Solutions for the 2018 Entrance Examination for the BSc Programmes at CMI

Part A

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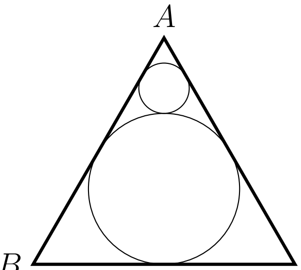
1. Consider an equilateral triangle ABC with *altitude* 3 centimeters. A circle is inscribed in this triangle, then another circle is drawn such that it is tangent to the inscribed circle and the sides AB, AC. Infinitely many such circles are drawn; each tangent to the previous circle and the sides AB, AC. The figure shows the construction after 2 steps.



Find the sum of the areas of all these circles.
Answer: The radius of the (first) inscribed circle is 1. **Its not hard to see that that as you go on inscribing the circles the corresponding radii decrease by** $1/3$. **Let** A **denote the total area of these circles then**

$$\begin{aligned} A &= \pi(1)^2 + \pi(1/3)^2 + \pi(1/9)^2 + \cdots \\ &= \pi + \pi(1/3)^2[1 + (1/3)^2 + (1/9)^2 + \cdots] \\ &= \pi + (1/9)A \\ &= (9/8)\pi. \end{aligned}$$

2. Consider the following function defined for all real numbers x

$$f(x) = \frac{2018}{10 + e^x}.$$

How many integers are there in the range of f?
Answer: 201. Note that the for all values of x **the function is strictly decreasing and the graph lies above** x **axis. As** x **goes far left the denominator approaches** 10 **and the function value approaches** 201.8. **On the other hand, as goes far right the denominator blows up and the function value approaches** 0. **Since this is a continuous function by the intermediate value theorem all values in the interval** $(0, 201.8)$ **are assumed.**

3. List every solution of the following equation. You need not simplify your answer(s).

$$\sqrt[3]{x+4} - \sqrt[3]{x} = 1.$$

Put $t = \sqrt[3]{x}$, **to get** $(t^3 + 4) = (1+t)^3$. **This leads to the quadratic** $t^2 + t - 1 = 0$. **Solve it and then take the cube root of the solutions. The answers are** $-2 \pm \sqrt{5}$.

4. Compute the following integral

$$\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}x}{(\sqrt{\sin x}+\sqrt{\cos x})^4}$$

Pull $\cos^2 x$ out from the denominator and then substitute u for $\sqrt{\tan x}+1$. The answer is $\frac{1}{3}$.

5. List in increasing order all positive integers $n \leq 40$ such that n cannot be written in the form $a^2 - b^2$, where a and b are positive integers.
Answer: $1, 4$ and all even numbers of the form $4k+2$

6. Consider the equation

$$z^{2018} = 2018^{2018} + i,$$

where $i = \sqrt{-1}$.

   (a) How many complex solutions does this equation have?

   (b) How many solutions lie in the first quadrant?

   (c) How many solutions lie in the second quadrant?

The equation has 2018 complex solutions. In the polar form the right hand side of the equation can be expressed as $re^{i\theta}$, where θ is a very small positive angle. Note that 2018 is 2 mod 4. Of the 2018 solutions of $x^2018 = r$, one each is on positive and negative X-axis. The remaining 2016 are divided equally in the four quadrants, 504 each. Now rotating these by the very tiny angle $\theta/2018$ gives 505 each in the first and third quadrant but still 504 in second and fourth.

7. Let $x^3 + ax^2 + bx + 8 = 0$ be a cubic equation with integer coefficients. Suppose both r and $-r$ are roots of this equation, where $r > 0$ is a real number. List all possible pairs of values (a, b).
Plugging in r and $-r$ in the equation we get $r^2 + b = 0$ and $ar^2 + 8 = 0$. Let the third root be s, then expanding $(x+r)(x-r)(x+s)$ and comparing it with the given equation tells us that $ab = 8$. So the possible values of a, b are $-1, -2, -4, -8$, i.e., both a, b negative such that $ab = 8$.

8. How many non-congruent triangles are there with *integer* lengths $a \leq b \leq c$ such that $a + b + c = 20$?
It is clear that $1 < a \leq b \leq c < 10$. Now, $c < a + b$ and $c = 20 - a - b$ implies $10 < a + b$; this also means that $b \geq a$ and $b \geq 11 - a$. Moreover, we also have $b \leq 20 - a - b$. One can further conclude that $a \leq 6$, otherwise $7 \leq b \leq 6$. So as a ranges from 2 to 6 we have that b takes the following values $a = 2, b = 9; a = 3, b = 8; a = 4, b \in \{7, 8\}; a = 5, b \in \{6, 7\}; a = 6, b \in \{6, 7\}$. The total number of possible triangles is 8.

