2017 Entrance Examination for the BSc Programmes at CMI

Solutions

Read the instructions on the front of the booklet carefully!

Part A. Write your final answers on page 3.

Part A is worth a total of ($4 \times 10 = 40$) points. Points will be given based only on clearly legible final answers filled in the correct place on page 3. Write all answers for a single question on the designated line and in the order in which they are asked, separated by commas.

Unless specified otherwise, each answer is either a number (rational/ real/ complex) or, where appropriate, one of the phrases "infinite"/"does not exist"/"not possible to decide". Write integer answers in the usual decimal form. Write non-integer rationals as ratios of two integers.

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1. Consider the following construction in a circle. Choose points A, B, C on the given circle such that $\angle ABC$ is $60°$ and $AB = BC$. Draw another circle that is tangential to the chords AB, BC and to the original circle.
   Do the above construction in the unit circle to obtain a circle S_1. Repeat the process in S_1 to obtain another circle S_2. What is the radius of S_2?

   Solution. Consider the center O and diameter BD of the unit circle. It is easy to see that S_1 passes through D and its center E lies between O and D. Let r be the radius of S_1, so length of ED is r. Consider the perpendicular from E to chord BA, meeting BA in point F. Then length of EF is also r and therefore in the 30-60-90 triangle BEF, the length of the hypotenuse BE is $2r$. Thus $2 = \text{BD} = \text{BE} + \text{ED} = 3r$, thus $r = \frac{2}{3}$. By similarity, the radius of S_2 is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

2. 10 oranges are to be placed in 5 distinct boxes labeled U, V, W, X, Y. A box may contain any number of oranges including no oranges or all the oranges. What is the number of ways to distribute the oranges so that exactly two of the boxes contain exactly two oranges each?

   Solution. From the five distinct boxes, there are 10 ways to pick the two boxes that will have 2 oranges each. We need to distribute the remaining 6 oranges in the remaining three boxes such that none of the three boxes gets exactly 2 oranges. The possible distributions are 6+0+0 (which can be done in 3 ways) or 5+1+0 (6 ways) or 4+1+1 (3 ways) or 3+3+0 (3 ways). Thus the required answer is $10 \times (3 + 6 + 3 + 3) = 150$.

3. Find the volume of the solid obtained when the region bounded by $y = \sqrt{x}, y = -x$ and the line $x = 4$ is revolved around the x-axis. (It may be useful to draw the specified region.)

   Solution. From $x = 0$ to $x = 1$ we have $\sqrt{x} \geq |-x|$, so from $x = 0$ to $x = 1$ the volume swept out by the part of the given region that lies below X-axis is included in the volume swept out by the part above X-axis. So from $x = 0$ and $x = 1$ we just have to take the volume obtained by revolving the area below $y = \sqrt{x}$. Similarly, from $x = 1$ to $x = 4$ we have $|-x| \geq \sqrt{x}$, so here we just have to take the volume obtained by the revolving the area below $y = x$. Thus the required volume is obtained by adding volumes of two solids of revolution around X-axis: area under $y = \sqrt{x}$ from $x = 0$ to $x = 1$ and area under $y = x$ from $x = 1$ to $x = 4$.

4. Positive integers a and b, possibly equal, are chosen randomly from among the divisors of 400. The numbers a, b are chosen independently, each divisor being equally likely to be chosen. Find the probability that $\gcd(a, b) = 1$ and $\operatorname{lcm}(a, b) = 400$.

   Solution. $400 = 5^2 \times 2^4$ has $(2 + 1) \times (4 + 1) = 15$ factors, so total number of pairs (a, b) is 225. For a, b to be coprime, they should have no prime factor in common and then their lcm is just their product, which is required to be 400. So there are only four allowed pairs: (1,400), (400,1), (25,16) and (16,25). The probability is $\frac{4}{225}$.

5. Find all complex solutions to the equation:

$$x^4 + x^3 + 2x^2 + x + 1 = 0.$$

   Solution. It is easy to see that $x^4 + x^3 + 2x^2 + x + 1 = (x^2 + 1)(x^2 + x + 1)$.

6. Let g be a function such that all its derivatives exist. We say g has an *inflection point* at x_0 if the second derivative g'' changes sign at x_0 i.e., if $g''(x_0 - \epsilon) \times g''(x_0 + \epsilon) < 0$ for all small enough positive ϵ.

