s-b)\cos\frac{C}{2}}{(s-c)\cos\frac{B}{2}},$$

which we recognize to equal $\frac{BI}{CI}$, as required by equation (1).

## 4582. *Proposed by Leonard Giugiuc.*

Let $k > 9$ be a fixed real number. Consider the following system of equations with $a \le b \le c \le d$:

$$\begin{cases} a+b+c+d = 3+k, \\ a^2+b^2+c^2+d^2 = 3+k^2, \\ abcd = k. \end{cases}$$

a) Find all solutions in positive reals.

b) Determine the number of real solutions.

*We received 5 correct solutions and one incomplete solution. We present 2 solutions.*

*Solution 1, by UC Lan Cyprus Problem Solving Group.*

(a) Since $4d \geq a+b+c+d = 3+k > 12$, $d > 3$. Since

$$(d-1)^2 \leq (a-1)^2+(b-1)^2+(c-1)^2+(d-1)^2 = (3+k^2)-2(3+k)+4 = (k-1)^2,$$

then $d \leq k$. Therefore

$$\begin{aligned}2(ab+bc+ca) &= (a+b+c)^2-(a^2+b^2+c^2) = (3+k-d)^2-(3+k^2-d^2)\\ &= 6-2(k-d)(d-3) \leq 6.\end{aligned}$$

Hence

$$3 \geq ab+bc+ca \geq 3(abc)^{2/3} = 3(k/d)^{2/3} \geq 3,$$

from which equalities hold everywhere, $k = d$ and $a = b = c = 1$. The sole solution in positive reals is $(a,b,c,d) = (1,1,1,k)$.

(b) Suppose that $k > 0$ and that $a, b, c, d$ satisfy the system of equations. Since

$$2(ab+bc+cd+da+ac+bd) = (3+k)^2-(3+k^2) = 6(1+k),$$

the polynomial

$$p(x) = x^4-(3+k)x^3+3(1+k)x^2-sx+k = (x-1)^3(x-k)-mx,$$

has these numbers as roots where $s = 3k+1+m = abc+bcd+cda+dab$.

Conversely, if $p(x)$ is a polynomial of this form with four distinct roots, then these roots will satisfy the system. Thus, we need to determine conditions on $m$ for which this is the case, *i.e.* for which $f(x) = m$ has four real solutions, where

$$f(x) = \frac{(x-1)^3(x-k)}{x} = (x-1)^3\left(1-\frac{k}{x}\right).$$

(The equation $f(x) = 0$ has four solutions counting multiplicity and gives us the solution $(a,b,c,d) = (1,1,1,k)$ found in (a).)

Now

$$\begin{aligned}f'(x) &= 3(x-1)^2\left(1-\frac{k}{x}\right)+(x-1)^3\left(\frac{k}{x^2}\right)\\ &= \frac{(x-1)^2}{x^2}(3x^2-2kx-k) = \frac{3(x-1)^2}{x^2}(x-u)(x-v),\end{aligned}$$

where $u = (1/3)(k-\sqrt{k^2+3k}) < 0 < v = (1/3)(k+\sqrt{k^2+3k})$. Therefore $f(x)$ increases on $(-\infty, u)$ and $(v, +\infty)$ and decreases on $(u, 0)$ and $(0, v)$. By examining its graph, we see that $f(x) = m$ has four real solutions if and only if $f(u) > f(v)$ and $m \in [f(v), f(u)]$.

Since $u+v=2k/3$ and $uv=-k/3<0$,

$$\begin{aligned} f(u)-f(v) &= (u-v)\left[(u+v)^2-vu-(k+3)(u+v)+3(k+1)-\frac{k}{uv}\right] \\ &= (u-v)\left[\frac{4k^2}{9}+\frac{k}{3}-\frac{2k^2}{3}-2k+3k+3+3\right] \\ &= \frac{(u-v)}{9}[-2k^2+12k+54] = \frac{2(v-u)}{9}(k-9)(k+3). \end{aligned}$$

When $k>9$, the system of equations has infinitely many solutions, one quadruple for any value of $m$ between $f(v)$ and $f(u)$ inclusive. When $0<k<9$, the system of equations has no solution. When $k=9$, the situation gets more interesting. We find that $u=3-2\sqrt{3}$, $v=3+2\sqrt{3}$, $m=f(u)=f(v)=-64$, and $s=28-64=-36$. Therefore

$$p(x)=x^4-12x^3+30x^2+36x+9=(x^2-6x-3)^2,$$

and we obtain the only solution $(a,b,c,d)=(3-2\sqrt{3},3-2\sqrt{3},3+2\sqrt{3},3+2\sqrt{3})$ other than $(1,1,1,9)$.

