Note. In this question-paper, $\mathbb{R}$ denotes the set of real numbers.

1. Consider a board having 2 rows and n columns. Thus there are $2n$ cells in the board. Each cell is to be filled in by 0 or 1.

   (a) In how many ways can this be done such that each row sum and each column sum is even?

   (b) In how many ways can this be done such that each row sum and each column sum is odd?

2. Consider the function

$$f(x) = \sum_{k=1}^{m} (x-k)^4, \quad x \in \mathbb{R},$$

   where $m > 1$ is an integer. Show that f has a unique minimum and find the point where the minimum is attained.

3. Consider the parabola $C : y^2 = 4x$ and the straight line $L : y = x + 2$. Let P be a variable point on L. Draw the two tangents from P to C and let Q_1 and Q_2 denote the two points of contact on C. Let Q be the mid-point of the line segment joining Q_1 and Q_2. Find the locus of Q as P moves along L.

4. Let $P(x)$ be an odd degree polynomial in x with real coefficients. Show that the equation $P(P(x)) = 0$ has at least as many *distinct* real roots as the equation $P(x) = 0$.

5. For any positive integer n, and $i = 1, 2$, let $f_i(n)$ denote the number of divisors of n of the form $3k + i$ (including 1 and n). Define, for any positive integer n,

$$f(n) = f_1(n) - f_2(n).$$

   Find the values of $f(5^{2022})$ and $f(21^{2022})$.

6. Consider a sequence $P_1, P_2, \ldots$ of points in the plane such that P_1, P_2, P_3 are non-collinear and for every $n \geq 4$, P_n is the mid-point of the line segment joining P_{n-2} and P_{n-3}. Let L denote the line segment joining P_1 and P_5. Prove the following:

   (a) The area of the triangle formed by the points P_n, P_{n-1}, P_{n-2} converges to zero as n goes to infinity.

   (b) The point P_9 lies on L.

7. Let

$$P(x) = 1 + 2x + 7x^2 + 13x^3, \; x \in \mathbb{R}.$$

Calculate for all $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \left(P\left(\frac{x}{n}\right)\right)^n.$$

8. Find the minimum value of

$$|\sin x + \cos x + \tan x + \cot x + \sec x + \operatorname{cosec} x|$$

for real numbers x not multiple of $\pi/2$.

9. Find the smallest positive real number k such that the following inequality holds

$$|z_1 + \ldots + z_n| \geq \frac{1}{k}\left(|z_1| + \ldots + |z_n|\right).$$

for every positive integer $n \geq 2$ and every choice $z_1, \ldots, z_n$ of complex numbers with non-negative real and imaginary parts.

[Hint: First find k that works for $n = 2$. Then show that the same k works for any $n \geq 2$.]