   (a) If $g''(x_0) = 0$ then g has an inflection point at x_0. True or False?

   (b) If g has an inflection point at x_0 then $g''(x_0) = 0$. True or False?

   (c) Find all values x_0 at which $x^4(x - 10)$ has an inflection point.

   Solution. In (c), $g''(x) = 20x^3 - 120x^2 = 20x^2(x - 6)$ and this changes sign only at $x = 6$. Note that for this function, $g''(0) = 0$ but g'' does not change sign at $x = 0$, thus (a) is FALSE. On the other hand (b) is TRUE: Suppose for some g, the double derivative g'' changes sign at x_0. Then $g''(x_0) = 0$ as g'' is continuous (because g'' is given to be differentiable).

7. Write the values of the following.

   (a) $\displaystyle\int_{-3}^{3} |3x^2 - 3| \, dx$.

   (b) $f'(1)$ where $f(t) = \displaystyle\int_{0}^{t} |3x^2 - 3| \, dx$.

   Solution. (a) By symmetry we can calculate the definite integral from 0 to 3 and double the answer. Note that $|3x^2 - 3| = 3x^2 - 3$ from $x = 1$ to 3 and $|3x^2 - 3| = 3 - 3x^2$ from $x = 0$ to 1. So break the calculation at $x = 1$ etc.

(b) $|3x^2 - 3|$ is a continuous function so by the fundamental theorem of calculus, $f'(1) = |3 \times 1^2 - 3| = 0$

8. For this question write your answers as a series of four letters (Y for Yes and N for No) in order. Is it possible to find a 2×2 matrix M for which the equation $M\vec{x} = \vec{p}$ has:

   (a) no solutions for some but not all $\vec{p}$; exactly one solution for all other $\vec{p}$?

   (b) exactly one solution for some but not all $\vec{p}$; more than one solution for all other $\vec{p}$?

   (c) no solutions for some but not all $\vec{p}$; more than one solution for all other $\vec{p}$?

   (d) no solutions for some $\vec{p}$, exactly one solution for some $\vec{p}$ and more than one solution for some $\vec{p}$?

   Solution. If M has nonzero determinant, then for any $\vec{p}$, we see that $M\vec{x} = \vec{p}$ has the unique solution $\vec{x} = M^{-1}\vec{p}$. If determinant of M is zero then we can make two cases. (i) If M is the zero matrix, then $M\vec{x} = \vec{p}$ has infinitely many solutions for $\vec{p} = \vec{0}$ and no solutions otherwise. (ii) If M is nonzero then it is easy to see that we are solving two equations in two variables whose left hand sides are proportional. So if the two right hand constants that make up $\vec{p}$ are in the same proportion, then we will have infinitely many solutions (because one of the variables can be arbitrary). If the constants are not in the same proportion, then the two equations will be inconsistent and we will have no solutions. Thus the answer is NNYN. It is also possible to think geometrically in terms of (at most) two lines, each moving in a parallel family. If the lines have the same slope they either coincide or don't intersect. Otherwise they have a unique point of intersection.

   Note: In general *linear algebra* gives the right tools to analyze matrix equations, e.g. in this problem we can say the following. If $M = 0$ then the space of solutions is either empty or two-dimensional. If $M \neq 0$ then either there is a unique solution (precisely when determinant $\neq 0$) or, when determinant is 0, the space of solutions is either empty or one-dimensional. For larger matrices the possibilities are more complicated, but they can be described precisely using the language provided by linear algebra.

9. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) that satisfies the following property: For every natural number n

$$f(n) = \text{ the smallest prime factor of } n.$$

   For example, $f(12) = 2$, $f(105) = 3$. Calculate the following.

   (a) $\lim_{x \to \infty} f(x)$.

   (b) The number of solutions to the equation $f(x) = 2016$.

   Solution. $f(x)$ will take value 2 for all even x. At the same time, primes provide an increasing infinite sequence of positive integers for which $f(x) = x$. Thus $\lim_{x \to \infty} f(x)$ does not exist. By intermediate value theorem, for each prime $p > 2016$ there is an x between p and $p + 1$ such that $f(x) = 2016$.

10. Consider the following function:

$$f(x) = \begin{cases} x^2 \cos(\frac{1}{x}), & x \neq 0, \\ a, & x = 0. \end{cases}$$

(a) Find the value of a for which f is continuous.

Use this value of a to calculate the following.

