Test Codes: UGA (Multiple-choice Type) and UGB (Short Answer Type), 2016

Questions will be set on the following and related topics.

Algebra: Sets, operations on sets. Prime numbers, factorization of integers and divisibility. Rational and irrational numbers. Permutations and combinations, Binomial Theorem. Logarithms. Polynomials: Remainder Theorem, Theory of quadratic equations and expressions, relations between roots and coefficients. Arithmetic and geometric progressions. Inequalities involving arithmetic, geometric & harmonic means. Complex numbers.

Geometry: Plane geometry. Geometry of 2 dimensions with Cartesian and polar coordinates. Equation of a line, angle between two lines, distance from a point to a line. Concept of a Locus. Area of a triangle. Equations of circle, parabola, ellipse and hyperbola and equations of their tangents and normals. Mensuration.

Trigonometry: Measures of angles. Trigonometric and inverse trigonometric functions. Trigonometric identities including addition formulae, solutions of trigonometric equations. Properties of triangles. Heights and distances.

Calculus: Sequences - bounded sequences, monotone sequences, limit of a sequence. Functions, one-one functions, onto functions. Limits and continuity. Derivatives and methods of differentiation. Slope of a curve. Tangents and normals. Maxima and minima. Using calculus to sketch graphs of functions. Methods of integration, definite and indefinite integrals, evaluation of area using integrals.

Reference (For more sample questions)
Test of Mathematics at the 10 + 2 level, Indian Statistical Institute. Published by Affiliated East-West Press Pvt. Ltd., 105, Nirmal Tower, 26 Barakhamba Road, New Delhi 110001.

Sample Questions for UGA

Instructions. UGA is a multiple choice examination. In each of the following questions, exactly one of the choices is correct. You get four marks for each correct answer, one mark for each unanswered question, and zero marks for each incorrect answer.

1 Define $a_n = (1^2 + 2^2 + \ldots + n^2)^n$ and $b_n = n^n(n!)^2$. Recall $n!$ is the product of the first n natural numbers. Then,

(A) $a_n < b_n$ for all $n > 1$ (B) $a_n > b_n$ for all $n > 1$

(C) $a_n = b_n$ for infinitely many n (D) None of the above

2 The sum of all distinct four digit numbers that can be formed using the digits $1, 2, 3, 4$, and 5, each digit appearing at most once, is

(A) 399900 (B) 399960 (C) 390000 (D) 360000

3 The last digit of $(2004)^5$ is

(A) 4 (B) 8 (C) 6 (D) 2

4 The coefficient of $a^3b^4c^5$ in the expansion of $(bc + ca + ab)^6$ is

(A) $\frac{12!}{3!4!5!}$ (B) $\binom{6}{3}3!$ (C) 33 (D) $3\binom{6}{3}$

5 Let $ABCD$ be a unit square. Four points E, F, G and H are chosen on the sides AB, BC, CD and DA respectively. The lengths of the sides of the quadrilateral $EFGH$ are α, β, γ and δ. Which of the following is always true?

(A) $1 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2\sqrt{2}$

(B) $2\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4\sqrt{2}$

(C) $2 \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 4$

(D) $\sqrt{2} \leq \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \leq 2 + \sqrt{2}$

6 If $\log_{10} x = 10^{\log_{100} 4}$ then x equals

(A) 4^{10} (B) 100 (C) $\log_{10} 4$ (D) none of the above

7 z_1, z_2 are two complex numbers with $z_2 \neq 0$ and $z_1 \neq z_2$ and satisfying $\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = 1$. Then $\frac{z_1}{z_2}$ is

(A) real and negative
(B) real and positive
(C) purely imaginary
(D) none of the above need to be true always

8 The set of all real numbers x satisfying the inequality $x^3(x+1)(x-2) \geq 0$ is
(A) the interval $[2, \infty)$ (B) the interval $[0, \infty)$
(C) the interval $[-1, \infty)$ (D) none of the above

9 The minimum area of the triangle formed by any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the coordinate axes is
(A) ab (B) $\frac{a^2+b^2}{2}$ (C) $\frac{(a+b)^2}{2}$ (D) $\frac{a^2+ab+b^2}{3}$

10 Let A be the fixed point $(0, 4)$ and B be a moving point $(2t, 0)$. Let M be the mid-point of AB and let the perpendicular bisector of AB meet the y-axis at R. The locus of the mid-point P of MR is
(A) $y + x^2 = 2$ (B) $x^2 + (y-2)^2 = 1/4$
(C) $(y-2)^2 - x^2 = 1/4$ (D) none of the above

11 The sides of a triangle are given to be $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$. Then the largest of the three angles of the triangle is

(A) $75°$ (B) $\left(\frac{x}{x+1}\pi\right)$ radians (C) $120°$ (D) $135°$

12 Two poles, AB of length two metres and CD of length twenty metres are erected vertically with bases at B and D. The two poles are at a distance not less than twenty metres. It is observed that $\tan \angle ACB = 2/77$. The distance between the two poles is

(A) $72m$ (B) $68m$ (C) $24m$ (D) $24.27m$

13 If A, B, C are the angles of a triangle and $\sin^2 A + \sin^2 B = \sin^2 C$, then C is equal to

(A) $30°$ (B) $90°$ (C) $45°$ (D) none of the above

14 In the interval $(-2\pi, 0)$, the function $f(x) = \sin\left(\frac{1}{x^3}\right)$
(A) never changes sign
(B) changes sign only once
(C) changes sign more than once, but finitely many times
(D) changes sign infinitely many times

15 The limit

$$\lim_{x\to 0}\frac{(e^x-1)\tan^2 x}{x^3}$$

(A) does not exist (B) exists and equals 0
(C) exists and equals 2/3 (D) exists and equals 1

16 Let $f_1(x) = e^x$, $f_2(x) = e^{f_1(x)}$ and generally $f_{n+1}(x) = e^{f_n(x)}$ for all $n \geq 1$. For any fixed n, the value of $\frac{d}{dx}f_n(x)$ is equal to

(A) $f_n(x)$ (B) $f_n(x)f_{n-1}(x)$
(C) $f_n(x)f_{n-1}(x)\cdots f_1(x)$ (D) $f_{n+1}(x)f_n(x)\cdots f_1(x)e^x$

17 If the function

$$f(x) = \begin{cases} \frac{x^2-2x+A}{\sin x} & \text{if } x \neq 0 \\ B & \text{if } x = 0 \end{cases}$$

is continuous at $x = 0$, then

(A) $A = 0$, $B = 0$ (B) $A = 0$, $B = -2$
(C) $A = 1$, $B = 1$ (D) $A = 1$, $B = 0$

18 A truck is to be driven 300 kilometres (kms.) on a highway at a constant speed of x kms. per hour. Speed rules of the highway require that $30 \leq x \leq 60$. The fuel costs ten rupees per litre and is consumed at the rate $2 + (x^2/600)$ litres per hour. The wages of the driver are 200 rupees per hour. The most economical speed (in kms. per hour) to drive the truck is

(A) 30 (B) 60 (C) $30\sqrt{3.3}$ (D) $20\sqrt{33}$

19 If $b = \int_0^1 \frac{e^t}{t+1}dt$ then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}dt$ is

(A) be^a (B) be^{-a} (C) $-be^{-a}$ (D) $-be^a$

20 In the triangle ABC, the angle $\angle BAC$ is a root of the equation

$$\sqrt{3}\cos x + \sin x = 1/2.$$

Then the triangle ABC is

(A) obtuse angled (B) right angled
(C) acute angled but not equilateral (D) equilateral

21 Let n be a positive integer. Consider a square S of side $2n$ units with sides parallel to the coordinate axes. Divide S into $4n^2$ unit squares by drawing $2n-1$ horizontal and $2n-1$ vertical lines one unit apart. A circle of diameter $2n-1$ is drawn with its centre at the intersection

of the two diagonals of the square S. How many of these unit squares contain a portion of the circumference of the circle?