*Solution 2, by Sergey Sadov.*

As in Solution 1, the system has a solution for each value of $m$ or $s$ for which the polynomial

$$p(x)=x^4-(3+k)x^3+3(k+1)x^2-sx+k=(x-1)^3(x-k)-mx$$

has four real roots. We note that

$$p'(x)=4x^3-3(3+k)x^2+6(k+1)x-s=(x-1)^2[4x-3k-1]-m$$

$$p''(x)=12x^2-6(3+k)x+6(k+1)=12(x-1)\left[x-\frac{k+1}{2}\right]$$

$p(0)=k$, $p'(0)=-s=-(3k+1+m)$, $p(k)=-mk$, and $p(1)=p'(1)=3k+1-s=-m$.

(a) Let $k>3$. If $p(1)=0$, then $p(x)=(x-1)^3(x-k)$ and we obtain the solution $(a,b,c,d)=(1,1,1,k)$.

Let $p(1)<0$. Then $m>0$ and $p(x)<0$ on $[1,k]$. Since $p'(x)$ has odd degree and increases to $p'(1)<0$ on $(-\infty,1]$, $p(x)$ must be decreasing and have one root in $(-\infty,1]$. Since $p(x)$ is convex on $[k,+\infty)$ and $p(k)<0$, it can have only one root in $[k,+\infty)$.

Let $p(1)>0$. Then $p(0)>0$, $p'(0)<0<p'(1)$, $p(x)$ is convex on $[0,1]$ and $[(k+1)/2,\infty)$ and concave on $[1,(k+1)/2]$. There is a unique point $w$ in $(0,1)$ at which $p'(w)=0$ and where $p(x)$ assumes its minimum value there. Since

$$p(w)=p(w)-wp'(w)=(w-1)^2(-3w^2+2kw+k)>(w-1)^2(-3+k)>0,$$

$p(x)$ has no root in $[0,1]$. Since $p(x)$ has at most two roots in $[1,+\infty)$, it cannot have four positive real roots.

It follows that the only solution to the system in positive reals is $(a,b,c,d) = (1,1,1,k)$

(b) Let $k>0$. Suppose that not all the roots of the polynomial $p(x)$ are positive. Since their product $k$ is positive, we must have $a \le b < 0 < c \le d$. Let $u = ac$ and $v = a+c$, so that $bd = k/u$ and $b+d = k+3-v$. Then $u<0$, $a^2+c^2 = v^2-2u$, $b^2+d^2 = (k+3-v)^2 - (2k/u)$ and

$$v^2 - 2u + (k+3-v)^2 - \frac{2k}{u} = k^2+3.$$

Thus

$$\left(v - \frac{3+k}{2}\right)^2 - \left(\frac{k^2-6k-3}{4}\right) = v^2 - (k+3)v + 3k + 3 = u + \frac{k}{u}$$
$$= -\left(|u| + \frac{k}{|u|}\right).$$

Let us use this equation relating $u$ and $v$ to solve the system of equations. For there to be a solution $(u,v)$ we require that the maximum value $M = -2\sqrt{k}$ of the right side be at least as great as the minimum value $N = -(k^2-6k-3)/4$ of the left side. Using

$$4(M-N) = k^2 - 6k - 8\sqrt{k} - 3 = (\sqrt{k}+1)^3(\sqrt{k}-3),$$

we can analyze the situation.

When $0<k<9$, $N>M$ and there is no solution. When $k=9$, $N=M=-6$ and there is a unique pair $(u,v) = (-3,6)$ yielding $a=b=3-2\sqrt{3}$ and $c=d=3+2\sqrt{3}$. When $k>9$, there are infinitely many pairs $(u,v)$ with the required relationship, each giving rise to a solution of the system since the quadratic polynomial $x^2-vx+u$ always has real nonzero roots of opposite sign.

*Editor's comment.* Some solvers located the conditions for $p(x)$ to have four real roots by using various functions, including the discriminant, of the coefficients. Another approach was to fix the value of $d$, thus obtaining a cubic polynomial with parameter $d$; then it was a matter of analyzing when it had three real roots. Both approaches involved daunting manipulations. Theo Koupelis also examined the situation for $-3 \le k < 9$ and showed that the solution was essentially unique.