(b) $f'(0)$.

(c) $\lim\limits_{x \to 0} f'(x)$.

Solution. $\cos(\frac{1}{x})$ is sandwiched between -1 and 1, so $\lim_{x\to 0} f(x) = 0 = a$ makes f continuous. Now $f'(0) = \lim_{h \to 0} \frac{h^2 \cos(\frac{1}{h}) - 0}{h} = \lim_{h\to 0} h \cos(\frac{1}{h})$ which is similarly 0. Finally, for nonzero x, calculate $f'(x) = 2x \cos(\frac{1}{x}) + \sin(\frac{1}{x})$, so $\lim_{x\to 0} f'(x)$ does not exist as $\lim_{x\to 0} 2x\cos(\frac{1}{x}) = 0$ and $\lim_{x \to 0} \sin(\frac{1}{x})$ does not exist.

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Solutions to Part **B**

1. Answer the following questions

(a) Evaluate

$$\lim_{x \to 0^+} (x^{x^x} - x^x).$$

First consider the limit

$$\begin{aligned} \lim_{x \to 0^+} x^x &= \lim_{x \to 0^+} (e^{\log_x x}) \\ &= \lim_{x \to 0^+} (e^{\frac{\log x}{1/x}}). \end{aligned} \tag{1}$$

Now consider the following limit

$$\begin{aligned} \lim_{x \to 0^+} \frac{\log x}{1/x} &= \lim_{x \to 0} \frac{1/x}{-1/x^2} \\ &= 0 \end{aligned} \tag{2}$$

substituting the value 0 **from (2) in equation (1) we get that the limit is** 1**. Now,**

$$\begin{aligned} \lim_{x \to 0^+} (x^{x^x} - x^x) &= \lim_{x \to 0^+} x^{x^x} - \lim_{x \to 0^+} x^x \\ &= \lim_{x \to 0^+} x^{\lim_{x \to 0^+} x^x} - \lim_{x \to 0^+} x^x \\ &= 0 - 1 \\ &= -1. \end{aligned}$$

(b) Let $A = \frac{2\pi}{9}$, i.e., $A = 40$ degrees. Calculate the following

$$1 + \cos A + \cos 2A + \cos 4A + \cos 5A + \cos 7A + \cos 8A.$$

There are many ways to arrive at the answer 1. Here are two approaches. Let S be the above sum. Then

$$\begin{aligned}
S &= 1 + \cos A + \cos 2A + \cos 4A \cos(2\pi - 4A) + \cos(2\pi - 2A) + \cos(2\pi - A) \\
&= 1 + 2(\cos A + \cos 2A + \cos 4A) \\
&= 1 + 2\left(2\cos(\frac{3A}{2})\cos(\frac{A}{2}) + \cos(\pi - \frac{A}{2})\right) \\
&= 1 + 2\left(2\cos(\frac{\pi}{3})\cos(\frac{A}{2}) - \cos(\frac{A}{2})\right) \\
&= 1 + 2\left(2 \times \frac{1}{2}\cos(\frac{A}{2}) - \cos(\frac{A}{2})\right) \\
&= 1.
\end{aligned}$$

Recall that $\cos nA$ is the real part of e^{inA}. Then

$$\begin{aligned}
S &= \sum_{n=0}^{8} \cos nA - \sum_{n=1}^{2} \cos(3nA) \\
&= \mathrm{Re}\left(\sum_{n=0}^{8} e^{inA} - \sum_{n=1}^{2} e^{in\frac{2\pi}{3}}\right) \\
&= \mathrm{Re}\left(0 - \omega - \omega^2\right) \\
&= 1.
\end{aligned}$$

Here ω is a complex cube root of unity.

(c) Find the number of solutions to $e^x = \frac{x}{2017} + 1$.

First, note that $x = 0$ is clearly a solution. Let $f(x) = e^x - \frac{x}{2017} - 1$. Then $x_0 = -\log 2017$ is the only critical point of $f(x)$. For all $x < x_0$ we have $f'(x) < 0$. Since $f(x) \to \infty$ as $x \to -\infty$ there is only one solution in the interval $(-\infty, x_0)$. For all $x > x_0$ we have $f'(x) > 0$ (i.e., $e^x > \frac{1}{2017}$). Hence there is only one solution in the interval (x_0, ∞). In total there are exactly two solutions.

