CHENNAI MATHEMATICAL INSTITUTE
Undergraduate Entrance Examination, May 2022
Time: 3.5 hours

Unless specified otherwise, in this exam all numbers are real and "function" means a function whose codomain as well as domain is the set of all real numbers (or an implied subset).

Part A instructions

- **Part A has 10 questions, each worth 4 points, for a total of 40.** Every question has a group of four statements. (These statements are numbered 1 to 40 for technical reasons.) For each statement, *independently* choose one of the three given options: True / False / No Attempt. In particular there is no guarantee that at least one of the four statements in a given question is true.

- **Grading scheme.** Each question will be graded as follows.

All 4 answers correct	4 points
3 correct and 1 No Attempt	2 points
2 correct and 2 No Attempt	1 point
Anything else	0 points

Note that getting even one of the four answers wrong will result in zero points for that question. So if you are not sure, you are advised to choose No Attempt instead of guessing.

- **Enter your answers to part A into the computer.** Points for part A will be given based only on answers entered into the computer. If you do not choose an option for a statement, it will be treated as No Attempt.

- **Part A will be used for screening.** Part B is assured to be graded if you meet *any one* of the following two conditions. (i) You score at least 24 in part A. (ii) You are among the top 400 students in part A. Thus part B will be graded for at least 400 students, more if enough students score at least 24 in part A.

Part B instructions

- **Part B has 6 problems worth a total of 80 points.** See each question for the break-up. You are advised to spend at least 2 hours on part B.

- **Clearly explain your entire reasoning**. No credit will be given without correct reasoning. Partial solutions may get partial credit. You may solve a later part of a problem by assuming a previous part, even if you could not do the earlier part.

- **Solve each part B problem on the designated pages** in the answer booklet. Use the blank pages at the end for rough work OR if you need extra space for any problem. Clearly label any such solution overflowing to last pages. For problems with multiple parts, clearly label your solution to each part separately.

CMI BSc entrance exam on May 22, 2022

Part A. Select True or False or No Attempt for each statement.

A1. Suppose $a_0, a_1, a_2, a_3, \dots$ is an arithmetic progression with a_0 and a_1 *positive integers*. Let $g_0, g_1, g_2, g_3, \dots$ be the geometric progression such that $g_0 = a_0$ and $g_1 = a_1$.

Statements

(1) We must have $(a_5)^2 \geq a_0 a_{10}$.

(2) The sum $a_0 + a_1 + \cdots + a_{10}$ must be a multiple of the integer a_5.

(3) If $\sum_{i=0}^{\infty} a_i$ is $+\infty$ then $\sum_{i=0}^{\infty} g_i$ is also $+\infty$.

(4) If $\sum_{i=0}^{\infty} g_i$ is finite then $\sum_{i=0}^{\infty} a_i$ is $-\infty$.

A2. Any two events X and Y are called *mutually exclusive* when the probability $P(X \textit{ and } Y) = 0$ and they are called *exhaustive* when $P(X \textit{ or } Y) = 1$. Suppose A and B are events and the probability of each of these two events is **strictly** between 0 and 1 (i.e., $0 < P(A) < 1$ and $0 < P(B) < 1$).

Statements

(5) A and B are mutually exclusive if and only if *not* A and *not* B are exhaustive.

(6) A and B are independent if and only if *not* A and *not* B are independent.

(7) A and B cannot be simultaneously independent and exhaustive.

(8) A and B cannot be simultaneously mutually exclusive and exhaustive.

A3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 10 & 20 & 30 \\ 11 & 22 & k \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, where k is a constant and x, y, z are variables.

Statements

(9) Regardless of the value of k, the matrix A is not invertible, i.e., there is no 3×3 matrix B such that $BA =$ the 3×3 identity matrix.

(10) There is a unique k such that determinant of A is 0.

(11) The set of solutions (x, y, z) of the matrix equation $A\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is either a line or a plane containing the origin.

(12) If the equation $A\mathbf{v} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$ has a solution, then it must be true that $q = 10p$.

A4. Consider the following conditions on a function f whose domain is the closed interval [0,1]. (For any condition involving a limit, at the endpoints, use the relevant one-sided limit.)

I. f is differentiable at each $x \in [0,1]$.

II. f is continuous at each $x \in [0,1]$.

III. The set $\{f(x) \mid x \in [0,1]\}$ has a maximum element and a minimum element.

Statements

(13) If I is true, then II is true.

