2021 CMI BSc entrance examination

Part A Answers with explanation

1. Consider the two equations numbered [1] and [2]:

$$\log_{2021} a = 2022 - a \qquad [1]$$
$$2021^b = 2022 - b \qquad [2]$$

   (a) Equation [1] has a unique solution.

   (b) Equation [2] has a unique solution.

   (c) There exists a solution a for [1] and a solution b for [2] such that $a = b$.

   (d) There exists a solution a for [1] and a solution b for [2] such that $a+b$ is an integer.

   Correct options: a,b,d **Wrong: c**

   $a = 2021, b = 1$ is the unique solution. (This can also be solved qualitatively, e.g., the graph of $y = 2022 - x$ is decreasing, with range the set of real numbers. It is easy to see that it must intersect each of the graphs $y = 2021^x$ and $y = \log_{2021} x$ (both of which are increasing) exactly once. Substituting $c = 2022 - a$, the first equation is equivalent to $2021^c = 2022 - c$, which is the same as the second equation, so the (unique) solutions c and b are equal, i.e., $b = c = 2022 - a$, so $a + b = 2022$. If $a = b$, both would need to be 1011, which is manifestly not a solution to either equation.)

2. A prime p is an integer ≥ 2 whose only positive integer factors are 1 and p.

   (a) For any prime p the number $p^2 - p$ is always divisible by 3.

   (b) For any prime $p > 3$ exactly one of the numbers $p - 1$ and $p + 1$ is divisible by 6.

   (c) For any prime $p > 3$ the number $p^2 - 1$ is divisible by 24.

   (d) For any prime $p > 3$ one of the three numbers $p + 1, p + 3$ and $p + 5$ is divisible by 8.

   Correct options: b,c **Wrong: a,d**

   (a) is false for any number that is 2 modulo 3, in particular for 2. To see that (b) and (c) are true, note that any prime p greater than 3 is not divisible by 3, so p is either 1 mod 3 (which makes $p - 1$ divisible by 3) or 2 mod 3 (which makes $p + 1$ divisible by 3). Note also that both $p - 1$ and $p + 1$ are even, which gives (b). In fact they are consecutive even numbers, so one of them is a multiple of 4, making their product $p^2 - 1$ a multiple of 8, giving (c). Finally, note that any prime p that is 1 mod 8 violates (d), e.g., $p = 17$.

3. We want to construct a triangle ABC such that angle A is $20.21°$, side AB has length 1 and side BC has length x where x is a positive real number. Let $N(x) =$ the number of pairwise noncongruent triangles with the required properties.

   (a) There exists a value of x such that $N(x) = 0$.

   (b) There exists a value of x such that $N(x) = 1$.

   (c) There exists a value of x such that $N(x) = 2$.

   (d) There exists a value of x such that $N(x) = 3$.

   Correct options: a,b,c **Wrong: d**

   Draw a ray with endpoint A. The point C will be chosen on this ray later on. Draw a segment AB of length 1 making an angle $20.21°$ with this ray. Now, to fulfil the required properties, a necessary and sufficient condition for C is that it is on the original ray as well as on the circle with center B and radius x. As x increases from 0, the number of intersections of the expanding circle with the ray goes from 0 to 1 (when ABC is a right angled triangle, i.e., when $x = \sin 20.21°$) to 2 and finally back to 1.

4. Consider polynomials of the form $f(x) = x^3 + ax^2 + bx + c$ where a, b, c are *integers*. Name the three (possibly non-real) roots of $f(x)$ to be p, q, r.

   (a) If $f(1) = 2021$, then $f(x) = (x-1)(x^2+sx+t)+2021$ where s, t must be integers.

   (b) There is such a polynomial $f(x)$ with $c = 2021$ and $p = 2$.

   (c) There is such a polynomial $f(x)$ with $r = \frac{1}{2}$.

   (d) The value of $p^2 + q^2 + r^2$ does not depend on the value of c.