When $k=0$, then one of the variables, say $d$, vanishes in which case we find that the remaining variables are equal to the roots of a polynomial of the form $(x-1)^3 - m$. This polynomial has real roots if and only if $m=0$ and we get the essentially unique solution $(a,b,c,d) = (1,1,1,0)$ along with its permutations.

When $k=-3$, then $(a,b,c,d)$ satisfies the system if and only if $(-a,-b,-c,-d)$ does.

**4583**. *Proposed by Daniel Sitaru.*

Let

$$A = \begin{pmatrix} \frac{a^2}{(a+b)^2} & \frac{2ab}{(a+b)^2} & \frac{b^2}{(a+b)^2} \\ \frac{c^2}{(b+c)^2} & \frac{b^2}{(b+c)^2} & \frac{2bc}{(b+c)^2} \\ \frac{2ca}{(c+a)^2} & \frac{a^2}{(c+a)^2} & \frac{c^2}{(c+a)^2} \end{pmatrix},$$

where $a$, $b$ and $c$ are positive real numbers. Find the value of the sum of all the entries of $A^n$, where $n$ is a natural number, $n \geq 2$.

*We received 20 submissions of which 15 were correct and complete. We present the solution by Michel Bataille.*

First, we remark that the entries of each row of $A$ sum to 1, that is,

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{1}$$

Second, the sum $s(M)$ of all the entries of any $3 \times 3$ matrix $M$ is

$$s(M) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} M \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

It follows that

$$s(A) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 3.$$

Now, assume that for some integer $n \geq 1$, we have $s(A^n) = 3$. Then, using (1), we obtain

$$s(A^{n+1}) = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \cdot A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} A^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = s(A^n) = 3.$$

By induction, we have $s(A^n) = 3$ for all positive integers $n$.

**4584**. *Proposed by Michel Bataille.*

For $n \in \mathbb{N}$, let

$$S_n = \sum_{k=1}^{n} \frac{k}{n + \sqrt{k + n^2}}.$$

Find real numbers $a, b$ such that $\lim_{n\to\infty} (S_n - an) = b$.

*We received 16 solutions. We present the solution by UCLan Cyprus Problem Solving Group.*

Note that if an $a \in \mathbb{R}$ exists such that $\lim_{n\to\infty}(S_n - an)$ is finite then it is the unique $a$ for which this limit is finite. We have

$$S_n = \sum_{k=1}^{n} \frac{k}{n+\sqrt{n^2+k}} = \sum_{k=1}^{n} \frac{k(\sqrt{n^2+k}-n)}{(n+\sqrt{n^2+k})(\sqrt{n^2+k}-n)}$$
$$= \sum_{k=1}^{n} \left(\sqrt{n^2+k}-n\right)$$
$$= \sum_{k=1}^{n} \left(\sqrt{n^2+k}-n-\frac{k}{2n}\right) + \frac{n(n+1)}{4n}\,.$$

Rearranging,

$$S_n - \frac{n}{4} = \frac{1}{4} - \sum_{k=1}^{n}\left(n+\frac{k}{2n}-\sqrt{n^2+k}\right). \tag{1}$$

We observe that

$$n+\frac{k}{2n}-\sqrt{n^2+k} = \sqrt{n^2+k+\frac{k^2}{4n^2}}-\sqrt{n^2+k}$$
$$= \frac{k^2}{4n^2\left(\sqrt{n^2+k+\frac{k^2}{4n^2}}+\sqrt{n^2+k}\right)}. \tag{2}$$

Since for $1 \leqslant k \leqslant n$ we have

$$n < \sqrt{n^2+k} < \sqrt{n^2+k+\frac{k^2}{4n^2}} = n+\frac{k}{2n} \leqslant n+\frac{1}{2}$$

from (2) we get

$$\frac{k^2}{4n^2(2n+1)} < n+\frac{k}{2n}-\sqrt{n^2+k} < \frac{k^2}{8n^3}.$$

Add up the inequalities for $k$ from 1 to $n$ and use the well-known formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

to get

$$\frac{n+1}{24n} < \sum_{k=1}^{n}\left(n+\frac{k}{2n}-\sqrt{n^2+k}\right) < \frac{(n+1)(2n+1)}{48n^2}.$$

By the Squeeze Theorem

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(n+\frac{k}{2n}-\sqrt{n^2+k}\right) = \frac{1}{24}\,,$$

and therefore from (1)

$$\lim_{n\to\infty}\left(S_n - \frac{n}{4}\right) = \frac{1}{4} - \frac{1}{24} = \frac{5}{24}\,.$$

The solution is thus $a = \frac{1}{4}$ and $b = \frac{5}{24}$.

