Lecture 7

Alternating series

$\sum_{n=1}^{\infty} u_n = u_1 - u_2 + u_3 - u_4 + u_5 \cdots ; u_i \geq 0$ is alternating series as alternating terms have opposite signs.

Leibnitz test

An alternating series converges if

1. Each term is numerically less than its preceding term $u_{n+1} < u_n \quad \forall \quad n$.
2. $\lim_{n\to\infty} U_n = 0$

Proof.

$$S_{2n} = u_1 - u_2 + u_3 - \cdots + u_{2n-1} - u_{2n}$$

$$= \underbrace{(u_1 - u_2)}_{+ve} + \underbrace{(u_3 - u_4)}_{+ve} + \cdots + (u_{2n-1} - u_{2n})$$

if $u_{n+1} < u_n \quad \forall \quad n$,

$$S_{2n+2} = S_{2n} + \underbrace{(U_{2n+1} - U_{2n+2})}_{+ve}$$

⇒ $\left\{ S_{2n} \right\}$ is monotonically increasing sequence.

$$S_{2n} = u_1 - u_2 + u_3 - u_4 + \cdots - u_{2n}$$

$$= u_1 - \underbrace{(u_2 - u_3)}_{+ve} - \underbrace{(u_4 - u_5)}_{+ve} \cdots - \underbrace{u_{2n}}_{+ve}$$

$$\Rightarrow S_{2n} < u_1 \quad \forall \quad n$$

$\Rightarrow \left\{ S_{2n} \right\}$ is convergent.

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to infty} S_{2n} + u_{2n+1} = \lim_{n\to\infty} S_{2n} + \lim_{n\to\infty} U_{2n+1}$$

$$\Rightarrow \lim_{n\to\infty} S_{2n+1} = L = \lim_{n\to\infty} S_{2n}$$

⇒ $\left\{S_{2n}\right\}$ & $\left\{S_{2n+1}\right\}$ are both convergent.

⇒ $\left\{S_n\right\}$ is also convergent.

⇒ $\sum_{n=1}^{\infty}(-1)^n U_n$ is convergent if $\{U_n\}$ is decreasing & $\lim_{n\to\infty} U_n = 0$.

Ex: $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^n}{n(n-1)}; \quad 0 < x < 1$ is an alternating series where

$$U_n = \frac{x^n}{n(n-1)}$$

$$U_{n+1} - U_n = \frac{x^{n+1}}{n(n+1)} - \frac{x^n}{n(n-1)}$$

$$= \frac{x^n}{n}\left[\frac{x(n-1)-(n+1)}{(n+1)(n-1)}\right]$$

$$= \frac{x^n}{n}\left[\frac{n(x-1)-(x+1)}{(n+1)(n-1)}\right]$$

since 0 < x < 1, $U_{n+1} - U_n < 0$

⇒ $\{U_n\}$ is monotonically decreasing.

$$\lim_{n\to\infty} U_n = \lim_{n\to\infty} \frac{x^n}{n(n-1)} = 0$$

⇒ By leibnitz test, $\sum_{n=2}^{\infty}(-1)^{n-1}\frac{x^n}{n(n-1)}$ is convergent.

Convergence of arbitrary series $\sum U_n$

1. **Absolutely convergent series** : A series is absolutely convergent if $\sum U_n$ is convergent.
2. **Conditionally convergent series** : A series is conditionally convergent if $\sum U_n$ is convergent in the given form but $\sum |U_n|$ is divergent (divergent in

the absolute form).

Ex: $\sum(-1)^n \frac{1}{n^2}$

$\sum \left| \frac{1}{n^2} \right|$ is convergent.

⇒ it is absolutely convergent series.

Ex: $\sum(-1)^n \frac{1}{n}$ is convergent by leibnitz test but,

$\sum \left|(-1)^n \frac{1}{n}\right| = \sum \frac{1}{n}$ is divergent.

⇒ It is convergent in the given form but divergent in absolute form.
⇒ It is conditionally convergent series.

> **Theorem.** Every absolutely convergent series is convergent but converse need not be true.

Reimann's rearrangement theorem

For a conditionally convergent series, for any real x there is an rearrangement of $\sum U_n$ which converges to x.

#semester-1 #mathematics #real-analysis