CHENNAI MATHEMATICAL INSTITUTE

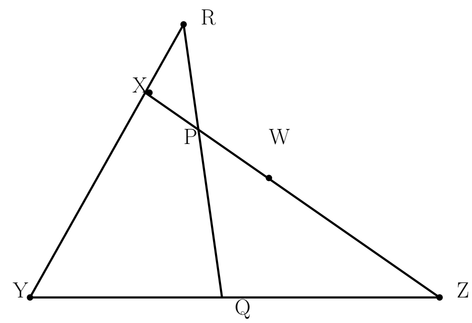
Undergraduate Programme in Mathematics and Computer Science/Physics
Solutions of the 22$^{\text{nd}}$ May 2022 exam

Note: The solutions below consist only of main steps and strategies and do not contain all the details expected in the exam.

B1. [**11 points**] Given $\triangle XYZ$, the following constructions are made: mark point W on segment XZ, point P on segment XW and point Q on segment YZ such that

$$\frac{WZ}{YX} = \frac{PW}{XP} = \frac{QZ}{YQ} = k.$$

See the schematic figure (not to scale). Extend segments QP and YX to meet at the point R as shown. Prove that $XR = XP$.



Solution: First a construction - mark V on XZ such that QV is parallel to YR. There are two cases here depending on whether V is between PW or WZ, however, the arguments are the same. We assume here that V is between PW. The aim is to show that $\triangle VPQ$ is isosceles and then show that it is similar to $\triangle XPR$. Use BPT to conclude that $VQ = \frac{k}{k+1}(XY)$. Using the given ratios find an expression for VZ and substitute it in $PV = PZ - VZ$ to conclude that $PV = VQ$.
One can also extend ZX to ZX' such that YX' is parallel to PQ. One can then show that $\triangle YXX'$ is isosceles and similar to $\triangle RXP$.
Another strategy is to use Menalaus theorem for $\triangle XYZ$ with segment QR as the transversal. We have:

$$\frac{XR}{RY}\frac{YQ}{QZ}\frac{ZP}{PX} = -1.$$

This leads to the following implications leading to the equality we want:

$$XR \cdot PZ = RY \cdot PW$$
$$\frac{XY + XR}{XR} = \frac{PW + WZ}{PW}$$
$$\frac{XY}{XR} = \frac{WZ}{PZ}$$
$$\frac{PW}{XR} = \frac{PW}{XP}.$$

B2. [**11 points**] In the XY plane, draw horizontal and vertical lines through each integer on both axes so as to get a grid of small 1×1 squares whose vertices have integer coordinates.

1. Consider the line segment D joining $(0, 0)$ with (m, n). Find the number of small 1×1 squares that D cuts through, i.e., squares whose interiors D intersect. For example, the line segment joining $(0, 0)$ and $(2, 3)$ cuts through 4 small squares.

2. Now let L be an arbitrary line. Find the maximum number of small 1×1 squares in an $n \times n$ grid that L can cut through.

Solution: Assume $\gcd(m, n) = 1$. The line D has to cross $m-1$ vertical as well as horizontal lines. Moreover, D doesn't pass through any grid points. Hence, together with the starting square, we see that D cuts through $m + n - 1$ squares.
Let $\gcd(m, n) = d$. The above argument is valid from $(0, 0)$ to $(m/d, n/d)$ and so on for d many sections of D. Therefore the total number of squares D cuts is $m + n - d$.
Note that in order for L to cut through maximum number of squares it should not pass through any internal grid point. This is possible for a line joining $(0, 0)$ with (x, n) where $n - 1 < x < n$. The required answer is $2n - 1$.

B3. [**14 points**] For a positive integer n, let $f(x) := 1 + x + x^2 \cdots + x^n$. Find the number of local maxima of $f(x)$. Find the number of local minima of $f(x)$. For each maximum/ minimum $(c, f(c))$, find the integer k such that $k \leq c < k + 1$.

Solution: We have $f'(x) = 1 + 2x + \cdots + nx^{n-1}$. For $x \geq 0$ the derivative is strictly positive, hence $f(x)$ is strictly increasing. Therefore, we should only analyze negative values of x. Write the derivative as the following rational function

$$f'(x) = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Note that there is no problem in the expression since we are assuming $x < 0$. Denote by $D(x)$ the denominator of the derivative.
The case when n is odd. For $x < 0$ the polynomial $D(x)$ is strictly positive. Hence there can't be any critical point.
The case when n is even. Observe that there could be only one critical point $c \in (-1, 0)$. Since $D(x) < 0$ for $x \leq -1$ and $D(0) = 1$. Moreover, $D'(x) > 0$ for $x < 0$ so $f'(x)$ is increasing on $(-\infty, 0)$ hence it vanishes exactly once. As the derivative changes sign from $-$ve to $+$ve passing through c, so there is exactly one global minimum at c (where, $-1 < c < 0$).