**4585**. *Proposed by George Stoica.*

Let $P(x)$ be a real polynomial of degree $n$ whose $n$ roots are all real. Then for all $k = 0, \dots, n-2$, prove that, for $c \in \mathbb{R}$:

$$P^{(k)}(c) \neq 0, P^{(k+1)}(c) = 0 \Rightarrow P^{(k+2)}(c) \neq 0.$$

*We received 6 submissions and all of them were correct and complete. All solutions received made use of the following fact in real analysis: if $P \in \mathbb{R}[x]$ splits over $\mathbb{R}$, then so does $P'$. Using this fact, it suffices to prove the required statement in the case $k = 0$. We present the following two solutions, slightly modified by the editor.*

*Solution 1, by Sergey Sadov and UCLan Cyprus Problem Solving Group (independently).*

Let us revisit the proof of this cute fact.

Let $x_1, x_2, \dots, x_k$ be the distinct roots of $P(x)$, and let $m_1, m_2, \dots, m_k$ be their respective multiplicities so that $m_1 + m_2 + \cdots + m_k = n$. Then $x_1, \dots, x_k$ are roots of $P'(x)$ with respective multiplicities $m_1 - 1, \dots, m_k - 1$. In addition, by Rolle's theorem, there is a root of $P'(x)$ in $(x_j, x_{j+1})$ for $j = 1, \dots, k-1$. Altogether, this provides $k - 1 + (m_1 - 1) + \cdots + (m_k - 1) = n - 1$ roots of $P'(x)$. Therefore, all the roots of $P'(x)$ are real. Moreover, from the proof, we could also conclude that if $P(c) \neq 0$ and $P'(c) = 0$, then this additional root $c$ of $P'$ must be a simple root (obtained from applying Rolle's theorem), and thus $P''(c) = 0$. This proves the required claim.

*Solution 2, by Walther Janous.*

Suppose there is $c \in \mathbb{R}$, such that $P(c) \neq 0, P'(c) = 0$. Without loss of generality, we may assume $c = 0$ (otherwise we can consider the polynomial $P(x - c)$). Let

$$P(x) = a \prod_{j=1}^{n} (x - x_j),$$

where $a \neq 0$, and $x_1, x_2, \dots, x_n \in \mathbb{R}$. Since $P(0) \neq 0$, all $x_j$'s are nonzero numbers. It is easy to verify that

$$P'(x) = P(x) \sum_{j=1}^{n} \frac{1}{x - x_j}, \quad P''(x) = P'(x) \sum_{j=1}^{n} \frac{1}{x - x_j} - P(x) \sum_{j=1}^{n} \frac{1}{(x - x_j)^2}.$$

Now $P'(0) = 0$ implies
$$-\frac{P''(0)}{P(0)} = \sum_{j=1}^{n} \frac{1}{x_j^2} > 0.$$
In particular, $P''(0) \neq 0$.

## 4586. *Proposed by Nguyen Viet Hung.*

Find all triples $(m, n, p)$ where $m, n$ are two non-negative integers and $p$ is a prime, satisfying the equation
$$m^4 = 4(p^n - 1).$$

*We received 16 submissions, of which 14 were complete and correct. We present the solution by Amit Kumar Basistha, slightly modified.*

Clearly $2 \mid m^4 \Rightarrow 2 \mid m$. Let $m = 2k$. Then $16k^4 = 4(p^n - 1)$ gives us
$$p^n = 4k^4 + 1 = (2k^2 + 1)^2 - (2k)^2 = (2k^2 + 2k + 1)(2k^2 - 2k + 1).$$

If $p \mid (2k^2 + 2k + 1)$ and $p \mid (2k^2 - 2k + 1)$ then $p$ divides their difference, so $p \mid 4k$, which is absurd as $p^n = 4k^4 + 1$.