(A) $4n - 2$ (B) $4n$ (C) $8n - 4$ (D) $8n - 2$

22 A lantern is placed on the ground 100 feet away from a wall. A man six feet tall is walking at a speed of 10 feet/second from the lantern to the nearest point on the wall. When he is midway between the lantern and the wall, the rate of change (in ft./sec.) in the length of his shadow is

(A) 2.4 (B) 3 (C) 3.6 (D) 12

23 An isosceles triangle with base 6 cms. and base angles $30°$ each is inscribed in a circle. A second circle touches the first circle and also touches the base of the triangle at its midpoint. If the second circle is situated outside the triangle, then its radius (in cms.) is

(A) $3\sqrt{3}/2$ (B) $\sqrt{3}/2$ (C) $\sqrt{3}$ (D) $4/\sqrt{3}$

24 Let n be a positive integer. Define

$$f(x) = \min\{|x-1|, |x-2|, \ldots, |x-n|\}.$$

Then $\displaystyle\int_0^{n+1} f(x)dx$ equals

(A) $\dfrac{(n+4)}{4}$ (B) $\dfrac{(n+3)}{4}$ (C) $\dfrac{(n+2)}{2}$ (D) $\dfrac{(n+2)}{4}$

25 Let $S = \{1, 2, \ldots, n\}$. The number of possible pairs of the form (A, B) with $A \subseteq B$ for subsets A and B of S is

(A) 2^n (B) 3^n (C) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$ (D) $n!$

26 The number of maps f from the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$ whenever $i < j$ is

(A) 60 (B) 50 (C) 35 (D) 30

27 Consider three boxes, each containing 10 balls labelled 1, 2, ..., 10. Suppose one ball is drawn from each of the boxes. Denote by n_i, the label of the ball drawn from the i-th box, $i = 1, 2, 3$. Then the number of ways in which the balls can be chosen such that $n_1 < n_2 < n_3$ is

(A) 120 (B) 130 (C) 150 (D) 160

28 Let a be a real number. The number of distinct solutions (x, y) of the system of equations $(x-a)^2 + y^2 = 1$ and $x^2 = y^2$, can only be

(A) $0, 1, 2, 3, 4$ or 5 (B) 0, 1 or 3
(C) $0, 1, 2$ or 4 (D) $0, 2, 3,$ or 4

29 The maximum of the areas of the isosceles triangles with base on the positive x-axis and which lie below the curve $y = e^{-x}$ is:

(A) $1/e$ (B) 1 (C) $1/2$ (D) e

30 Suppose a, b and n are positive integers, all greater than one. If $a^n + b^n$ is prime, what can you say about n?
(A) The integer n must be 2
(B) The integer n need not be 2, but must be a power of 2
(C) The integer n need not be a power of 2, but must be even
(D) None of the above is necessarily true

31 Water falls from a tap of circular cross section at the rate of 2 metres/sec and fills up a hemispherical bowl of inner diameter 0.9 metres. If the inner diameter of the tap is 0.01 metres, then the time needed to fill the bowl is

(A) 40.5 minutes (B) 81 minutes

(C) 60.75 minutes (D) 20.25 minutes

32 The value of the integral

$$\int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)} + e^{\tan^{-1}(\cos x)}}\, dx$$

equals (A) 1 (B) π (C) e (D) none of these

33 The set of all solutions of the equation $\cos 2\theta = \sin\theta + \cos\theta$ is given by
(A) $\theta = 0$
(B) $\theta = n\pi + \frac{\pi}{2}$, where n is any integer
(C) $\theta = 2n\pi$ or $\theta = 2n\pi - \frac{\pi}{2}$ or $\theta = n\pi - \frac{\pi}{4}$, where n is any integer
(D) $\theta = 2n\pi$ or $\theta = n\pi + \frac{\pi}{4}$, where n is any integer

34 The number

$$\left(\frac{2^{10}}{11}\right)^{11}$$

is
(A) strictly larger than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$
(B) strictly larger than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2$ but strictly smaller than $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$
(C) less than or equal to $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2$
(D) equal to $\binom{10}{1}^2\binom{10}{2}^2\binom{10}{3}^2\binom{10}{4}^2\binom{10}{5}$.

35 The value of

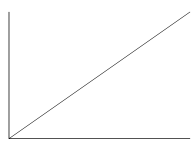
$$\sin^{-1}\cot\left[\sin^{-1}\left\{\frac{1}{2}\left(1-\sqrt{\frac{5}{6}}\right)\right\}+\cos^{-1}\sqrt{\frac{2}{3}}+\sec^{-1}\sqrt{\frac{8}{3}}\right]$$

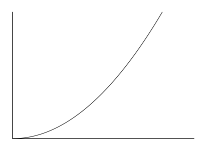
is

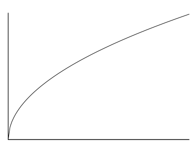
(A) 0 (B) $\pi/6$ (C) $\pi/4$ (D) $\pi/2$

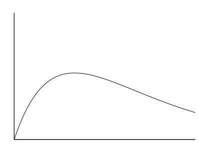
36 Which of the following graphs represents the function

$$f(x)=\int_0^{\sqrt{x}} e^{-u^2/x}du, \quad \text{for} \quad x>0 \quad \text{and} \quad f(0)=0?$$

(A) 

(B) 

(C) 

(D) 

37 If $a_n=\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\cdots\left(1+\frac{n^2}{n^2}\right)^n$, then

$$\lim_{n\to\infty} a_n^{-1/n^2}$$

is

(A) 0 (B) 1 (C) e (D) $\sqrt{e}/2$

38 The function $x(\alpha-x)$ is strictly increasing on the interval $0<x<1$ if and only if

(A) $\alpha\geq 2$ (B) $\alpha<2$

(C) $\alpha<-1$ (D) $\alpha>2$

39 Consider a circle with centre O. Two chords AB and CD extended intersect at a point P outside the circle. If $\angle AOC=43°$ and $\angle BPD=18°$, then the value of $\angle BOD$ is

(A) $36°$ (B) $29°$ (C) $7°$ (D) $25°$

40 A box contains 10 red cards numbered $1,\ldots,10$ and 10 black cards numbered $1,\ldots,10$. In how many ways can we choose 10 out of the 20 cards so that there are exactly 3 *matches*, where a *match* means a red card and a black card with the same number?

(A) $\binom{10}{3}\binom{7}{4}2^4$ (B) $\binom{10}{3}\binom{7}{4}$

(C) $\binom{10}{3}2^7$ (D) $\binom{10}{3}\binom{14}{4}$

41 Let P be a point on the ellipse $x^2 + 4y^2 = 4$ which does not lie on the axes. If the normal at the point P intersects the major and minor axes at C and D respectively, then the ratio $PC : PD$ equals

(A) 2 (B) 1/2 (C) 4 (D) 1/4

42 The set of complex numbers z satisfying the equation

$$(3 + 7i)z + (10 - 2i)\bar{z} + 100 = 0$$

represents, in the complex plane,

(A) a straight line
(B) a pair of intersecting straight lines
(C) a pair of distinct parallel straight lines
(D) a point

43 The number of triplets (a, b, c) of integers such that $a < b < c$ and a, b, c are sides of a triangle with perimeter 21 is

(A) 7 (B) 8 (C) 11 (D) 12.

44 Suppose a, b and c are three numbers in G.P. If the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then $\frac{d}{a}$, $\frac{e}{b}$ and $\frac{f}{c}$ are in

(A) A.P. (B) G.P. (C) H.P. (D) none of the above.

45 The number of solutions of the equation $\sin^{-1} x = 2\tan^{-1} x$ is

(A) 1 (B) 2 (C) 3 (D) 5.

46 Suppose $ABCD$ is a quadrilateral such that $\angle BAC = 50°$, $\angle CAD = 60°$, $\angle CBD = 30°$ and $\angle BDC = 25°$. If E is the point of intersection of AC and BD, then the value of $\angle AEB$ is

(A) 75° (B) 85° (C) 95° (D) 110°.

47 Let $\mathbb{R}$ be the set of all real numbers. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - 3x^2 + 6x - 5$ is

(A) one-to-one, but not onto
(B) one-to-one and onto
(C) onto, but not one-to-one
(D) neither one-to-one nor onto.

48 Let L be the point $(t, 2)$ and M be a point on the y-axis such that LM has slope $-t$. Then the locus of the midpoint of LM, as t varies over all real values, is

(A) $y = 2 + 2x^2$ (B) $y = 1 + x^2$

(C) $y = 2 - 2x^2$ (D) $y = 1 - x^2$.

49 Let $f : (0, 2) \cup (4, 6) \to \mathbb{R}$ be a differentiable function. Suppose also that $f"(x) = 1$ for all $x \in (0, 2) \cup (4, 6)$. Which of the following is ALWAYS true?

(A) f is increasing
(B) f is one-to-one
(C) $f(x) = x$ for all $x \in (0, 2) \cup (4, 6)$
(D) $f(5.5) - f(4.5) = f(1.5) - f(0.5)$

50 A triangle ABC has a fixed base BC. If $AB : AC = 1 : 2$, then the locus of the vertex A is

(A) a circle whose centre is the midpoint of BC
(B) a circle whose centre is on the line BC but not the midpoint of BC
(C) a straight line
(D) none of the above.

51 Let P be a variable point on a circle C and Q be a fixed point outside C. If R is the mid-point of the line segment PQ, then the locus of R is

(A) a circle (B) an ellipse

(C) a line segment (D) segment of a parabola

52 N is a 50 digit number. All the digits except the 26th from the right are 1. If N is divisible by 13, then the unknown digit is

(A) 1 (B) 3 (C) 7 (D) 9.

53 Suppose $a < b$. The maximum value of the integral

$$\int_a^b \left(\frac{3}{4} - x - x^2\right) dx$$

over all possible values of a and b is

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{2}{3}$.