   Correct options: a,d **Wrong: b,c**

   (a) is true by the remainder theorem. Long division automatically gives integers s, t. Uniqueness of quotient and remainder for polynomial long division means those are the only values of s, t that work. (b) is false by substituting $x = 2$ into $f(x)$ and noting that $c = 2021$ forces $f(2)$ to be odd, in particular nonzero. To see that (c) is false, substitute $x = \frac{1}{2}$ into $f(x)$, multiply by 8 to clear denominators and see that the leading term makes the integer $8f(\frac{1}{2})$ odd. So $f(\frac{1}{2})$ is nonzero. (General version of (b) and (c) that one gets by the similar reasoning: suppose a polynomial $p(x)$ with integer coefficients has a rational root $\frac{r}{s}$ written in lowest form. Then the leading coefficient of $p(x)$ is divisible by s and the constant term of $p(x)$ is divisible by r. Often used special case: for a polynomial $p(x) = x^n +$ lower terms with integer coefficients, any rational root must be an integer.) For (d) note that $p^2+q^2+r^2 = (p+q+r)^2 - 2(pq+pr+qr) = (-a)^2 - 2b$ does not depend on c.

5. For any *complex* number z define $P(z) =$ the cardinality of $\{z^k | k \text{ is a positive integer}\}$, i.e., the number of distinct positive integer powers of z. It may be useful to remember that π is an irrational number.

   (a) For each positive integer n there is a complex number z such that $P(z) = n$.

   (b) There is a *unique* complex number z such that $P(z) = 3$.

   (c) If $|z| \neq 1$, then $P(z)$ is infinite.

   (d) $P(e^i)$ is infinite.

   Correct options: a,c,d **Wrong: b**

   (a) is true by $z = e^{\frac{2\pi i}{n}}$ or any primitive n^{th} root of unity. (b) is false: there are 2 primitive third roots of 1, namely $\omega = e^{\frac{2\pi i}{3}}$ and ω^2. (c) is true because then each $|z^i|$ is a distinct positive real number. (d) is true because $P(z)$ is finite only if powers of z cycle back to 1, which happens for $z = re^{i\theta}$ only if ($r = 1$ and) the argument θ is a rational multiple of π. But for $z = e^i$, we have $\theta = 1$.

6. A stationary point of a function f is a real number r such that $f'(r) = 0$. A polynomial need not have a stationary point (e.g. $x^3 + x$ has none). Consider a polynomial $p(x)$.

   (a) If $p(x)$ is of degree 2022, then $p(x)$ must have at least one stationary point.

   (b) If the number of distinct *real* roots of $p(x)$ is 2021, then $p(x)$ must have at least 2020 stationary points.

   (c) If the number of distinct *real* roots of $p(x)$ is 2021, then $p(x)$ can have at most 2020 stationary points.

   (d) If r is a stationary point of $p(x)$ AND $p''(r) = 0$, then the point $(r, p(r))$ is neither a local maximum nor a local minimum point on the graph of $p(x)$.

   Correct options: a,b **Wrong: c,d**

   (a) $p'(x)$ is a polynomial of degree 2021, which is odd, so it has a root by intermediate value theorem by looking at behaviour as $x \to \pm\infty$. (b) The graph of $p(x)$ has to turn between any two consecutive zeros, giving a stationary point, in fact a local max/min (c) The graph of $p(x)$ can turn more than once between zeros, or turn outside extreme zeros or have stationary points that are not maxima or minima. (d) is false, e.g., $p(x) = x^4$.

7. Given three *distinct positive* constants a, b, c we want to solve the simultaneous equations

$$ax + by = \sqrt{2}$$
$$bx + cy = \sqrt{3}$$

(a) There exists a combination of values for a, b, c such that the above system has infinitely many solutions (x, y).

(b) There exists a combination of values for a, b, c such that the above system has exactly one solution (x, y).

(c) Suppose that for a combination of values for a, b, c, the above system has NO solution. Then $2b < a + c$.

(d) Suppose $2b < a + c$. Then the above system has NO solution.