9. Consider a sequence of polynomials with real coefficients defined by

$$p_0(x) = (x^2 + 1)(x^2 + 2) \cdots (x^2 + 1009)$$

with subsequent polynomials defined by $p_{k+1}(x) := p_k(x+1) - p_k(x)$ for $k \geq 0$. Find the least n such that

$$p_n(1) = p_n(2) = \cdots = p_n(5000).$$

Answer $n = 2018$. **Note that** $\deg p_0(x) = 2018$ **and** $\deg p_k(x) = 2018 - k$. **Define** $g_n(x) = p_n(x) - p_n(1)$, **hence** $g_n(x)$ **has degree** $2018 - n$ **and** 5000 **roots.**

10. Recall that $\arcsin(t)$ (also known as $\sin^{-1}(t)$) is a function with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Consider the function $f(x) := \arcsin(\sin(x))$ and answer the following questions as a series of four letters (T for True and F for False) in order.

    (a) The function $f(x)$ is well defined for all real numbers x.TRUE

    (b) The function $f(x)$ is continuous wherever it is defined. TRUE

    (c) The function $f(x)$ is differentiable wherever it is continuous. FALSE

    This is a periodic function with period 2π. **On** $[-\pi/2, \pi/2]$ **it is identity and on** $[\pi/2, 3\pi/2]$ **it is negative identity. Hence the function is well-defined and continuous everywhere. However, it is not differentiable at nonzero multiples of** $\pi/2$.

Answers to part B

If you need extra space for any problem,
continue on one of the colored blank pages at the end and write a note to that effect.

1. Answer the following questions-

   (a) A natural number k is called **stable** if there exist k *distinct* natural numbers $a_1, \ldots, a_k$, each $a_i > 1$, such that

   $$\frac{1}{a_1} + \cdots + \frac{1}{a_k} = 1.$$

   Show that if k is stable then $k+1$ is also stable. Using this or otherwise, find all stable numbers. [5 marks]
   It is clear that 1 and 2 are not stable. However, 3 is stable. Let $k \geq 3$ be stable hence there are $a_1, \ldots, a_k$ all distinct and $\sum \frac{1}{a_i} = 1$. This implies that $\frac{1}{2} + \sum \frac{1}{2a_i} = 1$. Hence all numbers except 2 are stable.

   (b) Let f be a differentiable function defined on a subset A of $\mathbb{R}$. Define

   $$f^*(y) := \max_{x \in A} \{yx - f(x)\},$$

   whenever the above maximum is finite. For the function $f(x) = -\ln(x)$, determine the set of points for which f^* is defined and find an expression for $f^*(y)$ involving only y and constants. [5 marks]
   First, note that the function $f(x)$ is defined only for the positive values of x. Now if $y \geq 0$ then the first derivative of $xy + \ln(x)$ is $y + \frac{1}{x}$ which is strictly positive for $x > 0$. Hence $xy + \ln(x)$ is an increasing function and consequently $f^*(y)$ is not defined.
   Now if $y < 0$ then $x = -\frac{1}{y}$ is the only critical point of $xy + \ln(x)$. Moreover, either of the derivative test tells us that it is in fact the maxima. Hence, the domain of $f^*(y)$ is $y < 0$ and

   $$f^*(y) = \ln(\frac{-1}{y}) - 1.$$

2. Answer the following questions

(a) Find all real solutions of the equation [6 marks]

$$\left(x^2 - 2x\right)^{x^2 + x - 6} = 1.$$

Explain why your solutions are the only solutions.
Answer $x = -3, 1, 1 \pm \sqrt{2}$ **are the only solutions. First, we want either** $x^2 + x - 6 = 0$ **or** $x^2 - 2x = 1$. **However, when** $x = 2$ **the base as well as the exponent are** 0 **giving us an indeterminate form. Hence** $x = 2$ **will not work. Moreover, when** $x = -3$ **the base is positive. Second, observe that when** $x = 1$ **we get** $(-1)^{-4}$ **which equals** 1.