54 For any $n \geq 5$, the value of $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1}$ lies between

(A) 0 and $\frac{n}{2}$ (B) $\frac{n}{2}$ and n

(C) n and $2n$ (D) none of the above.

55 Let ω denote a cube root of unity which is not equal to 1. Then the number of distinct elements in the set

$$\left\{(1+\omega+\omega^2+\cdots+\omega^n)^m \quad : \quad m,n=1,2,3,\cdots\right\}$$

is

(A) 4 (B) 5 (C) 7 (D) infinite.

56 The value of the integral

$$\int_2^3 \frac{dx}{\log_e x}$$

(A) is less than 2 (B) is equal to 2

(C) lies in the interval $(2,3)$ (D) is greater than 3.

57 The area of the region bounded by the straight lines $x=\frac{1}{2}$ and $x=2$, and the curves given by the equations $y=\log_e x$ and $y=2^x$ is

(A) $\frac{1}{\log_e 2}(4+\sqrt{2})-\frac{5}{2}\log_e 2+\frac{3}{2}$ (B) $\frac{1}{\log_e 2}(4-\sqrt{2})-\frac{5}{2}\log_e 2$

(C) $\frac{1}{\log_e 2}(4-\sqrt{2})-\frac{5}{2}\log_e 2+\frac{3}{2}$ (D) none of the above

58 In a win-or-lose game, the winner gets 2 points whereas the loser gets 0. *Six* players A, B, C, D, E and F play each other in a preliminary round from which the top *three* players move to the final round. After each player has played *four* games, A has 6 points, B has 8 points and C has 4 points. It is also known that E won against F. In the next set of games D, E and F win their games against A, B and C respectively. If A, B and D move to the final round, the final scores of E and F are, respectively,

(A) 4 and 2 (B) 2 and 4 (C) 2 and 2 (D) 4 and 4.

59 The number of ways in which one can select six distinct integers from the set $\{1,2,3,\cdots,49\}$, such that no two consecutive integers are selected, is

(A) $\binom{49}{6}-5\binom{48}{5}$ (B) $\binom{43}{6}$

(C) $\binom{25}{6}$ (D) $\binom{44}{6}$.

60 Let $n\geq 3$ be an integer. Assume that inside a big circle, exactly n small circles of radius r can be drawn so that each small circle touches the big circle and also touches both its adjacent small circles. Then, the radius of the big circle is

(A) $r\,\text{cosec}\frac{\pi}{n}$ (B) $r(1+\text{cosec}\frac{2\pi}{n})$

(C) $r(1+\text{cosec}\frac{\pi}{2n})$ (D) $r(1+\text{cosec}\frac{\pi}{n})$

61 If n is a positive integer such that $8n + 1$ is a perfect square, then
(A) n must be odd
(B) n cannot be a perfect square
(C) $2n$ cannot be a perfect square
(D) none of the above

62 Let $\mathbb{C}$ denote the set of all complex numbers. Define

$$A = \{(z, w) \mid z, w \in \mathbb{C} \text{ and } |z| = |w|\}$$
$$B = \{(z, w) \mid z, w \in \mathbb{C}, \text{ and } z^2 = w^2\}.$$

Then,
(A) $A = B$
(B) $A \subset B$ and $A \neq B$
(C) $B \subset A$ and $B \neq A$
(D) none of the above

63 Let $f(x) = a_0 + a_1|x| + a_2|x|^2 + a_3|x|^3$, where a_0, a_1, a_2, a_3 are constants. Then
(A) $f(x)$ is differentiable at $x = 0$ whatever be a_0, a_1, a_2, a_3
(B) $f(x)$ is not differentiable at $x = 0$ whatever be a_0, a_1, a_2, a_3
(C) $f(x)$ is differentiable at $x = 0$ only if $a_1 = 0$
(D) $f(x)$ is differentiable at $x = 0$ only if $a_1 = 0, a_3 = 0$

64 If $f(x) = \cos(x) - 1 + \frac{x^2}{2}$, then
(A) $f(x)$ is an increasing function on the real line
(B) $f(x)$ is a decreasing function on the real line
(C) $f(x)$ is increasing on $-\infty < x \leq 0$ and decreasing on $0 \leq x < \infty$
(D) $f(x)$ is decreasing on $-\infty < x \leq 0$ and increasing on $0 \leq x < \infty$

65 The number of roots of the equation $x^2 + \sin^2 x = 1$ in the closed interval $[0, \frac{\pi}{2}]$ is
(A) 0 (B) 1 (C) 2 (D) 3

66 The set of values of m for which $mx^2 - 6mx + 5m + 1 > 0$ for all real x is
(A) $m < \frac{1}{4}$
(B) $m \geq 0$
(C) $0 \leq m \leq \frac{1}{4}$
(D) $0 \leq m < \frac{1}{4}$

67 The digit in the unit's place of the number $1! + 2! + 3! + \ldots + 99!$ is
(A) 3 (B) 0 (C) 1 (D) 7

68 The value of $\lim_{n\to\infty} \frac{1^3+2^3+\ldots+n^3}{n^4}$ is:
(A) $\frac{3}{4}$ (B) $\frac{1}{4}$ (C) 1 (D) 4

69 For any integer $n \geq 1$, define $a_n = \frac{1000^n}{n!}$. Then the sequence $\{a_n\}$

(A) does not have a maximum

(B) attains maximum at exactly one value of n

(C) attains maximum at exactly two values of n

(D) attains maximum for infinitely many values of n

70 The equation $x^3y + xy^3 + xy = 0$ represents

(A) a circle (B) a circle and a pair of straight lines

(C) a rectangular hyperbola (D) a pair of straight lines

71 For each positive integer n, define a function f_n on $[0, 1]$ as follows:

$$f_n(x) = \begin{cases} 0 & \text{if} & x = 0 \\ \sin\dfrac{\pi}{2n} & \text{if} & 0 < x \leq \dfrac{1}{n} \\ \sin\dfrac{2\pi}{2n} & \text{if} & \dfrac{1}{n} < x \leq \dfrac{2}{n} \\ \sin\dfrac{3\pi}{2n} & \text{if} & \dfrac{2}{n} < x \leq \dfrac{3}{n} \\ \vdots & \vdots & \vdots \\ \sin\dfrac{n\pi}{2n} & \text{if} & \dfrac{n-1}{n} < x \leq 1. \end{cases}$$

Then, the value of $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$ is

(A) π (B) 1 (C) $\dfrac{1}{\pi}$ (D) $\dfrac{2}{\pi}$.

72 Let $d_1, d_2, \ldots, d_k$ be all the factors of a positive integer n including 1 and n. If $d_1 + d_2 + \ldots + d_k = 72$, then $\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_k}$ is:

(A) $\frac{k^2}{72}$ (B) $\frac{72}{k}$ (C) $\frac{72}{n}$ (D) none of the above

73 A subset W of the set of real numbers is called a ring if it contains 1 and if for all $a, b \in W$, the numbers $a - b$ and ab are also in W. Let $S = \left\{\frac{m}{2^n} \mid m, n \text{ integers}\right\}$ and $T = \left\{\frac{p}{q} \mid p, q \text{ integers, } q \text{ odd}\right\}$. Then

(A) neither S nor T is a ring (B) S is a ring T is not a ring

(C) T is a ring S is not a ring (D) both S and T are rings

74 A rod AB of length 3 rests on a wall. P is a point on AB such that $AP : PB = 1 : 2$. If the rod slides along the wall, then the locus of P lies on

(A) $2x + y + xy = 2$

(B) $4x^2 + y^2 = 4$

(C) $4x^2 + xy + y^2 = 4$
(D) $x^2 + y^2 - x - 2y = 0$.

75 Consider the equation $x^2 + y^2 = 2007$. How many solutions (x, y) exist such that x and y are positive integers?
(A) None
(B) Exactly two
(C) More than two but finitely many
(D) Infinitely many.

76 Consider the functions $f_1(x) = x, f_2(x) = 2 + \log_e x,\ x > 0$ (where e is the base of natural logarithm). The graphs of the functions intersect
(A) once in $(0, 1)$ and never in $(1, \infty)$
(B) once in $(0, 1)$ and once in (e^2, ∞)
(C) once in $(0, 1)$ and once in (e, e^2)
(D) more than twice in $(0, \infty)$.

77 Consider the sequence

$$u_n = \sum_{r=1}^{n} \frac{r}{2^r}, n \geq 1.$$

Then the limit of u_n as $n \to \infty$ is
(A) 1 (B) 2 (C) e (D) 1/2.

78 Suppose that z is any complex number which is not equal to any of $\{3, 3\omega, 3\omega^2\}$ where ω is a complex cube root of unity. Then

$$\frac{1}{z-3} + \frac{1}{z-3\omega} + \frac{1}{z-3\omega^2}$$

equals
(A) $\frac{3z^2+3z}{(z-3)^3}$ (B) $\frac{3z^2+3\omega z}{z^3-27}$ (C) $\frac{3z^2}{z^3-3z^2+9z-27}$ (D) $\frac{3z^2}{z^3-27}$.

