Lecture 5

D'Alembert's ratio test

In a positive term series $\sum U_n$, if $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lambda$ then the series convergest for $\lambda < 1$ and diverges for $\lambda > 1$, but this test fails for $\lambda = 1$.

Ex: We know that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

for $\sum_{n=1}^{\infty} \frac{1}{n}$, $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} = 1.$

for $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+\frac{1}{n})^2} = 1.$

Thus this test fails when $\lambda = 1$.

Proof.

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lambda$$

⇒ for $\epsilon > 0$ there exists a stage n_1 such that

$$\lambda - \epsilon < \frac{U_{n+1}}{U_n} < \lambda + \epsilon \quad \forall \quad n \geq n_1$$

Thus we need to show that $\sum_{n=n_1}^{\infty} U_n$ is convergent if $\lambda < 1$.

$$\begin{aligned}
&s_1 = u_1 \\
&s_2 = u_1 + u_2 = u_1 \left[1 + \frac{u_2}{u_1}\right] \\
&s_3 = u_1 + u_2 + u_3 = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_1}\right] = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1}\right] \\
&\vdots \\
&s_n = u_1 + u_2 + \cdots + u_n = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \cdots + \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \cdots \frac{u_2}{u_1}\right]
\end{aligned}$$

case (i) $\lambda < 1$, choose $\epsilon > 0$ such that $r = \lambda + \epsilon < 1$

$$\Rightarrow \frac{U_{n+1}}{U_n} < r < 1 \quad \forall \quad n \geq n_1$$

$s_1 = u_1$
$s_2 = u_1 \left[1 + \frac{u_2}{u_1}\right] < u_1(1 + r)$
$s_3 = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_1}\right] = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1}\right] < u_1(1 + r + r^2)$
$\vdots$
$s_n = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \cdots + \frac{u_n}{u_{n-1}} \cdot \frac{u_{n-1}}{u_{n-2}} \cdots \frac{u_2}{u_1}\right] < u_1(1 + r + r^2 + \cdots + r^{n-1})$

$$\lim_{n \to \infty} s_n < \lim_{n \to \infty} u_1(1 + r + r^2 + \cdots + r^{n-1})$$

$$\Rightarrow \lim_{n \to \infty} s_n < u_1 \left[\frac{1}{1 - r}\right]$$

⇒ s_n is monotonically increasing bounded above sequence.
⇒ s_n is convergent ⇒ $\sum U_n$ is convergent.

case (ii) $\lambda > 1$

$$\lambda - \epsilon < \frac{U_{n+1}}{U_n} < \lambda + \epsilon \quad \forall \quad n \geq n_1$$

choose $\epsilon > 0, r = \lambda - \epsilon > 1$

$$1 < r < \frac{U_{n+1}}{U_n} \quad \forall \quad n \geq n_1$$

$$\Rightarrow \frac{U_{n+1}}{U_n} > 1 \quad \forall \quad n \geq n_1$$

⇒ $s_n > u_1(1 + 1 + \cdots + 1) = nu_1$

⇒ $\lim_{n \to \infty} s_n \longrightarrow \infty$

⇒ $\sum U_n$ is divergent when $\lambda > 1$

Cauchy's root test

In a positive term series $\sum U_n$, if $\lim_{n \to \infty} (U_n)^{1/n} = \lambda$ then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$, but the test fails when $\lambda = 1$.

Proof.
case (i) $\lambda < 1$

There exists for $\epsilon > 0$, a stage n_1 such that $(U_n)^{1/n} < \lambda + \epsilon \quad \forall \quad n \geq n_1$.

choose $\epsilon > 0$ such that $r = \lambda + \epsilon < 1$

$$(U_n)^{1/n} < r \quad \forall \quad n \geq n_1$$

$$\Rightarrow U_n < r^n \quad \forall \quad n \geq n_1$$

By comparision test $\sum U_n$ is convergent since $\sum r^n$ is convergent for r < 1.

case (ii) $\lambda > 1$

$$\lambda - \epsilon < (U_n)^{1/n} \quad \forall \quad n \geq n_1$$

$$\Rightarrow (U_n)^{1/n} > 1 \quad \forall \quad n \geq n_1$$

but the necessary condition for convergent is $\lim_{n\to\infty} U_n = 0$ which is not possible when $U_n > 1$.

⇒ $\sum U_n$ is divergent.

Ex: Discuss the convergence of $\sum_{n=1}^{\infty} \frac{n^2}{3^n}$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^2 = \lambda = \frac{1}{3} < 1$$

$\lambda < 1 \Rightarrow \sum U_n$ is convergent.

Ex: Test the convergence of $\frac{2}{1} + \frac{2\cdot5\cdot8}{1\cdot5\cdot9} + \frac{2\cdot5\cdot8\cdot11}{1\cdot5\cdot9\cdot13} + \cdots$

$$= 2 + \sum_{n=1}^{\infty} U_n$$

$$U_{n+1} = U_n \left(\frac{3n+8}{4n+9}\right)$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{3+\frac{8}{n}}{4+\frac{9}{n}} = \frac{3}{4}$$

$\lambda < 1$ ⇒ series is convergent

#semester-1 #mathematics #real-analysis