Correct options: a,b,c **Wrong: d**

Each of the given equations defines a line in the XY plane. (a) One can arrange both lines to be identical by having each equation a scalar multiple of the other, e.g., $a = 1, b = \frac{\sqrt{3}}{\sqrt{2}}, c = \frac{3}{2}$. (b) There is a unique solution when the two lines are distinct and not parallel. (c) The two lines are given to be parallel. So slopes are equal, i.e., $b^2 = ac$. Thus b is the geometric mean of a and c, so $b <$ the arithmetic mean $\frac{a+c}{2}$. (Recall that a, b, c are distinct and positive.) (d) is absurd. Just ensure $b^2 \neq ac$.

8. Given two *distinct nonzero* vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in 3 dimensions, define a sequence of vectors by

$$\mathbf{v}_{n+2} = \mathbf{v}_n \times \mathbf{v}_{n+1} \text{ (so } \mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2, \mathbf{v}_4 = \mathbf{v}_2 \times \mathbf{v}_3 \text{ and so on).}$$

Let $S = \{\mathbf{v}_n | n = 1, 2, \ldots\}$ and $U = \{\frac{\mathbf{v}_n}{|\mathbf{v}_n|} | n = 1, 2, \ldots\}$. (**Note**: Here $\times$ denotes the cross product of vectors and $|\mathbf{v}|$ denotes the magnitude of the vector $\mathbf{v}$. The vector $\mathbf{0}$ with 0 magnitude, if it occurs in S, is counted. But in that case of course the $\mathbf{0}$ vector is not considered while listing elements of U.)

(a) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 2.

(b) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 3.

(c) There exist vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ for which the cardinality of S is 4.

(d) Suppose that for some $\mathbf{v_1}$ and $\mathbf{v_2}$, the set S is infinite. Then the set U is also infinite.

Correct options: b,c **Wrong: a,d**

It is easiest to do this geometrically, remembering that the cross product $\mathbf{p}\times\mathbf{q}$ of vectors $\mathbf{p}$ and $\mathbf{q}$ is perpendicular to both of them and $|\mathbf{p}\times\mathbf{q}| = |\mathbf{p}|\,|\mathbf{q}|\sin(\text{angle between } \mathbf{p}$ and $\mathbf{q}) = |\mathbf{p}|\,|\mathbf{q}|$ if $\mathbf{p}$ and $\mathbf{q}$ are perpendicular. The cross product of nonzero vectors is zero if and only if the vectors are collinear. It is easy to see that the only way the zero vector is in S is if $\mathbf{v}_3$ is zero, which will happen only when the nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are collinear, and in that case the sequence is zero all the way from $\mathbf{v}_3$ onwards.

As the starting vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are distinct and nonzero, the third vector $\mathbf{v}_3 = \mathbf{v}_1\times\mathbf{v}_2$, being perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, is distinct from them. This is true even if $\mathbf{v}_3$ is $\mathbf{0}$ due to $\mathbf{v}_1$ and $\mathbf{v}_2$ being collinear. So (a) is false.

Basic calculation. Taking $\mathbf{v}_1 = i$ and $\mathbf{v}_2 = j$, the sequence cycles: $i, j, k, i, j, k, \ldots$, so (b) is true. (c) is also true because we can arrange the sequence to be the following: a vector not in $\{i, j, k\}, j, k, i, j, k, \ldots$ (e.g., take $\mathbf{v}_1 = i + j$ and $\mathbf{v}_2 = j$. The basic calculation repeats after $\mathbf{v}_3$.) Note that any vector in the sequence depends only on the previous two vectors.

(d) is false. The set S can easily be infinite (e.g., if you start with i and $2j$, the magnitudes of subsequent vectors will keep increasing), but U is always finite. First note that the cycle of three vectors occurs whenever one starts with any two perpendicular vectors of unit length. Now U consists of unit vectors in the direction of each nonzero vector in S. So depending on the angle θ between $\mathbf{v}_1$ and $\mathbf{v}_2$, the cardinality of U is either 1 (when $\theta = 0$), 2 (when $\theta = \pi$), 3 (when $\theta = \pi/2$) or 4 (in all other cases, because $\mathbf{v}_2$ and $\mathbf{v}_3$ are still perpendicular).