79 Consider all functions $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ which are one-one, onto and satisfy the following property:

if $f(k)$ is odd then $f(k + 1)$ is even, $k = 1, 2, 3$.

The number of such functions is
(A) 4 (B) 8 (C) 12 (D) 16.

80 A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}}, & x > 0 \\ 0 & x \leq 0. \end{cases}$$

Then
(A) f is not continuous
(B) f is differentiable but f' is not continuous
(C) f is continuous but $f'(0)$ does not exist
(D) f is differentiable and f' is continuous.

81 The last digit of $9! + 3^{9966}$ is
(A) 3 (B) 9 (C) 7 (D) 1.

82 Consider the function

$$f(x) = \frac{2x^2+3x+1}{2x-1},\ 2 \le x \le 3.$$

Then

(A) maximum of f is attained inside the interval $(2,3)$
(B) minimum of f is $28/5$
(C) maximum of f is $28/5$
(D) f is a decreasing function in $(2,3)$.

83 A particle P moves in the plane in such a way that the angle between the two tangents drawn from P to the curve $y^2 = 4ax$ is always $90°$. The locus of P is

(A) a parabola (B) a circle (C) an ellipse (D) a straight line.

84 Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = |x^2 - 1|,\ x \in \mathbb{R}.$$

Then

(A) f has a local minima at $x = \pm 1$ but no local maximum
(B) f has a local maximum at $x = 0$ but no local minima
(C) f has a local minima at $x = \pm 1$ and a local maximum at $x = 0$
(D) none of the above is true.

85 The number of triples (a,b,c) of positive integers satisfying

$$2^a - 5^b 7^c = 1$$

is

(A) infinite (B) 2 (C) 1 (D) 0.

86 Let a be a fixed real number greater than -1. The locus of $z \in \mathbb{C}$ satisfying $|z - ia| = Im(z) + 1$ is

(A) parabola (B) ellipse (C) hyperbola (D) not a conic.

87 Consider the function $f : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{2\}$ given by

$$f(x) = \frac{2x}{x-1}.$$

Then

(A) f is one-one but not onto
(B) f is onto but not one-one
(C) f is neither one-one nor onto
(D) f is both one-one and onto.

88 Consider a real valued continuous function f satisfying $f(x+1) = f(x)$ for all $x \in \mathbb{R}$. Let

$$g(t) = \int_0^t f(x)\,dx, \quad t \in \mathbb{R}.$$

Define $h(t) = \lim_{n\to\infty} \frac{g(t+n)}{n}$, provided the limit exists. Then

(A) $h(t)$ is defined only for $t = 0$
(B) $h(t)$ is defined only when t is an integer

(C) $h(t)$ is defined for all $t \in \mathbb{R}$ and is independent of t

(D) none of the above is true.

89 Consider the sequence $a_1 = 24^{1/3}$, $a_{n+1} = (a_n + 24)^{1/3}, n \geq 1$. Then the integer part of a_{100} equals

(A) 2 (B) 10 (C) 100 (D) 24.

90 Let $x, y \in (-2, 2)$ and $xy = -1$. Then the minimum value of

$$\frac{4}{4 - x^2} + \frac{9}{9 - y^2}$$

is

(A) 8/5 (B) 12/5 (C) 12/7 (D) 15/7.

91 What is the limit of

$$\left(1 + \frac{1}{n^2 + n}\right)^{n^2 + \sqrt{n}}$$

as $n \to \infty$?

(A) e (B) 1 (C) 0 (D) ∞.

92 Consider the function $f(x) = x^4 + x^2 + x - 1, x \in (-\infty, \infty)$. The function

(A) is zero at $x = -1$, but is increasing near $x = -1$

(B) has a zero in $(-\infty, -1)$

(C) has two zeros in $(-1, 0)$

(D) has exactly one local minimum in $(-1, 0)$.

93 Consider a sequence of 10 A's and 8 B's placed in a row. By a run we mean one or more letters of the same type placed side by side. Here is an arrangement of 10 A's and 8 B's which contains 4 runs of A and 4 runs of B:

$$A\,A\,A\,B\,B\,A\,B\,B\,B\,A\,A\,B\,A\,A\,A\,A\,B\,B$$

In how many ways can 10 A's and 8 B's be arranged in a row so that there are 4 runs of A and 4 runs of B?

(A) $2\binom{9}{3}\binom{7}{3}$ (B) $\binom{9}{3}\binom{7}{3}$ (C) $\binom{10}{4}\binom{8}{4}$ (D) $\binom{10}{5}\binom{8}{5}$.

94 Suppose $n \geq 2$ is a fixed positive integer and

$$f(x) = x^n |x|, x \in \mathbb{R}.$$

Then

(A) f is differentiable everywhere only when n is even

(B) f is differentiable everywhere except at 0 if n is odd

(C) f is differentiable everywhere

(D) none of the above is true.

95 The line $2x + 3y - k = 0$ with $k > 0$ cuts the x axis and y axis at points A and B respectively. Then the equation of the circle having AB as diameter is

(A) $x^2 + y^2 - \frac{k}{2}x - \frac{k}{3}y = k^2$

(B) $x^2 + y^2 - \frac{k}{3}x - \frac{k}{2}y = k^2$

(C) $x^2 + y^2 - \frac{k}{2}x - \frac{k}{3}y = 0$

(D) $x^2 + y^2 - \frac{k}{3}x - \frac{k}{2}y = 0$.

96 Let $\alpha > 0$ and consider the sequence

$$x_n = \frac{(\alpha+1)^n + (\alpha-1)^n}{(2\alpha)^n}, n = 1, 2, \ldots.$$

Then $\lim_{n\to\infty} x_n$ is

(A) 0 for any $\alpha > 0$

(B) 1 for any $\alpha > 0$

(C) 0 or 1 depending on what $\alpha > 0$ is

(D) 0, 1 or ∞ depending on what $\alpha > 0$ is.

97 If $0 < \theta < \pi/2$ then

(A) $\theta < \sin\theta$

(B) $\cos(\sin\theta) < \cos\theta$

(C) $\sin(\cos\theta) < \cos(\sin\theta)$

(D) $\cos\theta < \sin(\cos\theta)$.

98 Assume the following inequalities for positive integer k:

$$\frac{1}{2\sqrt{k+1}} < \sqrt{k+1} - \sqrt{k} < \frac{1}{2\sqrt{k}}.$$

The integer part of

$$\sum_{k=2}^{9999} \frac{1}{\sqrt{k}}$$

equals

(A) 198 (B) 197 (C) 196 (D) 195.

99 Consider the sets defined by the inequalities

$$A = \{(x, y) \in \mathbb{R}^2 : x^4 + y^2 \leq 1\},\ B = \{(x, y) \in \mathbb{R}^2 : x^6 + y^4 \leq 1\}.$$

Then

(A) $B \subseteq A$

(B) $A \subseteq B$

(C) each of the sets $A - B, B - A$ and $A \cap B$ is non-empty

(D) none of the above is true.

100 The number of one-to-one functions from a set with 3 elements to a set with 6 elements is

(A) 20

(B) 120

(C) 216

(D) 720

101 The minimum value of the function $f(x) = x^2 + 4x + \frac{4}{x} + \frac{1}{x^2}$ where $x > 0$, is

(A) 9.5 (B) 10 (C) 15 (D) 20

102 The angle between the hyperbolas $xy = 1$ and $x^2 - y^2 = 1$ (at their point of intersection) is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

103 Given two complex numbers z, w with unit modulus (i.e., $|z| = |w| = 1$), which of the following statements will ALWAYS be correct?
(A) $|z+w| < \sqrt{2}$ and $|z-w| < \sqrt{2}$
(B) $|z+w| \leq \sqrt{2}$ and $|z-w| \geq \sqrt{2}$
(C) $|z+w| \geq \sqrt{2}$ or $|z-w| \geq \sqrt{2}$
(D) $|z+w| < \sqrt{2}$ or $|z-w| < \sqrt{2}$

Hints and Answers to selected problems.

There are also other ways to solve the problems apart from the ones sketched in the hints. Indeed, a student should feel encouraged upon finding a different way to solve some of these problems.

Hints and Answers to selected UGA Sample Questions.