Since $2k^2 - 2k + 1 < 2k^2 + 2k + 1$ the only option is that $2k^2 - 2k + 1 = 1 \Rightarrow k(k - 1) = 0 \Rightarrow k = 0$ or $k = 1$.

For $k = 0$ we get $m = 0$, so $p^n = 1$, which has solutions $n = 0$ and $p$ is any prime. For $k = 1$ we get $p^n = 5$ which has the only solution $p = 5$ and $n = 1$.

Therefore the solutions $(m, n, p)$ are $(2, 1, 5)$ and $(0, 0, p)$ where $p$ is any prime.

## 4587. *Proposed by Kai Wang.*

Find an elementary proof of $\tan \dfrac{2\pi}{7} = -\sqrt{7} + 4 \sin \dfrac{4\pi}{7}$.

*We received 24 solutions. They came with a remarkable variety of approaches accompanied by elegant presentations. With reluctance, we limit our selection to three of them.*

*Solution 1 is a composite of similar solutions by Brian Bradie, Richard Hess, and Walther Janous.*

The desired equality, namely $\tan \frac{2\pi}{7} = -\sqrt{7} + 4 \sin \frac{4\pi}{7}$, is equivalent to
$$\sqrt{7} = 8 \sin \frac{2\pi}{7} \cos \frac{2\pi}{7} - \frac{\sin \frac{2\pi}{7}}{\cos \frac{2\pi}{7}}.$$

Because $\frac{2\pi}{7} < \frac{\pi}{3}$, we have
$$\cos \frac{2\pi}{7} > \cos \frac{\pi}{3} = \frac{1}{2} \quad \text{and} \quad 8 \cos^2 \frac{2\pi}{7} - 1 > 0,$$

which means that it suffices to prove

$$\left(8\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}-\frac{\sin\frac{2\pi}{7}}{\cos\frac{2\pi}{7}}\right)^2=7. \tag{1}$$

This will follow quickly from the familiar identity

$$\sin 7x=\sin x(7-56\sin^2 x+112\sin^4 x-64\sin^6 x). \tag{2}$$

Let $s=\sin\frac{2\pi}{7}$ and $c=\cos\frac{2\pi}{7}$. Because $\sin\left(7\cdot\frac{2\pi}{7}\right)=0$, equation (2) becomes

$$0=64s^6-112s^4+56s^2-7,$$

which can be rewritten successively as

$$\begin{aligned}(64s^6-128s^4+64s^2)+(16s^4-16s^2)+s^2 &= 7-7s^2,\\ 64s^2\left(1-s^2\right)^2-16s^2(1-s^2)+s^2 &= 7(1-s^2),\\ 64s^2(1-s^2)-16s^2+\frac{s^2}{(1-s^2)} &= 7,\end{aligned}$$

and finally,

$$\left(8sc-\frac{s}{c}\right)^2=7,$$

which is equation (1).

*Solution 2 by Subhankar Gayen.*

Let $\alpha=\frac{2\pi}{7}$. Note that $\sin 7\alpha=0$. Note also that because

$$\begin{aligned}4\sin 2\alpha-\tan\alpha &= \sec\alpha(4\sin 2\alpha\cos\alpha-\sin\alpha)\\ &= \sec\alpha(2\sin 3\alpha+\sin\alpha)>0,\end{aligned}$$

it suffices to prove that

$$(4\sin 2\alpha-\tan\alpha)^2=7.$$

The desired result will follow from the known equality,

$$\cos 2\alpha+\cos 4\alpha+\cos 6\alpha=-\frac{1}{2},$$

which itself follows from the identity $2\sin x\cos kx=\sin(k+1)x-\sin(k-1)x$:

$$\cos 2\alpha+\cos 4\alpha+\cos 6\alpha=\frac{1}{2\sin\alpha}[(\sin 3\alpha-\sin\alpha)+(\sin 5\alpha-\sin 3\alpha)+(\sin 7\alpha-\sin 5\alpha)]=-\frac{1}{2}.$$

We are now ready for the main argument:

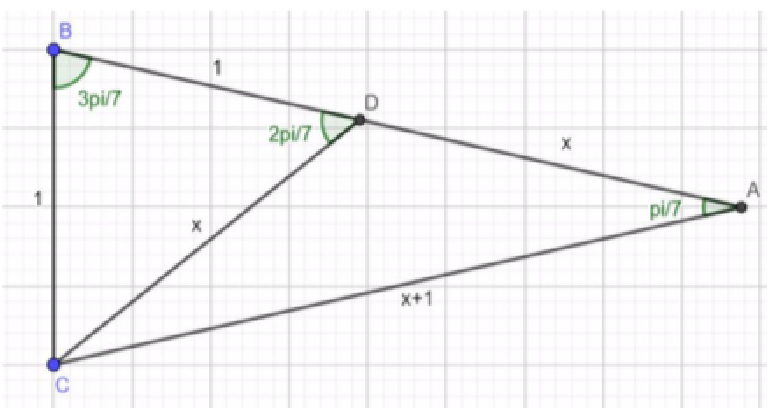
$$\begin{aligned}
7-(4\sin 2\alpha-\tan\alpha)^2 &= \sec^2\alpha[7\cos^2\alpha-(4\sin 2\alpha\cos\alpha-\sin\alpha)^2]\\
&= \sec^2\alpha[7\cos^2\alpha-(2\sin 3\alpha+\sin\alpha)^2]\\
&= \sec^2\alpha[7\cos^2\alpha-4\sin^2 3\alpha-4\sin 3\alpha\sin\alpha-\sin^2\alpha]\\
&= \frac{1}{2}\sec^2\alpha[7(1+\cos 2\alpha)-4(1-\cos 6\alpha)\\
&\quad -4(\cos 2\alpha-\cos 4\alpha)-1+\cos 2\alpha]\\
&= \sec^2\alpha[1+2(\cos 2\alpha+\cos 4\alpha+\cos 6\alpha)]\\
&= 0,
\end{aligned}$$

as desired.

*Solution 3 by Theo Koupelis.*

We consider the isosceles triangle $ABC$ as shown in the accompanying diagram. Let $\theta=\pi/7$. From triangle $CDA$ we get $\cos\theta=\frac{x+1}{2x}$, and from triangle $CBD$ we get $\cos(2\theta)=\frac{x}{2}$. Because $\cos(2\theta)=2\cos^2\theta-1$, we have $\frac{x}{2}=2\left(\frac{x+1}{2x}\right)^2-1$, so that

$$x^3+x^2-2x-1=0. \tag{3}$$




Also, using the law of sines in triangle $CBD$ we get $\sin(3\theta)=x\cdot\sin(2\theta)$, and therefore $\tan(2\theta)=\frac{2}{x^2}\sin(3\theta)$. Because $\sin(3\theta)=\sin(4\theta)$, the given equation is therefore equivalent to

$$\frac{2}{x^2}\sin(3\theta)=-\sqrt{7}+4\sin(3\theta)\iff \sin(3\theta)=\frac{\sqrt{7}x^2}{4x^2-2}. \tag{4}$$

However, using the law of cosines in triangle $CBD$ we get

$$\cos(3\theta)=\frac{2-x^2}{2}. \tag{5}$$

We note that $\pi/4 < 2\theta < \pi/3$ and thus $1 < x < \sqrt{2}$. Therefore, showing that (4) holds is equivalent to showing that

$$\left(\frac{2-x^2}{2}\right)^2 + \left(\frac{\sqrt{7}x^2}{4x^2-2}\right)^2 = 1. \tag{6}$$

Clearing denominators and simplifying, we find that equation (6) is equivalent to

$$4x^2(x^3 - x^2 - 2x + 1)(x^3 + x^2 - 2x - 1) = 0,$$

which is true because of (3).

*Editors Comment.* Problem 4587 deals implicitly with properties of the regular heptagon. For a discussion of such properties and yet one more solution to this problem, see the article "Further Properties of the Regular Heptagon" that appears on pages 194–198 of this issue.

**4588**. *Proposed by the Editorial Board.*

If $a$, $b$, $c$ are positive real numbers with $ab+bc+ca = 3$, prove that $a^n + b^n + c^n \geq 3$ for all integers $n$.