(b) The following expression is a rational number. Find its value. [9 marks]

$$\sqrt[3]{6\sqrt{3} + 10} - \sqrt[3]{6\sqrt{3} - 10}.$$

Answer : 2. Let the numbers be a, b **respectively. Note** $a^3 - b^3 = 20$ **and** $ab = 2$. **Putting it in** $(a - b)^3$ **we get** $(a - b)^3 = 20 - 6(a - b)$. **This cubic has one real solution** $a - b = 2$ **and two complex solutions.**

3. Let f be a function on the nonnegative integers defined as follows

$$f(2n) = f(f(n)) \quad \text{and} \quad f(2n+1) = f(2n) + 1.$$

(a) If $f(0) = 0$, find $f(n)$ for every n. [2 marks]

(b) Show that $f(0)$ cannot equal 1. [4 marks]

(c) For what nonnegative integers k (if any) can $f(0)$ equal 2^k? [9 marks]

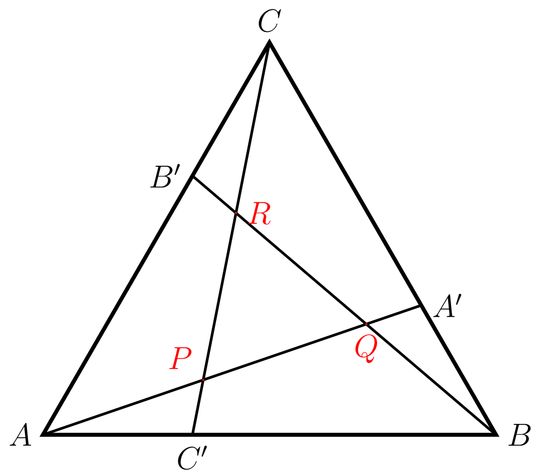
Answer

(a) Suppose $f(0) = 0$ **then** $f(1) = 1$ **and** $f(2) = f(f(1)) = f(1) = 1$. **It implies that** $f(3) = 1 + 1 = 2$ **and** $f(4) = f(1) = 1$. **The pattern continues and we get that if** $2k + 1 \geq 3$ **then** $f(2k + 1) = 2$. **On the other hand if** $2k \geq 4$ **then** $f(2k) = 1$.

(b) Suppose $f(0) = 1$. **We have** $f(0) = f(2 \cdot 0) = f(f(0)) = f(1)$. **But we also have** $f(1) = f(0) + 1$, **a contradiction.**

(c) Suppose $f(0) = 2^k$. **Then,** $2^k = f(2 \cdot 0) = f(f(0)) = f(2^k)$, **and** $f(2^k+1) = f(2^k)+1 = 2^k + 1$. **Notice that** $f(1) = f(0) + 1 = 2^k + 1$, **and** $f(2) = f(f(1)) = 2^k + 1$. **In this way, we see that for any** n, $f(2^n) = 2^k + 1$. **This contradicts that fact that** $f(2^k) = 2^k$.

4. Let ABC be an equilateral triangle with side length 2. Point A' is chosen on side BC such that the length of $A'B$ is $k < 1$. Likewise points B' and C' are chosen on sides CA and AB with $AC' = CB' = k$. Line segments are drawn from points A', B', C' to their corresponding opposite vertices. The intersections of these line segments form a triangle, labeled PQR in the interior.



Show that the triangle PQR is an equilateral triangle with side length $\frac{4(1-k)}{\sqrt{k^2-2k+4}}$.

Note that triangles ABA', CAC' and BCB' are congruent by the SAS test. Triangles $BA'Q$, $CB'R$ and $AC'P$ are also congruent. By using the property of opposite angles we get that all the three angles of the triangle PQR are the same. Hence it is an equilateral triangle.

Dropping the perpendicular bisector AO on the side BC we get the following:

$$\begin{aligned} AA'^2 &= AO^2 + A'A^2 \\ &= (1-k)^2 + (\sqrt{3})^2 \\ &= k^2 - 2k + 4. \end{aligned}$$

Observe that triangles ABA' and BQA' are similar by the AAA test: $A'QB$ and $A'BA$ are 60 degrees and $A'BQ$ and $A'AB$ are corresponding angles. Therefore:

$$\begin{aligned} \frac{AB}{BQ} &= \frac{BA'}{QA'} = \frac{A'A}{A'B} \\ \frac{2}{BQ} &= \frac{k}{QA'} = \frac{\sqrt{k^2-2k+4}}{k} \\ BQ &= \frac{2k}{\sqrt{k^2-2k+4}} \\ QA' &= \frac{k^2}{\sqrt{k^2-2k+4}}. \end{aligned}$$

Now using $AA' = AP + PQ + QA'$ we get

$$PQ = \frac{4(1-k)}{\sqrt{k^2-2k+4}}.$$

5. An alien language has n letters $b_1, \dots, b_n$. For some $k < n/2$ assume that all words formed by any of the k letters (written left to right) are meaningful. These words are called k-words. Such a k-word is considered **sacred** if:

   i) no letter appears twice and,

   ii) if a letter b_i appears in the word then the letters b_{i-1} and b_{i+1} do not appear. (Here $b_{n+1} = b_1$ and $b_0 = b_n$.)