9.

$$f(x) = \frac{x}{x + \sin x} \qquad \text{and} \qquad g(x) = \frac{x^4 + x^6}{e^x - 1 - x^2}\,.$$

(a) Limit as $x \to 0$ of $f(x)$ is $\frac{1}{2}$.

(b) Limit as $x \to \infty$ of $f(x)$ does not exist.

(c) Limit as $x \to \infty$ of $g(x)$ is finite.

(d) Limit as $x \to 0$ of $g(x)$ is 720.

Correct options: a,c **Wrong: b,d**

Calculate (a) and (c) using L'Hôpital's rule. (Or in (a) use that $\sin x$ behaves like x near 0 and in (c) the limit is 0 because e^x dominates any polynomial for large x.) In (b) the limit is 1 as $f(x)$ is sandwiched between $\frac{x}{x\pm 1}$, both of which $\to 1$. L'Hôpital's rule is not applicable as the expression one gets after attempting it does not have a limit as $x \to \infty$, so L'Hôpital's rule does not tell us anything. In (d) the limit is 0 by L'Hôpital's rule used correctly. Only one application is enough.

10. Let $f(u) = \tan^{-1}(u)$, a function whose domain in the set of all real numbers and whose range is $(-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g(v) = \int_0^v f(t)dt$.

   (a) $f(1) = \frac{\pi}{4}$.

   (b) $f(1) + f(2) + f(3) = \pi$.

   (c) g is an increasing function on the entire real line.

   (d) g is an odd function, i.e., $g(-x) = -g(x)$ for all real x.

   Correct options: a,b **Wrong: c,d**

   (a) is direct and (b) is a straightforward calculation using the formula for $\tan(A+B)$ keeping in mind the range of $\tan^{-1}$. By the fundamental theorem of calculus, $g'(x) = f(x)$, so g is increasing when f is positive, which is true only in $(0, \infty)$. g is an even function as its derivative f is odd. Note that $g(x)$ is defined for all real x as $\int_p^q f(t)dt = -\int_q^p f(t)dt$.

Part B Solutions

B1. Solve the following two independent problems on pages 2–3 of the answer booklet.

(i) Let f be a function from domain S to codomain T. Let g be another function from domain T to codomain U. For each of the blanks below choose a single letter corresponding to one of the four options listed underneath. (*It is not necessary that each choice is used exactly once.*) Write your answers on page 2 as a sequence of four letters in correct order. **Do NOT explain your answers**.

If $g \circ f$ is one-to-one then f __B__ and g __D__.

If $g \circ f$ is onto then f __D__ and g __C__.

Option A: must be one-to-one and must be onto.

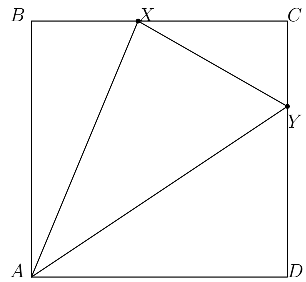
Option B: must be one-to-one but need not be onto.

Option C: need not be one-to-one but must be onto.

Option D: need not be one-to-one and need not be onto.

Recall: $g \circ f$ is the function defined by $g \circ f(a) = g(f(a))$. The function f is said to be one-to-one if, for any a_1 and any a_2 in S, $f(a_1) = f(a_2)$ implies $a_1 = a_2$. The function f is said to be onto if, for any b in T, there is an a in S such that $f(a) = b$.

(ii) In the given figure $ABCD$ is a square. Points X and Y, respectively on sides BC and CD, are such that X lies on the circle with diameter AY. What is the area of the square $ABCD$ if $AX = 4$ and $AY = 5$? (Figure is schematic and not to scale.)



Solution: AXY is a right angle, being an angle in a semicircle. Therefore by Pythagoras, XY = 3. Triangles ABX and XCY are similar, because both are right angled triangles and at the point X the three angles add to 180°, with the middle angle AXY being a right angle. We have the following three equations in three unknowns.

$$\frac{AB}{CX} = \frac{4}{3} \text{ by similarity} \qquad BX + CX = AB \qquad AB^2 + BX^2 = 16.$$

Solving these gives the answer, e.g., $CX = \frac{3}{4}AB$ by the first equation, so $BX = \frac{1}{4}AB$ by the second equation, so $AB^2 + \frac{1}{16}AB^2 = 16$ by the third equation, so area = $AB^2 = \frac{16^2}{17}$.