(14) If II is true, then III is true.

(15) If III is false, then I is false.

(16) No two of the three given conditions are equivalent to each other. (Two statements being equivalent means each implies the other.)

A5. Statements

(17) Let $a = \frac{1}{\ln 3}$. Then $3^a = e$.

(18) $\sin(0.02) < 2\sin(0.01)$.

(19) $\arctan(0.01) > 0.01$.

(20) $4\int_0^1 \arctan(x)dx = \pi - \ln 4$.

A6. Let

$$f(x) = \frac{1}{|\ln x|}\left(\frac{1}{x} + \cos x\right).$$

Statements

(21) As $x \to \infty$, the sign of $f(x)$ changes infinitely many times.

(22) As $x \to \infty$, the limit of $f(x)$ does not exist.

(23) As $x \to 1$, $f(x) \to \infty$.

(24) As $x \to 0^+$, $f(x) \to 1$.

A7. Let $f_0(x) = x$. For $x > 0$, define functions inductively by $f_{n+1}(x) = x^{f_n(x)}$. So $f_1(x) = x^x$, $f_2(x) = x^{x^x}$, etc. Note that $f_0(1) = f_0'(1) = 1$.

Statements

(25) As $x \to 0^+$, $f_1(x) \to 1$.

(26) As $x \to 0^+$, $f_2(x) \to 1$.

(27) $\int_0^1 f_3(x)dx = 1$.

(28) The derivative of f_{123} at $x = 1$ is 1.

A8. Let $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $L = \{a, b, c\}$.

Statements

(29) Suppose we arrange the 12 elements of $L \cup N$ in a line such that all three letters appear consecutively (in any order). The number such arrangements is less than $10! \times 5$.

(30) More than half of the functions from N to L have the element b in their range.

(31) The number of one-to-one functions from L to N is less than 512.

(32) The number of functions from N to L that do not map consecutive numbers to consecutive letters is greater than 512. (e.g., $f(1) = b$ and $f(2) = a$ or c is not allowed. $f(1) = a$ and $f(2) = c$ is allowed. So is $f(1) = f(2)$.)

A9. In this question z denotes a *non-real* complex number, i.e., a number of the form $a+ib$ (with a, b real) whose imaginary part b is nonzero. Let $f(z) = z^{222} + \frac{1}{z^{222}}$.

Statements

(33) If $|z| = 1$, then $f(z)$ must be real.

(34) If $z + \frac{1}{z} = 1$, then $f(z) = 2$.

(35) If $z + \frac{1}{z}$ is real, then $|f(z)| \leq 2$.

(36) If $f(z)$ is a real number, then $f(z)$ must be positive.

A10. Suppose that cards numbered $1, 2, \ldots, n$ are placed on a line in some sequence (with each integer $i \in [1, n]$ appearing exactly once). A *move* consists of interchanging the card labeled 1 with any other card. If it is possible to rearrange the cards in increasing order by doing a series of moves, we say that the given sequence can be *rectified.*

Statements

(37) The sequence 9 1 2 3 4 5 6 7 8 can be rectified in 8 moves and no fewer moves.

(38) The sequence 1 3 4 5 6 7 8 9 2 can be rectified in 8 moves and no fewer moves.

(39) The sequence 1 3 4 2 9 5 6 7 8 cannot be rectified.

(40) There exists a sequence of 99 cards that cannot be rectified.

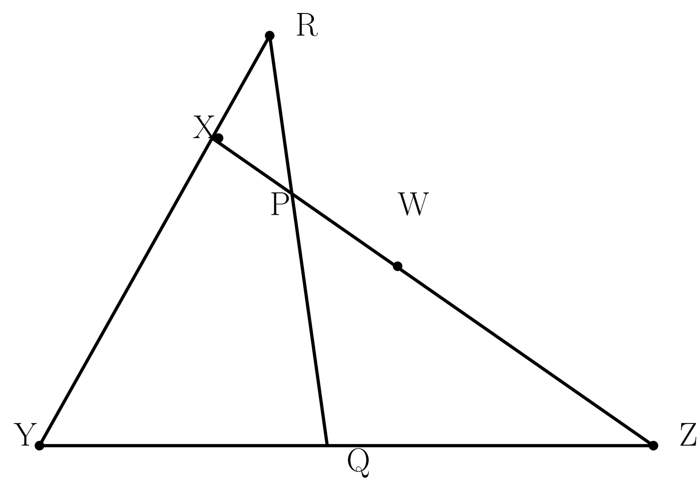
Part B. Explain your reasoning fully.