2. Let L be the line of intersection of the planes $x + y = 0$ and $y + z = 0$.

   (a) Write the vector equation of L, i.e., find (a, b, c) and (p, q, r) such that

   $$L = \{(a, b, c) + \lambda(p, q, r) \mid \lambda \text{ is a real number.}\}$$

   (b) Find the equation of a plane obtained by rotating $x + y = 0$ about L by $45°$.

Clearly the line L passes through the origin. Moreover L is in the direction perpendicular to the normals to the both the planes. The direction vector can be obtained by computing following cross product

$$(\hat{i} + \hat{j}) \times (\hat{j} + \hat{k}) = \hat{i} - \hat{j} + \hat{k}.$$

Hence L can be written as

$$L = \{(0, 0, 0) + \lambda(1, -1, 1) \mid \lambda \textbf{ is a real number } \}$$

First, note that the equation of any plane that contains the line L is given by

$$x + (1 + \lambda)y + \lambda z = 0.$$

Second, note that one can rotate the plane $x + y = 0$ in either clockwise or in anticlockwise direction. Consequently there are two such planes. The normal of one of the planes makes an angle of $45°$ with the normal of $x + y = 0$ and the other normal makes an angle of $135°$.

$$\begin{aligned}
(\hat{i} + \hat{j}) \cdot (\hat{i} + (1 + \lambda)\hat{j} + \lambda\hat{k}) &= \pm|\hat{i} + \hat{j}||\hat{i} + (1 + \lambda)\hat{j} + \lambda\hat{k}| \cos(\frac{\pi}{4}) \\
2 + \lambda &= \pm\sqrt{1 + (1 + \lambda)^2 + \lambda^2} \\
\lambda^2 - 2\lambda - 2 = 0 & \\
\lambda = 1 \pm \sqrt{3}. &
\end{aligned}$$

So the equation of the plane is

$$x + y + (1 \pm \sqrt{3})(y + z) = 0.$$

3. Let $p(x)$ be a polynomial of degree strictly less than 100 and such that it does not have $x^3 - x$ as a factor. If

$$\frac{d^{100}}{dx^{100}}\left(\frac{p(x)}{x^3-x}\right) = \frac{f(x)}{g(x)}$$

for some polynomials $f(x)$ and $g(x)$ then find the smallest possible degree of $f(x)$. Here $\frac{d^{100}}{dx^{100}}$ means taking the 100th derivative.

Using the division algorithm we have

$$\frac{p(x)}{x^3-x} = q(x) + \frac{r(x)}{x^3-x} \tag{3}$$

As the degree of $q(x)$ is strictly less than that of $p(x)$ its 100th derivative is certainly zero. As $x^3 - x$ is not a factor of $p(x)$ one may assume (without loss of generality) that $x^2 - 1$ is a divides $r(x)$. In that case we have

$$\begin{aligned}\frac{d^{100}}{dx^{100}}\left(\frac{p(x)}{x^3-x}\right) &= \frac{d^{100}}{dx^{100}}\left(\frac{k}{x}\right)\\ &= \frac{100!\,k}{x^{100}}\end{aligned}$$

Hence the least possible degree of $f(x)$ is 0.

If one assumes that $x^3 - x$ doesn't divide $p(x)$ then we have

$$\frac{r(x)}{x^3-x} = \frac{A'}{x} + \frac{B'}{x-1} + \frac{C'}{x+1}.$$

Consequently,

$$\begin{aligned}&\frac{f(x)}{g(x)} = \frac{A}{x^{101}} + \frac{B}{(x-1)^{101}} + \frac{C}{(x+1)^{101}}\\ &\therefore f(x) = A(x^2-1)^{101} + B(x^2+x)^{101} + C(x^2-x)^{101}\\ &\qquad = (A+B+C)x^{202} + 101(B-C)x^{201} + (\binom{101}{2}B + \binom{101}{2}C - 101A)x^{200} + \cdots.\end{aligned}$$

Choosing $B = C$ and $A+B+C = 0$ we see that the coefficient of x^{200} is $(101)(102) \neq 0$. Hence the least possible degree of $f(x)$ in this case is 200.

