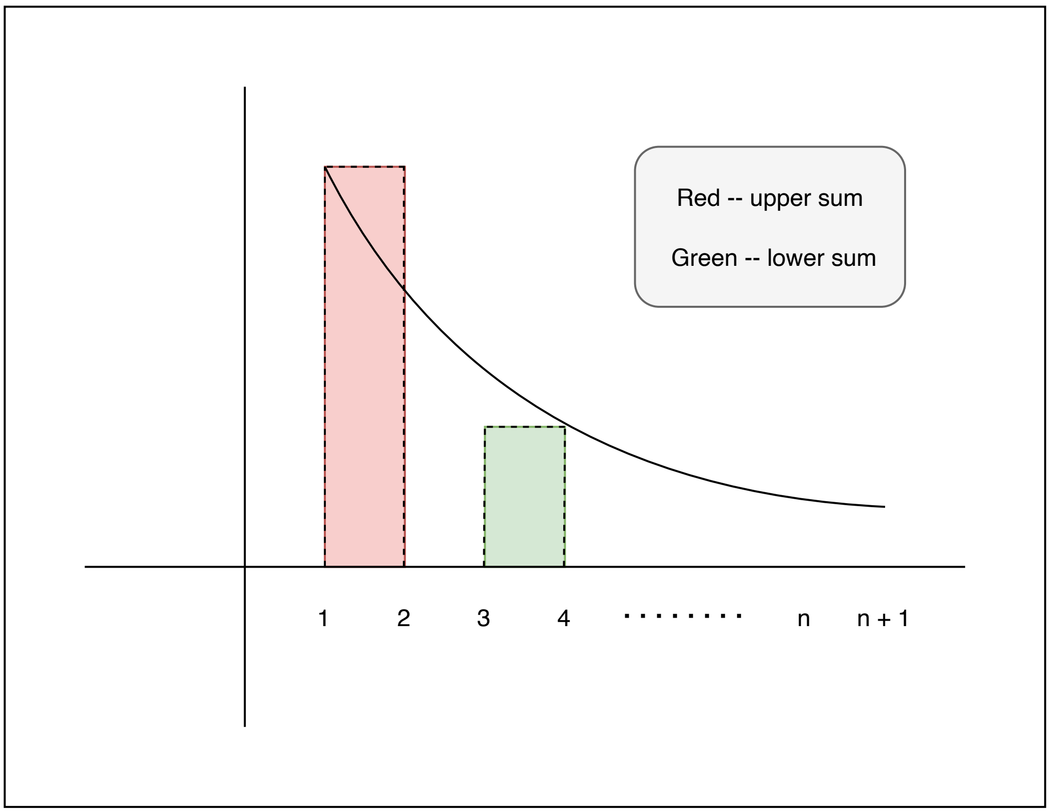
Lecture 4

Integral test

A positive term series $f(1) + f(2) + \cdots + f(n) + \cdots$ where $f(n)$ decreases as n increases, converges or diverges according as the integral $\int_1^\infty f(x)dx$ is finite or infinite.



Upper sum = $s_n = f(1) + f(2) + \cdots + f(n)$
$= f(1) \cdot 1 + f(2) \cdot 1 + \cdots + f(n) \cdot 1$

$$\Rightarrow s_n \geq \int_1^{n+1} f(x)dx$$

Also, the lower sum $s_{n+1} - f(1) = f(2) \cdot 1 + f(3) \cdot 1 + \cdots + f(n+1) \cdot 1$

$$\Rightarrow s_{n+1} - f(1) \leq \int_1^{n+1} f(x)dx$$

$$\Rightarrow s_{n+1} \leq \int_1^{n+1} f(x)dx + f(1)$$

as $n \longrightarrow \infty$ if $\int_1^\infty f(x)dx$ is finite

⇒ $s_{n+1} \leq L$
⇒ $\{s_n\}$ is monotonically increasing which is bounded above. Hence $\{s_n\}$ is convergent.
⇒ $\sum f(n)$ is convergent.

using the second relation, when $\int_1^\infty f(x)dx$ is infinite,

$$s_n \geq \int_1^{n+1} f(x)dx$$

$$\lim_{n \to \infty} s_n \quad \text{does not exists}$$

⇒ $\sum f(n)$ is divergent.

Infinite series of the form $\sum \frac{1}{n^p}$

Applying intergral test on $\sum \frac{1}{n}$.

$$\int_1^\infty \frac{1}{x}dx = log(x)\Big|_1^\infty$$

does not exists. ⇒ $\sum \frac{1}{n}$ is divergent.

For $\sum \frac{1}{n^p}$ when $p < 0, \lim_{n \to \infty} U_n \neq 0$
⇒ divergent.

For $\sum \frac{1}{n^p}$ applying intergral test.

$$\int_1^\infty \frac{1}{x^p}dx = \int_1^\infty x^{-p}dx = \frac{x^{-p+1}}{-p+1}\Big|_1^\infty$$

when p > 1,

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

⇒ finite.

when 0 < p < 1,

$$x^{-p+1} \longrightarrow \infty$$

⇒ integral is infinite.

$$\Rightarrow \int_1^\infty x^{-p} dx = \begin{Bmatrix} \frac{1}{1-p}; & \text{if } p > 1. \\ \infty; & \text{if } 0 < p < 1. \end{Bmatrix}$$

Thus $\sum \frac{1}{n^p}$ is convergent when $p > 1$ and divergent when $p \leq 1$.

Ex: Test the convergence of this infinite series $\frac{1}{1 \cdot 2 \cdot 3} + \frac{3}{2 \cdot 4 \cdot 5} + \frac{5}{3 \cdot 4 \cdot 5} + \cdots$

This is positive term series. The necessary condition for convergence is $\lim_{n \to \infty} U_n = 0$

$$\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)} > \frac{2n-1}{n^3}$$

⇒ $\lim_{n \to \infty} U_n = 0$

$$U_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{2 - \frac{1}{n}}{n^2(1+\frac{1}{n})(1+\frac{2}{n})}$$

according to limit form comparision test,

$$\text{if } \lim_{n \to \infty} \frac{U_n}{V_n} = L(\neq 0)$$

they will converge together.

taking $\sum V_n = \sum \frac{1}{n^2}$,

$$\lim_{n \to \infty} \frac{U_n}{V_n} = \lim_{n \to \infty} \frac{2 - \frac{1}{n}}{n^2(1+\frac{1}{n})(1+\frac{2}{n})} = 2(\neq 0)$$

we know that $\sum V_n$ is convergent series ⇒ $\sum U_n$ is also convergent.

#semester-1 #mathematics #real-analysis