B2. Solve the following two independent problems on pages 4–5 of the answer booklet.

(i) A mother and her two daughters participate in a game show. At first, the mother tosses a fair coin.

Case 1: If the result is heads, then all three win individual prizes and the game ends.

Case 2: If the result is tails, then *each* daughter separately throws a fair die and wins a prize if the result of *her* die is 5 or 6. (Note that in case 2 there are two independent throws involved and whether each daughter gets a prize or not is unaffected by the other daughter's throw.)

(a) Suppose the first daughter did not win a prize. What is the probability that the second daughter also did not win a prize?

Solution: Since the first daughter did not win a prize, the coin toss must have shown tails. Now the second daughter does not win precisely when she throws 1, 2, 3 or 4. The probability of this is unaffected by the first daughter's throw. So the desired probability is $\frac{4}{6} = \frac{2}{3}$. One can also do this more pedantically in a way similar to part (b), see below.

(b) Suppose the first daughter won a prize. What is the probability that the second daughter also won a prize?

Solution: Let T = the event that the coin toss gives tails. Similarly H = heads, F = first daughter wins, S = second daughter wins. We want P(S | F). Note that the outcome of the throw of each die is independent of that of the other die and is unaffected by the coin toss that preceded it.

$$\begin{aligned} \text{P(S} \mid \text{F)} &= \text{P(S \& F)/P(F)} = (\tfrac{5}{9})/(\tfrac{2}{3}) = \tfrac{5}{6} \text{ because} \\ \text{P(F)} &= \text{P(H)} + \text{P(T) P(first die} = 5 \text{ or } 6 \mid \text{T)} = \tfrac{1}{2} + \tfrac{1}{2}\tfrac{2}{6} = \tfrac{2}{3}, \text{ and} \\ \text{P(S \& F)} &= \text{P(H)} + \text{P(T) P(each die} = 5 \text{ or } 6 \mid \text{T)} = \tfrac{1}{2} + \tfrac{1}{2}(\tfrac{2}{6})^2 = \tfrac{5}{9}. \end{aligned}$$

We can also solve part (a) similarly to find the desired P(not S | not F).

$$\begin{aligned} \text{P(not S} \mid \text{not F)} &= \text{P((not S) \& (not F))/ P(not F)} = \tfrac{4}{6} \text{ because} \\ \text{P(not F)} &= \text{P(T) P(}1 \leq \text{first die} \leq 4 \mid \text{T)} = \tfrac{1}{2}\tfrac{4}{6}, \text{ and} \\ \text{P((not S) \& (not F))} &= \text{P(T) P(}1 \leq \text{both dice} \leq 4 \mid \text{T)} = \tfrac{1}{2}(\tfrac{4}{6})^2. \end{aligned}$$

(ii) Prove or disprove each of the following statements.

(a) $2^{40} > 20!$

Solution: False. In fact $2^{62} > 20! > 2^{61}$, so even crude estimation is enough to solve this. For example LHS $= 2^{40} = 4^{20} = 4 \times 4 \times \cdots \times 4$ (20 times). RHS $= 1 \times 2 \times \cdots \times 20$. Consider the ratio RHS/LHS and pair the 20 numbers in each product. The initial three fractions less than 1, namely $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}$ are easily overpowered by the remaining ones, e.g., they are individually matched by $\frac{16}{4}, \frac{8}{4}, \frac{6}{4}$. (OR, using $2 \times 3 > 2^2$, $4 \times \cdots \times 7 > 4^4 = 2^8$, $8 \times \cdots \times 15 > 8^8 = 2^{24}$ and $16 \times \cdots \times 20 > 16^5 = 2^{20}$, one gets $20! > 2^{54}$.)