B4. [**14 points**] For a continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$, define

- A_r = the area bounded by the graph of f, X-axis, $x = 1$ and $x = r$.
- B_r = the area bounded by the graph of f, X-axis, $x = r$ and $x = r^2$.

Find all continuous functions f for which $A_r = B_r$ for every positive number r.

Solution: We are given

$$\int_1^r f(x)dx = \int_r^{r^2} f(x)dx.$$

Applying d/dr, fundamental theorem of calculus and the chain rule to above equality we get

$$f(x) = xf(x^2) \quad \forall x \in \mathbb{R}_+.$$

Letting $g(x) = xf(x)$ we see that $g(x) = g(x^2)$ for all x in the domain. Hence $g(x) = g(x^{\frac{1}{2^n}})$ for all x and positive integers n. However, as n goes to infinity $x^{\frac{1}{2^n}}$ tends to 1 we have that $g(x)$ converges to $f(1)$. Hence $xf(x) = f(1)$ for all values of $x \in \mathbb{R}_+$.

B5. [**14 points**] Two distinct real numbers r and s are said to form a good pair (r, s) if

$$r^3 + s^2 = s^3 + r^2.$$

1. Find a good pair (a, l) with the largest possible value of l. Find a good pair (s, b) with the smallest value of s. For every good pair (c, d) other than the two you found, show that there is a third real number e such that (d, e) and (c, e) are good pairs.

2. Show that there are infinitely many good pairs of rational numbers.

Solution: Consider the function $f(x) = x^3 - x^2$. Therefore (r, s) is a good pair iff $f(r) = f(s)$.
Observe that $x = 0, \frac{2}{3}$ are the only critical points of f. The local maximum occurs at $x = 0$. The line $y = 0$ intersects the graph of $f(x)$ at $(0, 0)$ and $(1, 0)$. Hence the required good pair (a, l) with the largest l value is $(0, 1)$.
Note that the local minimum occurs at $x = \frac{2}{3}$. The line $y = f(\frac{2}{3}) = \frac{-4}{27}$ intersects the graph at $(\frac{-1}{3}, \frac{-4}{27})$ and $(\frac{2}{3}, \frac{-4}{27})$. Hence the required good pair is $(\frac{-1}{3}, \frac{2}{3})$.
For $k \in (\frac{-4}{27}, 0)$ the line $y = k$ intersects the graph at 3 points. Hence the last statement of the first part follows.
For the second part we need to show that there for every rational number $q \in (\frac{-4}{27}, 0)$ the equation

$$x^3 - x^2 - q = 0$$

has infinitely many rational solutions. However, this is true because there are infinitely many rationals satisfying $c + d + e = 1, cd + de + ce = 0, cde = q$.

B6. [**14 points**] Solve the following.

1. Let p be a prime. Show that $x^2 + x - 1$ has at most two roots modulo p. Find all primes p for which there is exactly one root.
2. Find all positive integers $n \leq 121$ such that $n^2 + n - 1$ is divisible by 121.
3. What can you say about the number of roots of this equation modulo p^2.

Solution: Let a, b be two distinct roots of the equation modulo p. Therefore, p divides $a^2 + a - 1 - (b^2 + b - 1)$, which is equivalent to saying that p divides either $a - b$ or $a + b + 1$. In the former case we will have $a = b$, which is not allowed. Since both a, b are between 1 and p we have $3 \leq a + b + 1 \leq 2p - 3$ which implies $a + b + 1 = p$. Thus $b = p - a - 1$ is uniquely determined.
Suppose a is the only root. Then $p - a - 1 = a$, i.e., $p = 2a + 1$. Therefore, $2a + 1$ divides $4(a^+ a - 1)$ and $(2a + 1)^2$. Subtracting we get that $2a + 1$ divides 5.
Part 2: Since 121 divides $n^+ n - 1$, 11 also divides it. Note that $n^2 + n - 1$ and $n^2 + n - 12$ are congruent modulo 11. So the roots of the equation are $7, 3$ modulo 11.
Consider $n = 3 + 11k$. Then $n^2 + n - 1$ is congruent to $77k + 11$ modulo 121. Then $k = 3$ works giving us $n = 36$. Now consider $n = 7 + 11k$. In that case, $n^+ n - 1$ is congruent to $165k + 55$ modulo 121. Which gives us $k = 7$ and $n = 84$.
For part (3), let a be a root modulo p. Then n is of the form $kp + a$ for some k between 0 and $p - 1$. We would like to solve for k the following equation

$$(kp + a)^2 + (kp + a) - 1$$

modulo p^2. This is equivalent to finding k such that p divides $k(2a + 1) + a^2 + a - 1$. If $2a + 1$ is not a multiple of p then $k = -(2a + 1)^{-1}(a^2 + a - 1)$. If p divides $(2a + 1)$ then it is 5 and there is no such n.