*This proposal was originally inspired by two inequalities proposed by George Apostolopoulos.*

*We received 23 solutions, three of which were incomplete. We present the solution by Ioan Viorel Codreanu.*

For $n \in \mathbb{Z}$, we consider the function $f : (0, \infty)$ defined by $f(x) = x^n$. Because $f''(x) = n(n-1)x^{n-2} \geq 0$ for all $x \geq 0$, for all integers $n$, $f$ is convex. Using Jensen's inequality, we get

$$f\left(\frac{\sum a}{3}\right) \leq \frac{\sum f(a)}{3},$$

that is,

$$\sum a^n \geq 3\left(\frac{a+b+c}{3}\right)^n.$$

We have

$$(a+b+c)^2 \geq 3(ab+bc+ca) = 9,$$

so that

$$a+b+c \geq 3.$$

It follows that

$$\sum a^n \geq 3.$$

*Editor's note.* Janous observed that if $0 < n < 1$, with $(a,b,c) \to \left(\sqrt{3}, \sqrt{3}, 0\right)$, subject to the constraint of the problem, the conclusion of the problem holds only for $n \geq 2 - \frac{2\ln 2}{\ln 3} \approx 0.738$. In the featured solution, convexity holds for all real numbers $n$ except those in the interval $(0, 1)$. Hence, the conclusion of the problem does not hold for all real numbers $n$, but it does hold for all real numbers $n$ outside of this interval.

**4589**. *Proposed by Lorian Saceanu.*

Let $a, b, c$ be real numbers, not all zero, such that $a^2 + b^2 + c^2 = 2(ab + bc + ca)$. Prove that

$$2 \leq \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \leq \frac{12}{5}.$$

*We received 16 submissions, all correct. We present the solution by Subhankar Gayen, slightly enhanced by the editor.*

First note that $a + b \neq 0$. To see this, suppose $a + b = 0$. Then $c \neq 0$ since if $c = 0$ then $a^2 + b^2 = 2ab \implies (a-b)^2 = 0 \implies a = b = 0$ so $a = b = c = 0$, a contradiction. Hence, $a + b \neq 0$. Similarly, $b + c \neq 0$ and $c + a \neq 0$. Due to homogeneity we may assume that $b + c = 2$. Let $x = bc$. We claim that $x \leq 1$. This is true if $b$ and $c$ are of opposite signs. If both are nonnegative, then it follows from the AM-GM inequality.

Next, from the given condition, we have

$$\begin{aligned} a^2 + (b+c)^2 - 2bc &= 2(a(b+c) + bc), && \text{or} \\ a^2 + 4 &= 4a + 4bc = 4a + 4x, && \text{so} \\ x &= \frac{a^2 - 4a + 4}{4} && (1) \end{aligned}$$

Since $x \leq 1$, we then have $0 \leq a \leq 4$. Using $b + c = 2$ and $x = bc$, we get

$$\begin{aligned} \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= \frac{a}{b+c} + \frac{ab + b^2 + c^2 + ca}{a^2 + a(b+c) + bc} \\ &= \frac{a}{b+c} + \frac{a(b+c) + (b+c)^2 - 2bc}{a^2 + 2a + bc} = \frac{a}{2} + \frac{2(a + 2 - x)}{a^2 + 2a + x} \end{aligned} \quad (2)$$

By (1), we have

$$2(a + 2 - x) = 2(a+2) - \frac{a^2 - 4a + 4}{2} = \frac{-a^2 + 8a + 4}{2} \quad (3)$$

and

$$a^2 + 2a + x = a^2 + 2a + \frac{a^2 - 4a + 4}{4} = \frac{5a^2 + 4a + 4}{4}. \quad (4)$$

Hence, from (2)-(4), we get

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = \frac{a}{2} + \frac{2(-a^2 + 8a + 4)}{5a^2 + 4a + 4} = \frac{5a^3 + 36a + 16}{2(5a^2 + 4a + 4)}. \quad (5)$$

By (5), it remains to show that

$$2 \leq \frac{5a^3 + 36a + 16}{2(5a^2 + 4a + 4)} \leq \frac{12}{5},$$

which is true since

$$\frac{5a^3 + 36a + 16}{2(5a^2 + 4a + 4)} - 2 = \frac{5a^3 - 20a^2 + 20a}{2(5a^2 + 4a + 4)} = \frac{5a(a-2)^2}{2(5a^2 + 4a + 4)} \geq 0$$

and

$$\begin{aligned} \frac{5a^3 + 36a + 16}{2(5a^2 + 4a + 4)} - \frac{12}{5} &= \frac{25a^3 - 120a^2 + 84a - 16}{10(5a^2 + 4a - 4)} \\ &= \frac{25a^2(a-4) - 4(5a-1)(a-4)}{10(5a^2 + 4a - 4)} \\ &= \frac{(a-4)(5a-2)^2}{10(5a^2 + 4a - 4)} \leq 0 \end{aligned}$$

For the left inequality, equality holds if and only if $a = 0$ and $b = c$, or any of its permutations. For the right inequality, equality holds if and only if $a = 4b = 4c$ or any of its permutations.

