Lecture 6

Raabe's test

In the positive term series $\sum U_n$ if $\lim_{n\to\infty} n(\frac{U_n}{U_{n+1}} - 1) = \lambda$ then the series converges for $\lambda > 1$ and diverges for $\lambda < 1$, but the test fails when $\lambda = 1$.

Logarithmic test

In a positive term series $\sum U_n$ if $\lim_{n\to\infty} nlog(\frac{U_n}{U_{n+1}}) = \lambda$ then the series converges for $\lambda > 1$ and diverges for $\lambda < 1$, but the test fails when $\lambda = 1$.

Ex: Test the convergence of $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^{2n}$

$$\frac{U_{n+1}}{U_n} = \frac{((n+1)!)^2}{(2n+2)!} x^{2n+2} \frac{(2n)!}{(n!)^2} \frac{1}{x^{2n}}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{(n+1)^2}{(2n+1)(2n+2)} x^2$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2 x^2}{(2+\frac{1}{n})(2+\frac{2}{n})} = \frac{x^2}{4}$$

By D'Alembert's ration test, this is convergent when $\lambda = \frac{x^2}{4} < 1$ and divergent when $\lambda = \frac{x^2}{4} > 1$.

when $\frac{x^2}{4} = 1$,

$$\frac{U_{n+1}}{U_n} = \frac{4(n+1)^2}{(2n+1)(2n+2)}$$

Using Raabe's test,

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}}-1\right) = \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{4(n+1)^2}-1\right]$$

$$= \lim_{n\to\infty} \frac{n}{4}\left[\frac{-2n-2}{(n+1)^2}\right] = \frac{-1}{2} < 1$$

⇒ $\sum U_n$ diverges when $x^2 = 4$ i.e. $x = \pm 2$

Thus, $\sum U_n \longrightarrow \begin{Bmatrix} \text{convergent}; & |x| < 2 \\ \text{divergent}; & |x| \geq 2 \end{Bmatrix}$

Ex: $x + \frac{2^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \cdots \quad ; \quad x > 0$

$$\sum U_n = \sum \frac{n^n}{n!}x^n$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!}x^{n+1} \times \frac{n!}{n^n}\frac{1}{x^n}$$

$$= \lim_{n\to\infty} \frac{x(n+1)^{n+1}}{(n+1)n^n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n x = ex$$

By ration test, when $ex < 1$ i.e.
$x < 1/e$ the series is convergent.
$x > 1/e$ the series is divergent.

when $x = 1/e$,

$$\frac{U_{n+1}}{U_n} = \frac{1}{e}\frac{(n+1)^n}{n^n}$$

applying log test,

$$\lim_{n\to infty} n \log \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} n \log \frac{en^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} n\left[\log e - \log\left(1+\frac{1}{n}\right)^n\right]$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \lim_{n\to\infty} n\left[\frac{1}{2n} - \frac{1}{3n^2} + \frac{1}{4n^3} - \cdots\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{3n}+\frac{1}{4n^2}-\cdots\right]=\frac{1}{2}<1$$

By logarithmic test, $\sum U_n$ is divergent when $x = 1/e$.

Thus $\sum U_n$ is $\begin{Bmatrix} \text{convergent} & ; & x < 1/e \\ \text{divergent} & ; & x \geq 1/e \end{Bmatrix}$

#semester-1 #mathematics #real-analysis