4. The domain of a function f is the set of natural numbers. The function is defined as follows:

$$f(n) = n + \lfloor \sqrt{n} \rfloor$$

where $\lfloor k \rfloor$ denotes the nearest integer smaller than or equal to k. For example, $\lfloor \pi \rfloor = 3, \lfloor 4 \rfloor = 4$. Prove that for every natural number m the following sequence contains at least one perfect square

$$m, f(m), f^2(m), f^3(m), \ldots$$

The notation f^k denotes the function obtained by composing f with itself k times, e.g., $f^2 = f \circ f$.

If m is itself a square then we are done. So assume that $m = k^2 + j$ for $1 \leq j \leq 2k$. Hence we have $f(m) = k^2 + j + k$. Consider the following two sets

$$A = \{m \text{ **a natural number** } \mid m = k^2 + j \text{ **and** } 0 \leq j \leq k\}.$$

$$B = \{m \text{ **a natural number** } \mid m = k^2 + j \text{ **and** } k + 1 \leq j \leq 2k\}.$$

Suppose m is in the set B. Then

$$\begin{aligned} f(m) &= k^2 + j + k \\ &= (k+1)^2 + (j - k - 1). \end{aligned}$$

Hence $f(m)$ is either a square or is in A. Thus it is enough to assume that $m \in A$. In that case $k^2 < f(m) < (k+1)^2$, so $\lfloor f(m) \rfloor = k$. Therefore

$$f^2(m) = (k+1)^2 + (j - 1).$$

This clearly implies that $f^{2j}(m) = (k + j)^2$.

5. Each integer is colored with exactly one of three possible colors - black, red or white - satisfying the following two rules: the negative of a black number must be colored white, and the sum of two white numbers (not necessarily distinct) must be colored black.

   (a) Show that the negative of a white number must be colored black and the sum of two black numbers must be colored white.

   (b) Determine all possible colorings of the integers that satisfy these rules.

   Suppose an integer n is colored white. Then $(n+n) = 2n$ is black, so $-2n$ is white, so $-2n+n=-n$ is black. Thus, the negative of a white number must be colored black. Now suppose the integers n and m are both colored black. Then $-n$ and $-m$ are both white, so $-n-m$ is black, so $n+m$ is white. Thus, the sum of two black numbers must be colored white.

   One possible coloring has all the integers colored red, since there are no conditions on red numbers.

   If that is not the case, let n be the smallest positive integer that is not colored red. Suppose the number n is colored black. Then we claim the remaining colors are all fully determined. Namely, the numbers of the form $(3k+1)n$ will be black, the numbers of the form $(3k+2)n$ will be white, and the numbers of the form $(3k)n$ will be red, for all integers k. And all remaining colors will be red. On the other hand, if the number n is colored white to begin with, then the remaining numbers will be determined by the same rules, but with black and white switched. Thus we have listed all possible colorings.

   In order to prove the above claim, we first prove one more rule the colors must obey. Namely, that (*) The sum of a black number and a white number must be colored red. Suppose n is black and m is white, and that $n+m$ is black. But then $(n+m)+(-m)$ is the sum of two black numbers, and must be colored white, which is a contradiction. Similarly, the sum of n and m cannot be white. Therefore it must be red.

   Using this rule, it is easy to see that the numbers of the form $(3k+1)n$ will be black, the numbers of the form $(3k+2)n$ will be white, and the numbers of the form $(3k)n$ will be red, for all integers k. It remains to show that all numbers that are not multiples of n are colored red.

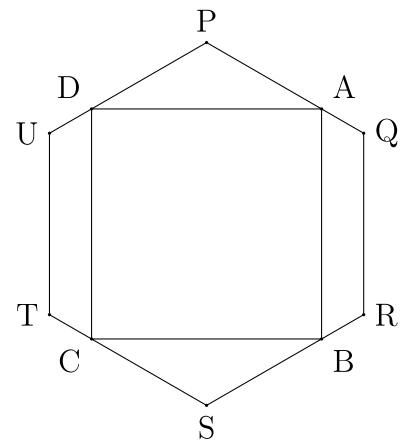
   We can prove this by contradiction. As before n is the smallest positive integer that is not *red*, and it is colored black. Suppose m is the smallest positive integer that is neither red nor a multiple of n. Then $m = qn + r$, where $0 < r < n$ is the remainder when m is divided by n. We know this remainder is nonzero, since m is not a multiple of n. We also know that $q > 0$, since $m > n$. Suppose m is white. Then, because $-n$ is white, we know $m-n = (q-1)n+r$ is black, which gives us a smaller non-red positive integer that's not a multiple of n. On the other hand, suppose m is colored black. Then $-2n$ is black, so $m-2n = (q-2)n+r$ is white. If q ¿ 1, this gives us a smaller positive non-red integer that's not a multiple of n, which is a contradiction, provided $q > 1$. But if q = 1, and $m-2n = -n+r$ is white, that means that $n-r$ is black, another contradiction.