   For example, if $n = 7$ and $k = 3$ then $b_1b_3b_6, b_3b_1b_6, b_2b_4b_6$ are sacred 3-words. On the other hand $b_1b_7b_4, b_2b_2b_6$ are not sacred. What is the total number of sacred k-words? Use your formula to find the answer for $n = 10$ and $k = 4$.

   We will count the sacred words starting with b_1. Since b_1 is chosen b_n and b_2 are out of the picture. In order to fill the remaining $k-1$ positions we have to choose non-consecutive b_i's. Note that, specifying these b_i's is equivalent to specifying the gaps between them. For example, let $n = 7, k = 3$ and we would like to choose b_1, b_3, b_6. Then the triple $(1, 2, 1)$ specifies that leave one alphabet after b_1, drop two after b_3 and drop one after b_6. Hence, in general let $x_1, x_2, \dots, x_k$ be these gaps. It is clear that each of this gap is at least 1 and they add up to $n-k$. So our counting problem is now - in how many different ways one can choose k natural numbers, each of which is at least 1, that add up to $n-k$. The answer is $\binom{n-k-1}{k-1}$. In fact, this is equivalent to counting the number of ways one can choose $k-1$ 'plus' signs from $n-k-1$of them when $n-k$ is written as a sum of 1's ($n-k$ of them). However, note that we haven't assigned positions to these letters yet. t This can be done in $(k-1)!$ ways. Hence the answer is

$$(k-1)!\binom{n-k-1}{k-1}.$$

   In order to count the total number of sacred words we just need to multiply the above number by n. The final answer is

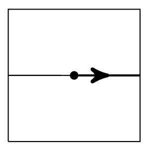
$$\begin{aligned} n(k-1)!\binom{n-k-1}{k-1} &= n(k-1)!\ \frac{(n-k-1)!}{(n-2k)!(k-1)!} \\ &= n\frac{(n-k-1)!}{(n-2k)!} \\ &= n\ (n-k-1)(n-k-2)\cdots(n-2k+1). \end{aligned}$$

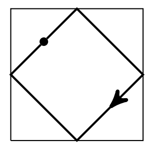
   For $n = 10$ and $k = 4$ the answer is 600.

6. Imagine the unit square in the plane to be a carrom board. Assume the *striker* is just a point, moving with no friction (so it goes forever), and that when it hits an edge, the angle of reflection is equal to the angle of incidence, as in real life. When it hits another edge it bounces again similarly and so on. If the striker ever hits a corner it falls into the pocket and disappears. The trajectory of the striker is completely determined by its starting point (x, y) and its initial velocity $\overrightarrow{(p, q)}$.

   If the striker eventually returns to its initial state (i.e. initial position *and* initial velocity), we define its *bounce number* to be the number of edges it hits before returning to its initial state for the first time.

   For example, the trajectory with initial state $[(.5, .5); \overrightarrow{(1, 0)}]$ has bounce number 2 and it returns to its initial state for the first time in 2 time units. And the trajectory with initial state $[(.25, .75); \overrightarrow{(1, 1)}]$ has bounce number 4.





   (a) Suppose the striker has initial state $[(.5, .5); \overrightarrow{(p, q)}]$. If $p > q \geq 0$ then what is the velocity after it hits an edge for the first time? What if $q > p \geq 0$? [2 marks]

   (b) Draw a trajectory with bounce number 5 or justify why it is impossible. [3 marks]

   (c) Consider the trajectory with initial state $[(x, y); \overrightarrow{(p, 0)}]$ where p is a positive integer. In how much time will the striker first return to its initial state? [2 marks]

   (d) What is the bounce number for the initial state $[(x, y); \overrightarrow{(p, q)}]$ where p, q are relatively prime positive integers, assuming the striker never hits a corner? [8 marks]

   (a) If $p > q$ then the striker will hit the vertical edge first, and its new velocity will be $\overrightarrow{(-p, q)}$. If $p < q$ then the striker will hit the horizontal edge first, and its new velocity will be $\overrightarrow{(p, -q)}$.

   (b) No, it is not possible. If the striker has bounce number 5, then it must have an odd number of vertical edge bounces or horizontal edge bounces. In the former case, when the striker returns to its initial state, the x-component of its velocity will be wrong, by the formula in part (a). In the latter case the y component will be wrong.

   (c) It will take $\frac{2}{p}$ time to return to its initial state.

   (d) The bounce number is $2p + 2q$. At time 2, the striker will have completed p horizontal round-trips and q vertical round trips, and will have returned to its initial state. To see this, note that from part (c) it will take time $\frac{2}{p}$ for each horizontal round trip and time $\frac{2}{q}$ for each vertical round trip. Since p and q are relatively prime, it will only be at time 2 that an integer number of vertical and horizontal round trips have been completed.