(b) $1 - \frac{1}{x} \leq \ln x \leq x - 1$ for all $x > 0$.

Solution: True. Let $f(x) = x - 1 - \ln x$. By analyzing the sign of $f'(x) = 1 - \frac{1}{x}$ (or by looking at the sign of $f''(x)$), see that $f(x)$ has a global minimum at $x = 1$ and that this minimum value is 0, giving $\ln x \leq x - 1$. For the other inequality, substitute $x = \frac{1}{t}$ in $\ln x \leq x - 1$ to get $\ln \frac{1}{t} = -\ln t \leq \frac{1}{t} - 1$, i.e., $1 - \frac{1}{t} \leq \ln t$ for all t such that $\frac{1}{t} > 0$, which is equivalent to having the same inequality for all $t > 0$. So we may replace t by x, giving the desired result. (Of course, it is also possible to repeat the earlier logic by analyzing $\ln x - 1 + \frac{1}{x}$.)

B3. You are supposed to create a 7-character long password for your mobile device.

(i) How many 7-character passwords can be formed from the 10 digits and 26 letters? (Only lowercase letters are taken throughout the problem.) Repeats are allowed, e.g., 0001a1a is a valid password.

Solution: For each character there are 36 choices. So number of passwords $= 36^7$.

(ii) How many of the passwords contain at least one of the 26 letters *and* at least one of the 10 digits? Write your answer in the form: (Answer to part i) − (something).

Solution: From 36^7 remove 26^7 passwords containing only letters and 10^7 passwords containing only digits. Required number $= 36^7 - (26^7 + 10^7)$.

(iii) How many of the passwords contain at least one of the 5 vowels, at least one of the 21 consonants *and* at least one of the 10 digits? Extend your method for part ii to write a formula and explain your reasoning.

Solution: Apply the inclusion exclusion principle or use a Venn diagram. Out of 36^7 passwords, 31^7 contain no vowels (V), 15^7 contain no consonants (C) and 26^7 contain no digits (D). As first step we take $36^7-(31^7$ missing V + 15^7 missing C + 26^7 missing D). But this removes the passwords without two types of characters (i.e., those containing only one type of character) *twice*. So we need to add these back so as to effectively remove them only once. So we need to add (10^7 missing VC + 21^7 missing VD + 5^7 missing CD). So the final answer is

$$36^7 - (31^7 + 15^7 + 26^7) + (10^7 + 21^7 + 5^7).$$

(iv) Now suppose that in addition to the lowercase letters and digits, you can also use 12 special characters. How many 7-character passwords are there that contain at least one of the 5 vowels, at least one of the 21 consonants, at least one of the 10 digits *and* at least one of the 12 special characters? Write *only the final formula* analogous to your answer to part iii. **Do NOT explain.**

Solution: The answer is

$$48^7-(43^7+27^7+38^7+36^7)+(22^7+33^7+17^7+31^7+15^7+26^7)+(5^7+21^7+10^7+12^7).$$

A password missing one type of character is subtracted only in the second term. A password missing two types of character is subtracted twice in the second term, so added once in the third term. A password missing three types of character is subtracted thrice in the second term, added back $\binom{3}{2} = 3$ times in the third term and subtracted once in the last term. (This is easier to understand as an application of the inclusion exclusion principle. Venn diagram gets harder to keep track of as there are more possibilities for overlaps.)

B4. Show that there is no polynomial $p(x)$ for which $\cos(\theta) = p(\sin\theta)$ for all angles θ *in some nonempty interval.*

Hint: Note that x and $|x|$ are different functions but their values are equal on an interval (as $x = |x|$ for all $x \geq 0$). You may want to show as a first step that this cannot happen for two polynomials, i.e., if polynomials f and g satisfy $f(x) = g(x)$ for all x in some interval, then f and g must be equal as polynomials, i.e., in each degree they must have the same coefficient.

Solution: To prove the assertion in the hint, note that the polynomial $f - g$ would have infinitely many roots and hence must be the zero polynomial.

