Lecture 2

Continuity

$f(x, y)$ is continuous at (x_0, y_0) if

1. $f(x_0, y_0)$ is well defined.
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$ exists.
3. $L = f(x_0, y_0)$

Mathematically,
If for every $\epsilon > 0 \quad \exists \quad \delta > 0$ such that $|f(x, y) - f(x_0, y_0)| < \epsilon \quad \forall \quad (x, y)$ such that $0 < |(x, y) - (x_0, y_0)| < \delta$

Ex: $Z = f(x, y) = x^2$

at $(x_0, y_0) = (0, 0)$,

$$f(0, 0) = 0$$

⇒ the function is well defined.

$$\lim_{(x,y)\to(0,0)} x^2 = 0$$

$$\&$$

$$f(0, 0) = L$$

⇒ The function is continuous at (0, 0).

Ex: $Z = \begin{Bmatrix} \frac{x^3-y^3}{x^3+y^3} & , & (x, y) \neq 0 \\ 0 & , & (x, y) = 0 \end{Bmatrix}$ Check the continuity at (0, 0).

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^3 + y^3}$$

Approaching from path 1: (first y → 0, then x → 0)

$$= \lim_{x\to 0}\left(\lim_{y\to 0}\frac{x^3-y^3}{x^3+y^3}\right) = \lim_{x\to 0} 1 = 1$$

Approaching from path 2: (first x → 0, then y → 0)

$$= \lim_{y\to 0}\left(\lim_{x\to 0}\frac{x^3-y^3}{x^3+y^3}\right) = \lim_{x\to 0} -1 = -1$$

Approaching from path 3: (along the line y = mx)

$$= \lim_{x\to 0}\frac{x^3-m^3x^3}{x^3+m^3x^3} = \lim_{x\to 0}\frac{1-m^3}{1+m^3} = \frac{1-m^3}{1+m^3}$$

⇒ limit does not exists at (0, 0)

⇒ The function is not continuous at (0, 0).

Derivative

We use the notation $\frac{\partial f}{\partial x}$ to represent partial derivative of $Z = f(x, y)$ with respect to x.

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0}\frac{(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y} = \lim_{\Delta y\to 0}\frac{(x, y+\Delta y) - f(x,y)}{\Delta y}$$

- $\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}$ are called first order partial derivatives.
- $\frac{\partial^2 Z}{\partial x^2}, \frac{\partial^2 Z}{\partial y^2}, \frac{\partial^2 Z}{\partial x\partial y}, \frac{\partial^2 Z}{\partial y\partial x}$ are second order partial derivatives.
- $\frac{\partial^2 Z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial Z}{\partial y}\right)$
- If $\frac{\partial^2 Z}{\partial x\partial y} = \frac{\partial^2 Z}{\partial y\partial x}$ then function is continuous & differentiable.
- Similarly we can define k^{th} power derivative $\frac{\partial^k Z}{\partial x^r\partial y^{k-r}}$

Ex: $Z = x^2 + y^2$

$$\frac{\partial Z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + y^2 - (x^2+y^2)}{\Delta x}$$

$$= \lim_{\Delta x = 0} \frac{\Delta x^2 + 2x\Delta x}{\Delta x} = \lim_{\Delta x \to 0} \Delta x + 2x = 2x$$

$$\Rightarrow \frac{\partial Z}{\partial x} = 2x$$

$$\frac{\partial Z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim_{\Delta y \to 0} \frac{x^2 + (y+\Delta y)^2 - (x^2+y^2)}{\Delta y} = 2y$$

$$\Rightarrow \frac{\partial Z}{\partial y} = 2y$$

Ex: $Z = x^3 + y^3 - 3axy$, a is constant

$$\frac{\partial Z}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial Z}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 Z}{\partial x^2} = 6x$$

$$\frac{\partial^2 Z}{\partial y^2} = 6y$$

$$\frac{\partial^2 Z}{\partial x \partial y} = -3a$$

#semester-1 #mathematics #calculus