B1. [11 points] Given $\triangle XYZ$, the following constructions are made: mark point W on segment XZ, point P on segment XW and point Q on segment YZ such that

$$\frac{WZ}{YX} = \frac{PW}{XP} = \frac{QZ}{YQ} = k\,.$$

See the schematic figure (not to scale). Extend segments QP and YX to meet at the point R as shown. Prove that $XR = XP$.

Hint (use this or your own method): A suitable construction may help in calculations.



B2. [11 points] In the XY plane, draw horizontal and vertical lines through each integer on both axes so as to get a grid of small 1×1 squares whose vertices have integer coordinates.

(i) Consider the line segment D joining (0,0) with (m, n). Find the number of small 1×1 squares that D cuts *through*, i.e., squares whose *interiors* D intersects. (Interiors consist of points for which *both* coordinates are non-integers.) For example, the line segment joining (0,0) and (2,3) cuts through 4 small squares, as you can check by drawing.

(ii) Now L is allowed to be an arbitrary line in the plane. Find the maximum number of small 1×1 squares in an $n \times n$ grid that L can cut *through*, i.e., we want L to intersect the interiors of maximum possible number of small squares inside the square with vertices $(0, 0), (n, 0), (0, n)$ and (n, n).

B3. [**14 points**] For a positive integer n, let $f(x) = \sum_{i=0}^{n} x^i = 1 + x + x^2 + \cdots + x^n$. Find the number of local maxima of $f(x)$. Find the number of local minima of $f(x)$. For each maximum/minimum $(c, f(c))$, find the integer k such that $k \leq c < k + 1$.

Hints (use these or your own method): It may be helpful to (i) look at small n, (ii) use the definition of f as well as a closed formula, and (iii) treat $x \geq 0$ and $x < 0$ separately.

B4. [**14 points**] Let $\mathbb{R}_+$ denote the set of *positive* real numbers. For a *continuous* function $f : \mathbb{R}_+ \to \mathbb{R}_+$, define

$A_r =$ the area bounded by the graph of f, X-axis, $x = 1$ and $x = r$
$B_r =$ the area bounded by the graph of f, X-axis, $x = r$ and $x = r^2$.

Find all *continuous* $f : \mathbb{R}_+ \to \mathbb{R}_+$ for which $A_r = B_r$ for every positive number r.

Hints (use these or your own method): Find an equation relating $f(x)$ and $f(x^2)$. Consider the function $xf(x)$. Suppose a sequence x_n converges to b where all x_n and b are in the domain of a continuous function g. Then $g(x_n)$ must converge to $g(b)$. E.g., $g(3^{\frac{1}{n}}) \to g(1)$.

B5. [**15 points**] Two *distinct* real numbers r and s are said to form a *good pair* (r, s) if

$$r^3 + s^2 = s^3 + r^2.$$

(i) Find a good pair (a, ℓ) with the largest possible value of ℓ. Find a good pair (s, b) with the smallest possible value s. For every good pair (c, d) other than the two you found, show that there is a third real number e such that (d, e) and (c, e) are also good pairs.

(ii) Show that there are infinitely many good pairs of *rational* numbers.

Hints (use these or your own method): The function $f(x) = x^3 - x^2$ may be useful. If (r, s) is good pair, can you express s in terms of r? You may use that there are infinitely many right triangles with integer sides such that no two of these triangles are similar to each other.

B6. [**15 points**] Solve the following. You may do (i) and (ii) in either order.

(i) Let p be a prime number. Show that $x^2 + x - 1$ has at most two roots modulo p, i.e.,

$$\text{the cardinality of } \{n \mid 1 \leq n \leq p \text{ and } n^2 + n - 1 \text{ is divisible by } p\} \text{ is } \leq 2.$$

Find all primes p for which this set has cardinality 1.

(ii) Find all positive integers $n \leq 121$ such that $n^2 + n - 1$ is divisible by 121.

(iii) What can you say about the number of roots of $x^2 + x - 1$ modulo p^2 for an arbitrary prime p, i.e., the cardinality of

$$\{n \mid 1 \leq n \leq p^2 \text{ and } n^2 + n - 1 \text{ is divisible by } p^2\}\,?$$

You do NOT need to repeat any reasoning from previous parts. You may simply refer to any such relevant reasoning and state your conclusion with a brief explanation.