1 **(B)**. Take the nth root of a_n and b_n and use A.M.$\geq$ G.M.
3 **(A)**. As 2004 = 2000 +4, the last digits of $(2004)^5$ and 4^5 are equal.
4 **(D)** Use binomial expansion of $(bc + a(b+c))^6$.
6 **(B)** Let $y = \log_{10} x$. Then $\log_{10} y = \log_{100} 4$. Hence $y = 2$.
8 **(D)** Check for 'test points'.
14 **(D)** $\sin\left(\frac{1}{x^3}\right)$ changes sign at the points $(n\pi)^{\frac{-1}{3}}$ for all $n \geq 1$.
15 **(D)** Observe that $\frac{(e^x-1)\tan^2 x}{x^3} = \frac{(e^x-1)}{x} \cdot \frac{\sin^2 x}{x^2} \cdot \frac{1}{\cos^2 x}$.
16 **(C)** Use induction and chain rule of differentiation.
22 **(B)** Show that the height function is $\frac{60}{t}$.
26 **(C)** Compute the number of maps such that $f(3) = 5$, $f(3) = 4$ etc.. Alternatively, define $g : \{1,2,3\} \to \{1,2,\ldots,7\}$ by $g(i) = f(i) + (i-1)$. Then, g is a strictly increasing function and its image is a subset of size 3 of $\{1, 2, \ldots 7\}$.
28 **(D)** Draw graphs of $(x+y)(x-y) = 0$ and $(x-a)^2 + y^2 = 1$.
38 **(A)** Differentiate.
51 **(A)** Compute for $C = \{x^2 + y^2 = 1\}$ and $Q = (a, 0)$ for some $a > 1$.
57 **(C)** Compute the integral $\int\limits_{1/2}^{2} 2^x dx - \int\limits_{1/2}^{2} \log x dx$.
60 **(D)** Let s be distance between the centre of the big circle and the centre of (any) one of the small circles. Then there exists a right angle triangle with hypoteneuse s, side r and angle $\frac{\pi}{n}$.
61**(C)** If $8n + 1 = m^2$, then $2n$ is a product of two consecutive integers.
62 **(C)** $z^2 = w^2 \Rightarrow z = \pm w \Rightarrow B \subseteq A$. But $|i| = 1$ and $i^2 \neq 1$.
63 **(C)** Amongst $1, |x|, |x|^2, |x|^3$, only $|x|$ is not differentiable at 0.
64 **(D)** Look at the derivative of f.
65 **(B)** Draw graphs of $y = \cos x$ and $y = \pm x$ and find the number of points of intersections.
66 **(D)** Calculate the discriminant $(b^2 - 4ac)$ of the given quadratic.

67 **(A)** The unit digit of all numbers $n!$ with $n \geq 5$ is 0.

68 **(B)** Use the formula for $\sum\limits_{i=1}^{n} i^3$.

69 **(C)** Find out the first values of n for which $\frac{a_{n+1}}{a_n}$ becomes < 1.

70 **(D)** The equation is $xy(x^2 + y^2 + 1) = 0$.

72 **(C)** Multiply the given sum by n.

73 (**D**) Verify using the given definition of a ring.

75 (**A**) Observe that one of x, y is odd and the other one is even. Square of an even number is divisible by 4 whereas square of an odd number leaves remainder 1 when divided by 4. Compare this with the right hand side.

76 (**C**) Check that $f_1(1) < f_2(1)$, $f_1(e) < f_2(e)$ and $f_1(e^2) > f_2(e^2)$.

83 (**D**) Note that a tangent to the parabola $y^2 = 4ax$ has equation of the form $y = mx + \frac{a}{m}$. Coordinates of P satisfy two equations: $y = mx + \frac{a}{m}$ and $y = -\frac{x}{m} - ma$. Eliminate m.

84 (**C**) The function f is non-negative and it vanishes only at 1 and -1. The derivative vanishes at $x = 0$ and it does not exist at $x = 1$, $x = -1$.

91 (**A**) Write $\left(1 + \frac{1}{n^2+n}\right)^{n^2+\sqrt{n}} = \left(\left(1 + \frac{1}{n^2+n}\right)^{n^2+n}\right)^{\frac{n^2+\sqrt{n}}{n^2+n}}$.

92 (**D**) As $f^{''} = 12x^2 + 2 > 0$, the function $f^{'}$ is increasing. Now $f^{'}(-1) < 0$ whereas $f^{'}(0) > 0$.

Sample Questions for UGB

Instructions UGB consists of questions that will require you to provide answers with appropriate justification.

1 Find the sum of all distinct four digit numbers that can be formed using the digits 1, 2, 3, 4, 5, each digit appearing at most once.

2 How many natural numbers less than 10^8 are there, with sum of digits equal to 7?

3 Consider the squares of an 8×8 chessboard filled with the numbers 1 to 64 as in the figure below. If we choose 8 squares with the property that there is exactly one from each row and exactly one from each column, and add up the numbers in the chosen squares, show that the sum obtained is always 260.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

4 Consider the function

$$f(x) = \lim_{n\to\infty} \frac{\log_e(2+x) - x^{2n}\sin x}{1+x^{2n}}$$

defined for $x > 0$. Is $f(x)$ continuous at $x = 1$? Justify your answer. Show that $f(x)$ does not vanish anywhere in the interval $0 \leq x \leq \frac{\pi}{2}$. Indicate the points where $f(x)$ changes sign.

5 An isosceles triangle with base 6 cms. and base angles 30^o each is inscribed in a circle. A second circle, which is situated outside the triangle, touches the first circle and also touches the base of the triangle at its midpoint. Find its radius.

6 Suppose a is a complex number such that

$$a^2 + a + \frac{1}{a} + \frac{1}{a^2} + 1 = 0.$$

If m is a positive integer, find the value of

$$a^{2m} + a^m + \frac{1}{a^m} + \frac{1}{a^{2m}}.$$

7 Let $a_n = 1\ldots1$ with 3^n digits. Prove that a_n is divisible by $3a_{n-1}$.

8 Let $f(u)$ be a continuous function and, for any real number u, let $[u]$ denote the greatest integer less than or equal to u. Show that for any

$x > 1$,

$$\int_1^x [u]([u]+1)f(u)du = 2\sum_{i=1}^{[x]} i \int_i^x f(u)du.$$

9 If a circle intersects the hyperbola $y = 1/x$ at four distinct points $(x_i, y_i), i = 1, 2, 3, 4$, then prove that $x_1x_2 = y_3y_4$.

10 Two intersecting circles are said to be orthogonal to each other if the tangents to the two circles at any point of intersection are perpendicular to each other. Show that every circle through the points $(2, 0)$ and $(-2, 0)$ is orthogonal to the circle $x^2 + y^2 - 5x + 4 = 0$.

11 Show that the function $f(x)$ defined below attains a unique minimum for $x > 0$. What is the minimum value of the function? What is the value of x at which the minimum is attained?

$$f(x) = x^2 + x + \frac{1}{x} + \frac{1}{x^2} \quad \text{for} \quad x \neq 0.$$

Sketch on plain paper the graph of this function.

12 Show that there is exactly one value of x which satisfies the equation

$$2\cos^2(x^3 + x) = 2^x + 2^{-x}.$$

13 Let $S = \{1, 2, \ldots, n\}$. Find the number of unordered pairs $\{A, B\}$ of subsets of S such that A and B are disjoint, where A or B or both may be empty.

14 An oil-pipe has to connect the oil-well O and the factory F, between which there is a river whose banks are parallel. The pipe must cross the river perpendicular to the banks. Find the position and nature of the shortest such pipe and justify your answer.

15 Find the maximum value of $x^2 + y^2$ in the bounded region, including the boundary, enclosed by $y = \frac{x}{2}, y = -\frac{x}{2}$ and $x = y^2 + 1$.

16 Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ where $x_1, \cdots, x_n, y_1, \cdots, y_n$ are real numbers. We write $x > y$ if either $x_1 > y_1$ or for some k, with $1 \le k \le n-1$, we have $x_1 = y_1, \ldots, x_k = y_k$, but $x_{k+1} > y_{k+1}$. Show that for $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$, if $u > v$ and $w > z$, then $u + w > v + z$.

17 How many real roots does $x^4 + 12x - 5$ have?

18 For any positive integer n, let $f(n)$ be the remainder obtained on dividing n by 9. For example, $f(263) = 2$.

(a) Let n be a three-digit number and m be the sum of its digits. Show that $f(m) = f(n)$.

(b) Show that $f(n_1 n_2) = f(f(n_1) \cdot f(n_2))$ where n_1, n_2 are any two positive three-digit integers.

19 Find the maximum among $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots$.

20 Show that it is not possible to have a triangle with sides a, b and c whose medians have lengths $\frac{2}{3}a$, $\frac{2}{3}b$ and $\frac{4}{5}c$.

21 For real numbers x, y and z, show that

$$|x| + |y| + |z| \le |x + y - z| + |y + z - x| + |z + x - y|.$$

22 Let X, Y, Z be the angles of a triangle.

(i) Prove that

$$\tan\frac{X}{2}\tan\frac{Y}{2} + \tan\frac{X}{2}\tan\frac{Z}{2} + \tan\frac{Z}{2}\tan\frac{Y}{2} = 1.$$

(ii) Using (i) or otherwise prove that

$$\tan\frac{X}{2}\tan\frac{Y}{2}\tan\frac{Z}{2} \le \frac{1}{3\sqrt{3}}.$$

23 Let α be a real number. Consider the function

$$g(x) = (\alpha + |\ x\ |)^2\, e^{(5-|x|)^2},\ -\infty < x < \infty.$$

(i) Determine the values of α for which g is continuous at all x.