**4590**. *Proposed by George Apostolopoulos.*

Let $ABC$ be an acute angled triangle. Prove that

$$\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} \leq \frac{9}{2},$$

where the sum is taken over all cyclic permutations of $(A, B, C)$.

*We received 13 correct solutions. One other solution that made use of* Maple *was not admitted since more straightforward and elementary methods were available. After some preliminary discussion, we present two solutions.*

Most of the solutions used at least one of the following two identities for angles $A, B, C$ in a triangle:

(1) $\cos A + \cos B + \cos C \leq \frac{3}{2}$;

(2) $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$.

For (1), let $s = \sin \frac{C}{2}$, and note that

$$\begin{aligned} &\frac{3}{2} - (\cos A + \cos B + \cos C) = \frac{3}{2} - \left(2\cos\frac{A-B}{2}\cos\frac{A+B}{2} + 1 - 2\sin^2\frac{C}{2}\right) \\ &= 2s^2 - 2\cos\frac{A-B}{2}s + \frac{1}{2} \geq \frac{1}{2}(4s^2 - 4s + 1) = \frac{1}{2}(2s-1)^2 \geq 0, \end{aligned}$$

with equality if and only if $A = B = C = 60^\circ$.

For (2),

$$
\begin{aligned}
&\cos^2 A + \cos^2 B + \cos^2 C \\
&\qquad = \cos^2 + \cos^2 B + (1 - \sin^2 C) \\
&\qquad = 1 + \cos^2 A + \cos^2 B - \sin^2(A+B) \\
&\qquad = 1 + \cos^2 A + \cos^2 B - \sin^2 A \cos^2 B - \sin^2 B \cos^2 A - 2\cos A \cos B \sin A \sin B \\
&\qquad = 1 + \cos^2 A(1 - \sin^2 B) + \cos^2 B(1 - \sin^2 A) - 2\cos A \cos B \sin A \sin B \\
&\qquad = 1 + 2\cos A \cos B(\cos A \cos B - \sin A \sin B) = 1 + 2\cos A \cos B \cos(A+B) \\
&\qquad = 1 - 2\cos A \cos B \cos C.
\end{aligned}
$$

*Solution 1, by Theo Koupelis and Kee-Wai Lau (independently).*

$$
\begin{aligned}
\cos^2 B + \cos^2 C &= 1 + \frac{1}{2}(\cos 2B + \cos 2C) \\
&= 1 + \cos(B-C)\cos(B+C) \\
&= 1 - \cos(B-C)\cos A \geq 1 - \cos A,
\end{aligned}
$$

with similar inequalities for other permutations of the angles. Hence

$$
\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} \leq \sum \frac{1 - \cos^2 A}{1 - \cos A} = \sum (1 + \cos A) = 3 + \sum \cos A \leq \frac{9}{2}.
$$

*Solution 2, by Michel Bataille, Marie-Nicole Gras and Nikos Ntorvas (done independently).*

Observe that $\cos^2 B + \cos^2 C \geq 2\cos B \cos C$, *etc.* Then

$$
\begin{aligned}
\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} &= \sum \frac{1 - \cos^2 A}{\cos^2 B + \cos^2 C} \\
&= \sum \frac{\cos^2 B + \cos^2 C + 2\cos A \cos B \cos C}{\cos^2 B + \cos^2 C} \\
&\leq 3 + \sum \frac{2\cos A \cos B \cos C}{2\cos B \cos C} \\
&= 3 + \sum \cos A \leq \frac{9}{2}.
\end{aligned}
$$

When the triangle is not acute, then $\cos A \cos B \cos C \leq 0$ and

$$
\sum \frac{\sin^2 A}{\cos^2 B + \cos^2 C} = \sum \frac{\sin^2 A}{\sin^2 A - 2\cos A \cos B \cos C} \leq \sum \frac{\sin^2 A}{\sin^2 A} = 3 < \frac{9}{2}.
$$

*Editor's comments.* Several solvers used Jensen's Inequality. Some converted the trigonometric functions into expressions involving the sidelengths, circumradius and inradius of the triangle.