Suppose a polynomial p satisfies $\cos(\theta) = p(\sin\theta)$ for θ in an interval. Let $t = \sin\theta$. Then the following equality is true for the (infinitely many) values of t in some nonempty interval:

$$p(t)^2 = \cos^2(\theta) = 1 - \sin^2(\theta) = 1 - t^2.$$

By the hint, this forces the polynomial $1 - x^2$ to be the square of the polynomial $p(x)$. But $1 - x^2$ is not a square because, e.g., the square of the leading coefficient would need to be -1, which cannot happen. (Or p would need to be a linear polynomial $ax + b$, etc.)

Note: The *italicized part "in some nonempty interval"* at the end of the problem statement was missing in the actual exam due to oversight. As sin of two angles can be equal without their cos being equal, a very easy solution now becomes possible, e.g., just plug in $\theta = 0$ and $\theta = \pi$ to get $p(0) = 1$ as well as $p(0) = -1$. So in fact there cannot be any *function* (not just polynomial) p such that $\cos(\theta) = p(\sin\theta)$ for ALL angles θ. In the exam, everyone who gave any correct solution to the problem as stated there was given full credit.

B5. Define a function f as follows: $f(0) = 0$ and, for any $x > 0$,

$$f(x) = \lim_{L\to\infty} \int_{\frac{1}{x}}^{L} \frac{1}{t^2}\cos(t)dt \ \Big(\text{or, in simpler notation, the improper integral} \int_{\frac{1}{x}}^{\infty} \frac{1}{t^2}\cos(t)dt\Big).$$

(i) Show that the definition makes sense for any $x > 0$ by justifying why the limit in the definition exists, i.e., why the improper integral converges.

Solution: As $|\cos(t)| \le 1$, the integral defining $f(x)$ is in fact absolutely convergent.

$$\int_{\frac{1}{x}}^{\infty} \Big|\frac{1}{t^2}\cos(t)\Big|dt \le \int_{\frac{1}{x}}^{\infty} \frac{1}{t^2}dt = -\frac{1}{t}\Big|_{\frac{1}{x}}^{\infty} = x.$$

(ii) Find $f'(\frac{1}{\pi})$ if it exists. Clearly indicate the basic result(s) you are using.

Solution: For $x \neq 0$, let $u = \frac{1}{x}$. By the fundamental theorem of calculus[1], $\frac{df}{du} = -\frac{1}{u^2}\cos(u) = -x^2\cos(\frac{1}{x})$. Since $\frac{du}{dx} = -\frac{1}{x^2}$, by chain rule $\frac{df}{dx} = \cos(\frac{1}{x})$, so $f'(\frac{1}{\pi}) = \cos(\pi) = -1$.

(iii) Using the hint or otherwise, find $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h}$, i.e., the right hand derivative of f at $x = 0$. We can take the limit only from the right hand side because $f(x)$ is undefined for negative values of x.

Hint: Break $f(h)$ into two terms by using a standard technique with an appropriate choice. Then separately analyze the resulting two terms in the derivative.

[1] To use the standard version of the fundamental theorem where the *lower* endpoint is a fixed *finite* number, take some positive constant K. Then $f(x) = -\int_K^{\frac{1}{x}} \frac{1}{t^2}\cos(t)dt$ + a constant (by part i).

Solution: Integrate by parts using $u = \frac{1}{t^2}$ and $dv = \cos(t)dt$. Then $du = -\frac{2}{t^3}dt$ and $v = \sin(t)$. (The method below does not work with $u = \cos(t)$ and $dv = \frac{1}{t^2}dt$.) So

$$\frac{f(h)-f(0)}{h} = \frac{1}{h}\int_{\frac{1}{h}}^{\infty} \frac{1}{t^2}\cos(t)dt = \frac{1}{h}\frac{\sin(t)}{t^2}\Big|_{\frac{1}{h}}^{\infty} + \frac{1}{h}\int_{\frac{1}{h}}^{\infty} \frac{2\sin(t)}{t^3}dt.$$