(ii) Determine the values of α for which g is differentiable at all x.

24 Write the set of all positive integers in a triangular array as

$$\begin{array}{ccccccc}
1 & 3 & 6 & 10 & 15 & . & . \\
2 & 5 & 9 & 14 & . & . & . \\
4 & 8 & 13 & . & . & . & . \\
7 & 12 & . & . & . & . & . \\
11 & . & . & . & . & . & .
\end{array}$$

Find the row number and column number where 20096 occurs. For example 8 appears in the third row and second column.

25 Show that the polynomial $x^8 - x^7 + x^2 - x + 15$ has no real root.

26 Let m be a natural number with digits consisting entirely of 6's and 0's. Prove that m is not the square of a natural number.

P.T.O.

27 Let $0 < a < b$.

(i) Show that amongst the triangles with base a and perimeter $a+b$, the maximum area is obtained when the other two sides have equal length $\frac{b}{2}$.

(ii) Using the result of (i) or otherwise show that amongst the quadrilateral of given perimeter the square has maximum area.

28 Let $n \geq 1$, $S = \{1, 2, \ldots, n\}$. For a function $f : S \to S$, a subset $D \subset S$ is said to be invariant under f, if $f(x) \in D$ for all $x \in D$. Note that the empty set and S are invariant for all f. Let $\deg(f)$ be the number of subsets of S invariant under f.

(i) Show that there is a function $f : S \to S$ such that $\deg(f) = 2$.

(ii) Further show that for any k such that $1 \leq k \leq n$ there is a function $f : S \to S$ such that $\deg(f) = 2^k$.

29 Let

$$P(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

be a polynomial with integer coefficients, such that $P(0)$ and $P(1)$ are odd integers. Show that:

(a) $P(x)$ does not have any even integer as root.

(b) $P(x)$ does not have any odd integer as root.

30 Let $N = \{1, 2, \ldots, n\}$ be a set of elements called voters. Let $\mathcal{C} = \{S : S \subseteq N\}$ be the set of all subsets of N. Members of $\mathcal{C}$ are called coalitions. Let f be a function from $\mathcal{C}$ to $\{0, 1\}$. A coalition $S \subseteq N$ is said to be *winning* if $f(S) = 1$; it is said to be a *losing* coalition if $f(S) = 0$. A pair $\langle N, f \rangle$ as above is called a voting game if the following conditions hold.

(a) N is a winning coalition.

(b) The empty set $\emptyset$ is a losing coalition.

(c) If S is a winning coalition and $S \subseteq S'$, then S' is also winning.

(d) If both S and S' are winning coalitions, then $S \cap S' \neq \emptyset$, i.e., S and S' have a common voter.

Show that the maximum number of winning coalitions of a voting game is 2^{n-1}. Find a voting game for which the number of winning coalitions is 2^{n-1}.

31 Suppose f is a real-valued differentiable function defined on $[1, \infty)$ with $f(1) = 1$. Suppose, moreover, that f satisfies $f'(x) = 1/(x^2 + f^2(x))$. Show that $f(x) \leq 1 + \pi/4$ for every $x \geq 1$.

32 If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ at some point makes an angle θ with the X-axis, show that the equation of the normal is

$$y \cos\theta - x \sin\theta = a \cos 2\theta.$$

33 Suppose that a is an irrational number.

(a) If there is a real number b such that both $(a+b)$ and ab are rational numbers, show that a is a quadratic surd. (a is a quadratic surd if it is of the form $r+\sqrt{s}$ or $r-\sqrt{s}$ for some rationals r and s, where s is not the square of a rational number).

(b) Show that there are two real numbers b_1 and b_2 such that

(i) $a+b_1$ is rational but ab_1 is irrational.

(ii) $a+b_2$ is irrational but ab_2 is rational.

(Hint: Consider the two cases, where a is a quadratic surd and a is not a quadratic surd, separately).

34 Let A, B, and C be three points on a circle of radius 1.

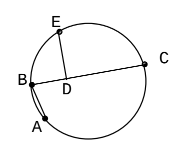
(a) Show that the area of the triangle ABC equals

$$\frac{1}{2}\left(\sin(2\angle ABC)+\sin(2\angle BCA)+\sin(2\angle CAB)\right).$$

(b) Suppose that the magnitude of $\angle ABC$ is fixed. Then show that the area of the triangle ABC is maximized when $\angle BCA=\angle CAB$.

(c) Hence or otherwise show that the area of the triangle ABC is maximum when the triangle is equilateral.

35 In the given figure, E is the midpoint of the arc $ABEC$ and ED is perpendicular to the chord BC at D. If the length of the chord AB is l_1, and that of BD is l_2, determine the length of DC in terms of l_1 and l_2



36 (a) Let $f(x)=x-xe^{-1/x}, x>0$. Show that $f(x)$ is an increasing function on $(0,\infty)$, and $\lim_{x\to\infty} f(x)=1$.

(b) Using part (a) and calculus, sketch the graphs of $y=x-1$, $y=x$, $y=x+1$, and $y=xe^{-1/|x|}$ for $-\infty<x<\infty$ using the same X and Y axes.

37 For any integer n greater than 1, show that

$$2^n < \binom{2n}{n} < \frac{2^n}{\prod_{i=0}^{n-1}(1-\frac{i}{n})}.$$

38 Show that there exists a positive real number $x\neq 2$ such that $\log_2 x=\frac{x}{2}$. Hence obtain the set of real numbers c such that

$$\frac{\log_2 x}{x}=c$$

has only one real solution.

39 Find a four digit number M such that the number $N=4\times M$ has the following properties.

(a) N is also a four digit number.

(b) N has the same digits as in M but in the reverse order.

40 Consider a function f on nonnegative integers such that $f(0) = 1$, $f(1) = 0$ and $f(n) + f(n-1) = nf(n-1) + (n-1)f(n-2)$ for $n \geq 2$. Show that

$$\frac{f(n)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

41 Of all triangles with a given perimeter, find the triangle with the maximum area. Justify your answer.

42 A 40 feet high screen is put on a vertical wall 10 feet above your eye-level. How far should you stand to maximize the angle subtended by the screen (from top to bottom) at your eye?

43 Study the derivatives of the function

$$y = \sqrt{x^3 - 4x}$$

and sketch its graph on the real line.

44 Suppose P and Q are the centres of two disjoint circles C_1 and C_2 respectively, such that P lies outside C_2 and Q lies outside C_1. Two tangents are drawn from the point P to the circle C_2, which intersect the circle C_1 at points A and B. Similarly, two tangents are drawn from the point Q to the circle C_1, which intersect the circle C_2 at points M and N. Show that $AB = MN$.

45 Evaluate: $\displaystyle\lim_{n\to\infty} \frac{1}{2n} \log \binom{2n}{n}$.

46 Consider the equation $x^5 + x = 10$. Show that

(a) the equation has only one real root;

(b) this root lies between 1 and 2;

(c) this root must be irrational.

47 In how many ways can you divide the set of eight numbers $\{2, 3, \ldots, 9\}$ into 4 pairs such that no pair of numbers has g.c.d. equal to 2?

48 Suppose S is the set of all positive integers. For $a, b \in S$, define

$$a * b = \frac{\text{l.c.m}(a, b)}{\text{g.c.d}(a, b)}$$

For example, $8 * 12 = 6$.

Show that ***exactly two*** of the following three properties are satisfied :

(a) If $a, b \in S$ then $a * b \in S$.

(b) $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$.

(c) There exists an element $i \in S$ such that $a * i = a$ for all $a \in S$.

Hints and Answers to selected UGB Sample Questions.

1. The answer is 399960. For each $x \in \{1,2,3,4,5\}$, there are 4! such numbers whose last digit is x. Thus the digits in the unit place of all the 120 numbers add up to $4!\,(1+2+3+4+5)$. Similarly the numbers at ten's place add up to 360 and so on. Thus the sum is $360\,(1+10+100+1000)$.

3. Let the chosen entries be in the positions (i, a_i), $1 \le i \le 8$. Thus $a_1, \ldots, a_8$ is a permutation of $\{1, \ldots, 8\}$. The entry in the square corresponding to (i,j)th place is $i + 8\,(j-1)$. Hence the required sum is $\sum_{i=1}^{8} (i + 8\,(a_j - 1))$.

5. Radius is $\frac{3\sqrt{3}}{2}$. Use trigonometry.

7. Observe that $a_n = a_{n-1}\left(1 + t + t^2\right)$ where $t = 10^{3^n}$

9. Substitute $y = \frac{1}{x}$ in the equation of a circle and clear denominator to get a degree 4 equation in x. The product of its roots is the constant term, which is 1.

11. The function $f\,(x) - 4$ is a sum of squares and hence non-negative. So the minimum is 4 which is attained at $x = 1$.