6. You are given a regular hexagon. We say that a square is inscribed in the hexagon if it can be drawn in the interior such that all the four vertices lie on the perimeter of the hexagon.

   (a) A line segment has its endpoints on opposite edges of the hexagon. Show that it passes through the center of the hexagon if and only if it divides the two edges in the same ratio.

   (b) Suppose a square $ABCD$ is inscribed in the hexagon such that A and C are on the opposite sides of the hexagon. Prove that center of the square is same as that of the hexagon.

   (c) Suppose the side of the hexagon is of length 1. Then find the length of the side of the inscribed square whose one pair of opposite sides is parallel to a pair of opposite sides of the hexagon.



   (d) Show that, up to rotation, there is a unique way of inscribing a square in a regular hexagon.

(a) Suppose a segment AC meets with opposite sides PQ and TS of a hexagon and O is the midpoint of AC. We show that

$$\frac{PA}{AQ} = \frac{TC}{CS} \iff O \text{ is the center of the hexagon.}$$

If O is the center of the hexagon, consider triangles OAQ and OCS. By the SAS test these are congruent. Similarly, triangles OAP and OCT are congruent.

Conversely, suppose $\frac{PA}{AQ} = \frac{TC}{CS} = k$ (say), then

$$PQ = TS \implies PA + AQ = TC + CS \implies AQ(k+1) = CS(k+1) \implies AQ = CS.$$

So $\triangle AQO \cong \triangle CTO$, so that $OQ = OT$. Also, $\angle AOQ = \angle COT$ and $\angle AOP = \angle COS$, so Q, O and T are collinear.

(b) Next suppose we have inscribed a square $ABCD$ in a hexagon $PQRSTU$, with A on PQ, B on QR, C on ST and D on TU. We claim that $\triangle AQB$ is

congruent to $\triangle CTD$. This will prove that both diagonals pass through the center of the hexagon (using the criterion proved above).

Proof: We know that $PA \parallel ST$ and AC is a transversal. So $\angle QAC = \angle TCA$, also $\angle BAC = \angle DCA = 45°$. So $\angle QAB = \angle DCT$.

Similarly, $\angle QBA = \angle CDT$. Also, $\angle AQB = \angle CTD$, since they are angles in a regular hexagon. Moreover, $AB = CD$. As a result we get that $\triangle QBA \cong \triangle TDC$.

So we have $QB = TD$ and $QA = TC$. This in turn implies that $BR = DU$ and $PA = CS$ Thus,

$$\frac{QB}{BR} = \frac{TD}{DU} \text{ and } \frac{PA}{AQ} = \frac{SC}{CT}.$$

Hence AC and BD pass through the center of the hexagon.

(c) Let $DU = x$ so $DP = 1 - x$. Observe that $DC = 2x \sin 30$ and $DA = 2(1 - x) \sin 60$. Since $DC = DA$ we solving the equations for x we get $x = \frac{2}{\sqrt{3}+1}$. Consequently the side $DC = \sqrt{3}(\sqrt{3} - 1)$.

(d) Finally we want to show that there is a unique way of inscribing a square in a regular hexagon.

Proof: It will be enough to show that the ratios $\frac{QB}{BR}$ and $\frac{QA}{AP}$ are equal.

Suppose on the contrary that these ratios aren't equal.

Let $\angle QAB = \alpha$ and $\angle QBA = \beta$. Note that then $\angle OAQ = 45° + \alpha$ and $\angle OBQ = 45° + \beta$. Also, $\alpha + \beta = 60°$, since $\angle AQB = 120°$.

Let A' be a point on QR such that $\frac{QA'}{A'R} = \frac{QA}{AP}$. Since $\triangle BOA'$ is isosceles, $\angle OBA'$ equals $\angle OA'B$, so that

$$180° = \angle OBA' + \angle OBQ = \angle OBQ + \angle OA'B = \angle OBQ + \angle OAQ = 45° + \beta + 45° + \alpha,$$

so $\alpha + \beta = 0°$, a contradiction since $\alpha + \beta = 60°$.