The first term in the sum $= \frac{1}{h}\frac{\sin(t)}{t^2}\Big|_{\frac{1}{h}}^{\infty} = -h\sin(\frac{1}{h}) \to 0$ as $h \to 0^+$, where we have twice used that $|\sin(t)| \le 1$ for all t. For the second term, by logic similar to part (i) we get

$$\Big|\frac{1}{h}\int_{\frac{1}{h}}^{\infty} \frac{2\sin(t)}{t^3}dt\Big| \le \frac{1}{h}\int_{\frac{1}{h}}^{\infty} \frac{2}{t^3}dt = -\frac{1}{h}t^{-2}\Big|_{\frac{1}{h}}^{\infty} = h \to 0 \text{ as } h \to 0^+.$$

So the desired limit is 0.

Note: Substituting $s = \frac{1}{t}$ gives $f(x) =$ the simpler looking improper integral $\int_0^x \cos(\frac{1}{s})ds$. This makes parts (i) and (ii) more transparent. Now $x < 0$ is also ok in the improper integral $\int_0^x \cos(\frac{1}{s})ds$. For the function $g(x)$ defined by this new integral, the above analysis gives one way to show that $g'(0)$ exists (see math.stackexchange.com/questions/2127903). g is differentiable everywhere (the only case requiring work being $x = 0$) but the derivative is not continuous at $x = 0$, as g' does not have a limit at $x = 0$.

B6. n and k are positive integers, not necessarily distinct. You are given two stacks of cards with a number written on each card, as follows.

Stack A has n cards. On each card a number in the set $\{1, \ldots, k\}$ is written.
Stack B has k cards. On each card a number in the set $\{1, \ldots, n\}$ is written.

Numbers may repeat in either stack. From this, you play a game by constructing a sequence $t_0, t_1, t_2, \ldots$ of integers as follows. Set $t_0 = 0$. For $j > 0$, there are two cases:

If $t_j \le 0$, draw the top card of stack A. Set $t_{j+1} = t_j +$ the number written on this card.
If $t_j > 0$, draw the top card of stack B. Set $t_{j+1} = t_j -$ the number written on this card.

In either case discard the taken card and continue. The game ends when you try to draw from an empty stack. *Example*: Let $n = 5$, $k = 3$, stack $A = 1, 3, 2, 3, 2$ and stack $B = 2, 5, 1$. You can check that the game ends with the sequence $0, 1, -1, 2, -3, -1, 2, 1$ (and with one card from stack A left unused).

(i) Prove that for every j we have $-n + 1 \le t_j \le k$.

(ii) Prove that there are at least two distinct indices i and j such that $t_i = t_j$.

(iii) Using the previous parts or otherwise, prove that there is a nonempty subset of cards in stack A and another subset of cards in stack B such that the sum of numbers in both the subsets is same.

Solution: (i) Induction. Base case is true as $t_0 = 0$. Assume the result up to t_j. Now there are two cases. If $t_j \in [-n+1, 0]$ then $t_{j+1} = t_j +$ a number from stack A, which is between 1 and k, so $t_{j+1} \in [(-n+1) + \text{lowest possibility } 1, 0 + \text{highest possibility } k] \subset [-n+1, k]$. If $t_j \in [1, k]$ then $t_{j+1} = t_j -$ a number from stack B, which is between 1 and n, so $t_{j+1} \in [1 - \text{highest possibility } n, k - \text{lowest possibility } 1] \subset [-n+1, k]$.

(ii) Suppose the game ends when we try to draw from stack B. As there are k cards in stack B, in all we must have tried to draw $k+1$ times from stack B. At each one of these attempts, the value of t_j must have been positive and by part (i) each one of these $k+1$ numbers is between 1 and k (inclusive). So there must be a repeat among these $k+1$ numbers.

If the game ends when we try to draw from stack A, the argument is parallel: there must have been $n+1$ attempts to draw from stack A, each one resulting from a value of t_j that lies among the n numbers from $-(n-1)$ to 0, so there must be a repeat.

(iii) Suppose $t_i = t_j$. Then, from the set of cards drawn at steps $i+1, \ldots, j$ the sum of the cards from stack A must equal the sum of the cards from stack B.