13. The number is $\frac{3^n+1}{2}$. An ordered pair (A, B) of disjoint subsets of S is determined by 3 choices for every element of S (either it is in A, or in B or in neither of them). Hence such pairs are 3^n in number. An unordered pair will be counted twice in this way, except for the case A and B both empty. Hence the number is $1 + \frac{3^n-1}{2}$.

15. Answer is 5. The maximum is attained at points $(2, 1)$ and $(2, -1)$.

17. Answer is 2. Let f be the given polynomial. Then $f\,(0)$ is negative and f is positive as x tends to $\pm\infty$. Hence it has at least 2 real roots. Since the derivative of f is zero only at $\sqrt[3]{-3}$, it cannot have more than two real roots.

19. Maximum is $\sqrt[3]{3}$. Either check the maximum of the function $x^{\frac{1}{x}}$, or compare $\sqrt[3]{3}$ with $\sqrt[n]{n}$.

21. Rewrite the given inequality in terms of the new variables $\alpha = x + y - z$, $\beta = y + z - x$, $\gamma = x + z - y$, and use the triangle inequality.

22. (i) Using the additive formula for $\tan(A/2 + B/2)$ and observing that $\tan(A/2 + B/2) = \tan(90^\circ - C/2) = \cot(C/2)$, it is easy to prove (i).

(ii) Note that $\tan(A/2), \tan(B/2), \tan(C/2)$ are positive numbers. So the arithmetic mean of $\tan(A/2)\tan(B/2), \tan(C/2)\tan(B/2), \tan(A/2)\tan(C/2)$ is greater than or equal to its geometric mean. Use this together with (i).

23. For any α, , $h(x) = (\alpha + x)^2 e^{(5-x)^2}, -\infty < x < \infty$ is continuous and differentiable at all x and note that $f(x) = h(|\, x\,|), -\infty < x < \infty$.

(i) As $| x |$ is a continuous function, f is a continuous function at all x for any real number α.
(ii) As $| x |$ is a differentiable function at all $x \neq 0$ f is differentiable at all $x \neq 0$ for any real number α. At $x = 0$, find the right hand and the left hand derivatives of f. Check that $f'(0)$ exists if and only if $\alpha = 0$ or $\alpha = \frac{1}{5}$.

24. The top row has the n-th triangular number $n(n+1)/2$ at the n-th place. Now $(200 \times 201)/2 = 20100 > 20096 > 19900 = (199 \times 200)/2$. So, 20100 occurs on the first row and 200-th column and is on the first row of this 200-th slanted line. Therefore $20096 = 20100 - 4$ occurs on the $200 - 4 = 196$-th column and on the 5-th row.

25. The polynomial can be re-written as $x^7(x-1) + x(x-1) + 15$. It it easily seen that the polynomial is strictly positive when $x \leq 0$ and $x \geq 1$. Further if $0 < x < 1$, then $| x^7(x-1) | < 1$ and $| x(x-1) | < 1$. This implies that the polynomial is strictly positive for all real numbers x.

26. Suppose n is a perfect square. If $10|n$, then $10^2|n$. Hence n ends in an even (possibly zero) number of 0's. So, for some k (possibly zero), the number $\frac{n}{10^{2k}}$ ends in 6 and has digits consisting of 0's and 6's. Therefore $\frac{n}{10^{2k}}$ ends in 66 or in 06. In both cases the number is even and leaves a remainder 2 on division by 4. This is a contradiction as an even perfect square is a multiple of 4.

28. (i) If $f(i) = i+1$ for all $i = 1, \cdots, n-1$, with $f(n) = 1$ then $\deg(f) = 2$.
(ii) Consider disjoint subsets $A_1, A_2, \cdots, A_k$ of S such that each A_i has more than one element. Let f be a cyclic function in each A_i. Then $\deg(f) = 2^k$.

A Model Question Paper for B.Math/B.Stat

Booklet No. | Test Code : **UGA**
Forenoon

Questions : 30 Time : 2 hours

Write your Name, Registration Number, Test Centre, Test Code and the Number of this Booklet in the appropriate places on the Answersheet.

This test contains 30 questions in all. For each of the 30 questions, there are four suggested answers. Only one of the suggested answers is correct. You will have to identify the correct answer in order to get full credit for that question. Indicate your choice of the correct answer by darkening the appropriate oval completely on the answersheet.

You will get
4 marks for each correctly answered question,
0 marks for each incorrectly answered question and
1 mark for each unattempted question.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET ONLY.
YOU ARE NOT ALLOWED TO USE CALCULATOR.

WAIT FOR THE SIGNAL TO START.

1 The system of inequalities

$$a - b^2 \geq \frac{1}{4},\ b - c^2 \geq \frac{1}{4},\ c - d^2 \geq \frac{1}{4},\ d - a^2 \geq \frac{1}{4}$$

(where a, b, c, d are real numbers) has

(A) no solutions (B) exactly one solution

(C) exactly two solutions (D) infinitely many solutions.

2 Let $\log_{12} 18 = a$. Then $\log_{24} 16$ is equal to

(A) $\frac{8-4a}{5-a}$ (B) $\frac{1}{3+a}$ (C) $\frac{4a-1}{2+3a}$ (D) $\frac{8-4a}{5+a}$.

3 The number of solutions of the equation $\tan x + \sec x = 2\cos x$, where $0 \leq x \leq \pi$, is

(A) 0 (B) 1 (C) 2 (D) 3.

4 Using only the digits 2, 3 and 9, how many six digit numbers can be formed which are divisible by 6?

(A) 41 (B) 80 (C) 81 (D) 161

5 What is the value of the following integral?

$$\int_{\frac{1}{2014}}^{2014} \frac{\tan^{-1} x}{x}\, dx$$

(A) $\frac{\pi}{4} \log 2014$ (B) $\frac{\pi}{2} \log 2014$ (C) $\pi \log 2014$ (D) $\frac{1}{2} \log 2014$

6 A light ray travelling along the line $y = 1$, is reflected by a mirror placed along the line $x = 2y$. The reflected ray travels along the line

(A) $4x - 3y = 5$ (B) $3x - 4y = 2$ (C) $x - y = 1$ (D) $2x - 3y = 1$.

7 For a real number x, let $[x]$ denote the greatest integer less than or equal to x. Then the number of real solutions of $\big|\, 2x - [x] \,\big| = 4$ is

(A) 1 (B) 2 (C) 3 (D) 4.

8 What is the ratio of the areas of the regular pentagons inscribed inside and circumscribed around a given circle?

(A) $\cos 36^\circ$ (B) $\cos^2 36^\circ$ (C) $\cos^2 54^\circ$ (D) $\cos^2 72^\circ$

9 Let z_1, z_2 be nonzero complex numbers satisfying $|z_1 + z_2| = |z_1 - z_2|$. The circumcentre of the triangle with the points z_1, z_2, and the origin as its vertices is given by

(A) $\frac{1}{2}(z_1 - z_2)$ (B) $\frac{1}{3}(z_1 + z_2)$ (C) $\frac{1}{2}(z_1 + z_2)$ (D) $\frac{1}{3}(z_1 - z_2)$.

10 In how many ways can 20 identical chocolates be distributed among 8 students so that each student gets at least one chocolate and exactly two students get at least two chocolates each?

(A) 308 (B) 364 (C) 616 (D) $\binom{8}{2}\binom{17}{7}$

11 Two vertices of a square lie on a circle of radius r, and the other two vertices lie on a tangent to this circle. Then, each side of the square is

(A) $\frac{3r}{2}$ (B) $\frac{4r}{3}$ (C) $\frac{6r}{5}$ (D) $\frac{8r}{5}$.

12 Let P be the set of all numbers obtained by multiplying five distinct integers between 1 and 100. What is the largest integer n such that 2^n divides at least one element of P?

(A) 8 (B) 20 (C) 24 (D) 25

13 Consider the function $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers with $a > 0$. If f is strictly increasing, then the function $g(x) = f^{'}(x) - f^{''}(x) + f^{'''}(x)$ is

(A) zero for some $x \in \mathbb{R}$ (B) positive for all $x \in \mathbb{R}$

(C) negative for all $x \in \mathbb{R}$ (D) strictly increasing.

14 Let A be the set of all points (h, k) such that the area of the triangle formed by (h, k), $(5, 6)$ and $(3, 2)$ is 12 square units. What is the least possible length of a line segment joining $(0, 0)$ to a point in A?

(A) $\frac{4}{\sqrt{5}}$ (B) $\frac{8}{\sqrt{5}}$ (C) $\frac{12}{\sqrt{5}}$ (D) $\frac{16}{\sqrt{5}}$

15 Let $P = \{ab\,c : a, b, c$ positive integers, $a^2 + b^2 = c^2$, and 3 divides $c\}$. What is the largest integer n such that 3^n divides every element of P?

