Lecture 3

$Z = f(x_1, x_2, \cdots, x_n); \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a function in n variables and requires n + 1 dimensional space to represent it graphically.

Total derivative

$Z = f(x, y)$ and $x = \phi(t), y = \psi(t)$ where $t \in [a, b]$

$\frac{dZ}{dt}$ is called the total derivative as all terms depend on t.

$$\frac{dZ}{dt} = \lim_{\Delta t \to 0} \frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta t}$$

as $\Delta t \to 0, \quad \Delta x \to 0$

$$\Delta f = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x, y)$$

$$\Rightarrow \frac{\Delta f}{\Delta t} = \frac{f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y)}{\Delta x} \cdot \frac{\Delta x}{\Delta t} + \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \cdot \frac{\Delta y}{\Delta t}$$

$$\frac{dZ}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\Rightarrow \frac{dZ}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

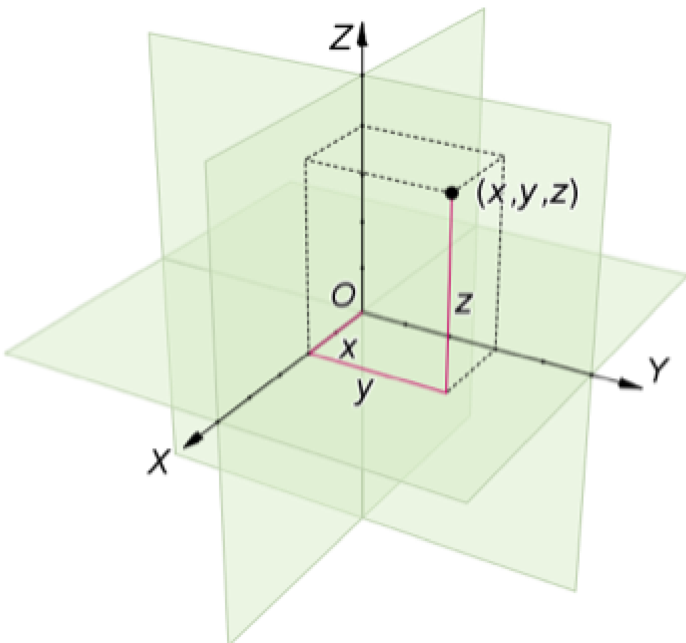
Thus if $Z = f(x_1, x_2, \cdots, x_n), \quad x_i = \phi_i(t); \quad t \in [a, b]$ then,

$$\frac{dZ}{dt} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{dt}$$

Coordinate systems in 3D

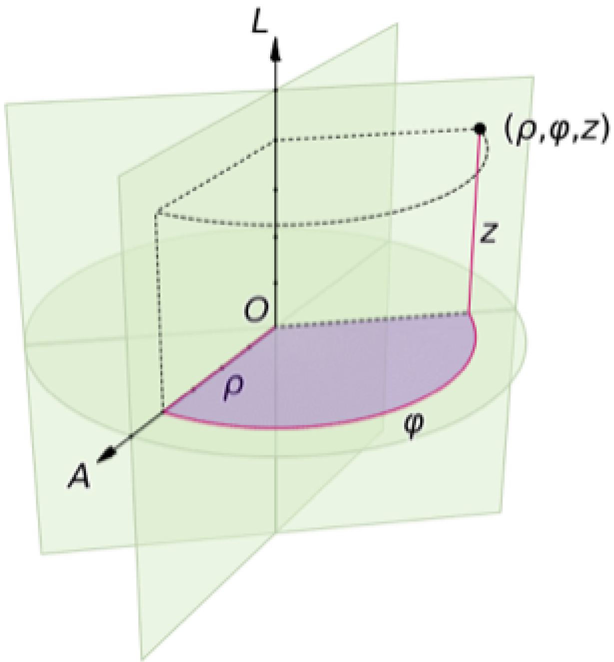
Cartesian coordinate system

Any point in the 3D space can be represented using the cartesian coordinates (x, y, z) where x, y, z represent the values of point at x-axis, y-axis & z-axis respectively.



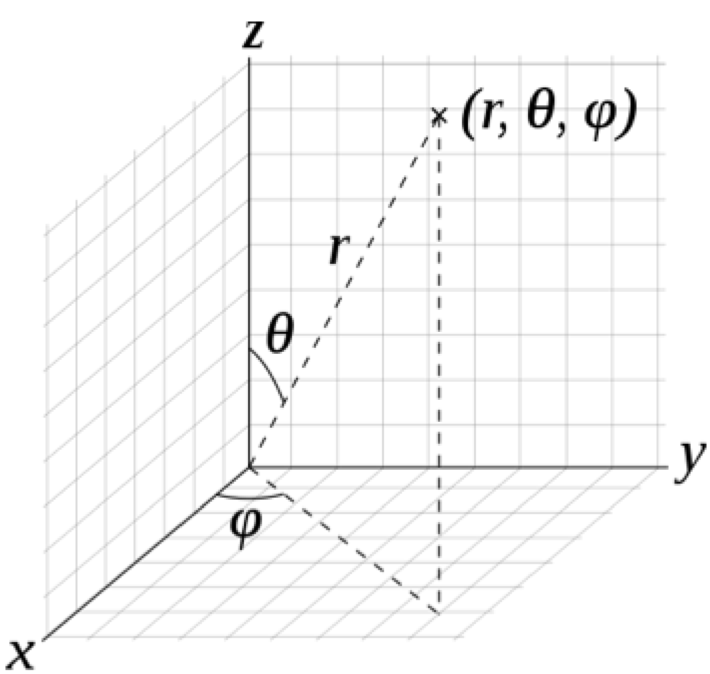
Cylindrical coordinate system

In this system we represent points using the parameters (r, θ, z) where r is the radial distance from the origin, θ is the angle from x-axis and z is the value at the z-axis.



Spherical coordinate system

We can represent any point in 3D using polar coordinates like (r, θ, ϕ) where r is the radial distance of the point from the origin, θ is the angle from z-axis and ϕ is the angle from x-axis.



Chain Rule

If $Z = x^2 + y^2$ where x & y are functions of (r, θ), $\quad X = x(r,\theta), Y = y(r,\theta)$ then to write $\frac{\partial Z}{\partial \theta}$ or $\frac{\partial Z}{\partial r}$ we use the chain rule.

$$\frac{\partial Z}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

similarly,

$$\frac{\partial Z}{\partial \theta} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

Ex: $Z = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta$

$$\frac{\partial Z}{\partial r} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= 2x \cdot \cos\theta + 2y \cdot \sin\theta$$

$$= 2r\cos^2\theta + 2r\sin^2\theta$$

$$= 2r(\cos^2\theta + \sin^2\theta) = 2r$$

$$\frac{\partial Z}{\partial \theta} = \frac{\partial Z}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial Z}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= 2x(-r\sin\theta) + 2y(r\cos\theta)$$

$$= 2(r\cos\theta)(-r\sin\theta) + 2(r\sin\theta)(r\cos\theta)$$

$$= -2r^2\cos\theta\sin\theta + 2r^2\cos\theta\sin\theta = 0$$

#semester-1 #mathematics #calculus