(A) 1 (B) 2 (C) 3 (D) 4

16 Let $A_0 = \emptyset$ (the empty set). For each $i = 1, 2, 3, \ldots$, define the set $A_i = A_{i-1} \cup \{A_{i-1}\}$. The set A_3 is

(A) $\emptyset$ (B) $\{\emptyset\}$ (C) $\{\emptyset, \{\emptyset\}\}$ (D) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

17 Let $f(x) = \frac{1}{x-2}$. The graphs of the functions f and f^{-1} intersect at

(A) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(1-\sqrt{2}, 1-\sqrt{2})$

(B) $(1+\sqrt{2}, 1+\sqrt{2})$ and $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$

(C) $(1-\sqrt{2}, 1-\sqrt{2})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

(D) $(\sqrt{2}, -1-\frac{1}{\sqrt{2}})$ and $(-\sqrt{2}, -1+\frac{1}{\sqrt{2}})$

18 Let N be a number such that whenever you take N consecutive positive integers, at least one of them is coprime to 374. What is the smallest possible value of N?

(A) 4 (B) 5 (C) 6 (D) 7

19 Let $A_1, A_2, \ldots, A_{18}$ be the vertices of a regular polygon with 18 sides. How many of the triangles $\triangle A_iA_jA_k$, $1 \le i < j < k \le 18$, are isosceles but not equilateral?

(A) 63 (B) 70 (C) 126 (D) 144

20 The limit $\lim_{x\to 0} \dfrac{\sin^\alpha x}{x}$ exists *only when*

(A) $\alpha \geq 1$ (B) $\alpha = 1$

(C) $|\alpha| \leq 1$ (D) α is a positive integer.

21 Consider the region $R = \{(x, y) : x^2+y^2 \leq 100, \sin(x+y) > 0\}$. What is the area of R?

(A) 25π (B) 50π (C) 50 (D) $100\pi - 50$

22 Consider a cyclic trapezium whose circumcentre is on one of the sides. If the ratio of the two parallel sides is $1 : 4$, what is the ratio of the sum of the two oblique sides to the longer parallel side?

(A) $\sqrt{3} : \sqrt{2}$ (B) $3 : 2$ (C) $\sqrt{2} : 1$ (D) $\sqrt{5} : \sqrt{3}$

23 Consider the function $f(x) = \left\{\log_e\left(\dfrac{4+\sqrt{2x}}{x}\right)\right\}^2$ for $x > 0$. Then,

(A) f decreases upto some point and increases after that

(B) f increases upto some point and decreases after that

(C) f increases initially, then decreases and then again increases

(D) f decreases initially, then increases and then again decreases.

24 What is the number of ordered triplets (a, b, c), where a, b, c are positive integers (not necessarily distinct), such that $abc = 1000$?

(A) 64 (B) 100 (C) 200 (D) 560

25 Let $f : (0, \infty) \to (0, \infty)$ be a function differentiable at 3, and satisfying $f(3) = 3f'(3) > 0$. Then the limit

$$\lim_{x\to\infty}\left(\frac{f\left(3+\frac{3}{x}\right)}{f(3)}\right)^x$$

(A) exists and is equal to 3 (B) exists and is equal to e

(C) exists and is always equal to $f(3)$ (D) need not always exist.

26 Let z be a non-zero complex number such that $\left|z - \dfrac{1}{z}\right| = 2$. What is the maximum value of $|z|$?

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $1+\sqrt{2}$.

27 The minimum value of

$$\left|\sin x + \cos x + \tan x + \operatorname{cosec} x + \sec x + \cot x\right| \text{ is}$$

(A) 0 (B) $2\sqrt{2}-1$ (C) $2\sqrt{2}+1$ (D) 6

28 For any function $f : X \to Y$ and any subset A of Y, define

$$f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

Let A^c denote the complement of A in Y. For subsets A_1, A_2 of Y, consider the following statements:

(i) $f^{-1}(A_1^c \cap A_2^c) = (f^{-1}(A_1))^c \cup (f^{-1}(A_2))^c$

(ii) If $f^{-1}(A_1) = f^{-1}(A_2)$ then $A_1 = A_2$.

Then,

(A) both (i) and (ii) are always true

(B) (i) is always true, but (ii) may not always be true

(C) (ii) is always true, but (i) may not always be true

(D) neither (i) nor (ii) is always true.

29 Let f be a function such that $f''(x)$ exists, and $f''(x) > 0$ for all $x \in [a, b]$. For any point $c \in [a, b]$, let $A(c)$ denote the area of the region bounded by $y = f(x)$, the tangent to the graph of f at $x = c$ and the lines $x = a$ and $x = b$. Then

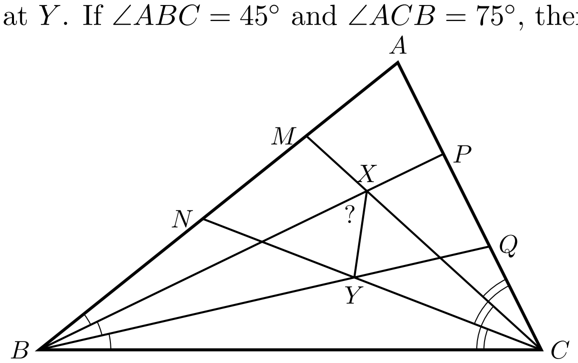
(A) $A(c)$ attains its minimum at $c = \frac{1}{2}(a + b)$ for any such f

(B) $A(c)$ attains its maximum at $c = \frac{1}{2}(a + b)$ for any such f

(C) $A(c)$ attains its minimum at both $c = a$ and $c = b$ for any such f

(D) the points c where $A(c)$ attains its minimum depend on f.

30 In $\triangle ABC$, the lines BP, BQ trisect $\angle ABC$ and the lines CM, CN trisect $\angle ACB$. Let BP and CM intersect at X and BQ and CN intersect at Y. If $\angle ABC = 45°$ and $\angle ACB = 75°$, then $\angle BXY$ is



(A) $45°$ (B) $47\frac{1}{2}°$ (C) $50°$ (D) $55°$

A Model Question Paper for B.Math/B.Stat

BOOKLET No. TEST CODE : UGB

Afternoon Session

There are 3 pages in this booklet.
The exam has 8 questions.
Answer as many as you can.

Time : 2 hours

Write your Name, Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK IS TO BE DONE ON THIS BOOKLET
AND/OR THE ANSWER-BOOKLET.
CALCULATORS ARE NOT ALLOWED.

STOP! WAIT FOR THE SIGNAL TO START.

P.T.O.

1 A class has 100 students. Let a_i, $1 \leq i \leq 100$, denote the number of friends the i-th student has in the class. For each $0 \leq j \leq 99$, let c_j denote the number of students having *at least* j friends. Show that

$$\sum_{i=1}^{100} a_i = \sum_{j=1}^{99} c_j \,.$$

2 It is given that the graph of $y = x^4 + ax^3 + bx^2 + cx + d$ (where a, b, c, d are real) has at least 3 points of intersection with the x-axis. Prove that either there are *exactly* 4 distinct points of intersection, or one of those 3 points of intersection is a local minimum or maximum.

3 Consider a triangle PQR in $\mathbb{R}^2$. Let A be a point lying on $\triangle PQR$ or in the region enclosed by it. Prove that, for any function $f(x, y) = ax + by + c$ on $\mathbb{R}^2$,

$$f(A) \leq \max\{f(P), f(Q), f(R)\} \,.$$

4 Let f and g be two non-decreasing twice differentiable functions defined on an interval (a, b) such that for each $x \in (a, b)$, $f''(x) = g(x)$ and $g''(x) = f(x)$. Suppose also that $f(x)g(x)$ is linear in x on (a, b). Show that we must have $f(x) = g(x) = 0$ for all $x \in (a, b)$.

5 Show that the sum of 12 consecutive integers can never be a perfect square. Give an example of 11 consecutive integers whose sum is a perfect square.

6 Let A be the region in the xy-plane given by

$$A = \{(x, y) : x = u + v, y = v, u^2 + v^2 \leq 1\} \,.$$

Derive the length of the longest line segment that can be enclosed inside the region A.

7 Let $f : [0, \infty) \to \mathbb{R}$ be a non-decreasing continuous function. Show then that the inequality

$$(z - x) \int_y^z f(u)du \geq (z - y) \int_x^z f(u)du$$

holds for any $0 \leq x < y < z$. **[P. T. O]**

8 Consider $n\,(> 1)$ lotus leaves placed around a circle. A frog jumps from one leaf to another in the following manner. It starts from some selected leaf. From there, it skips exactly one leaf in the clockwise direction and jumps to the next one. Then it skips exactly two leaves in the clockwise direction and jumps to the next one. Then it skips three leaves again in the clockwise direction and jumps to the next one, and so on. Notice that the frog may visit the same leaf more than

once. Suppose it turns out that if the frog continues this way, then all the leaves are visited by the frog sometime or the other. Show that n *cannot* be odd.