Lecture 3

Infinite series

$$\sum_{n=1}^{\infty} x_n = \{x_1, x_2, \cdots\}$$

is called infinite series.

Ex: {1, 2, 3, $\cdots$ }

$$\sum_{n=1}^{\infty} n = 1 + 2 + \cdots \longrightarrow \infty$$

Sequence of partial sums

$\left\{s_n\right\}_{n=1}^{\infty}$ is called the sequence of partial sum of the infinite series $\sum_{n=1}^{\infty} U_n$ if

$s_1 = u_1$
$s_2 = u_1 + u_2$
$s_3 = u_1 + u_2 + u_3$
$\vdots$
$s_n = u_1 + u_2 + u_3 + \cdots + u_n$

- $\sum U_n$ is convergent if the sequence of partial sum converges.
  ⇒ $\{s_1, s_2, \cdots, s_n\}$ should be convergent.

Ex: $\sum n = 1 + 2 + \cdots$

$s_1 = 1$
$s_2 = 1 + 2 = 3$
$s_3 = 1 + 2 + 3 = 6$

$\vdots$

$s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

$\left\{ s_n = \frac{n(n+1)}{2} \right\}_{n=1}^{\infty}$ is not convergent

⇒ $\sum n$ is divergent.

Ex: $\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots$

$s_1 = 1$
$s_2 = 1 + r$
$s_3 = 1 + r + r^2$
$\vdots$
$s_n = 1 + r + r^2 \cdots + r^n = \frac{1-r^n}{1-r}$

case (i) $\quad r = 0$

$$\sum_{n=0}^{\infty} r^n = 1$$

⇒ it is convergent for r = 0.

case (ii) $\quad r = 1$

$$\sum_{n=0}^{\infty} r^n = \sum_{n=0}^{\infty} n = 1 + 1 + \cdots$$

⇒ it is divergent for r = 1

case (iii) $\quad 0 < r < 1$

$$s_n = \frac{1 - r^n}{1 - r}$$

$$\Rightarrow \lim_{n \to \infty} s_n = \frac{1}{1 - r}$$

$$\Rightarrow \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

⇒ it is convergent for 0 < r < 1

case (iv) $r > 1$

$$s_n = \frac{1 - r^n}{1 - r} \longrightarrow \infty$$

⇒ divergent for r > 1

case (v) $-1 < r < 0$

$$s_n = \frac{1 - r^n}{1 - r} = \frac{1}{1 - r}$$

⇒ convergent for -1 < r < 0

case (vi) $r < -1$

$$s_n = \frac{1 - r^n}{1 - r} \longrightarrow \infty$$

⇒ divergent for r < -1

Thus we can conclude that

$$\sum_{n=0}^{\infty} r^n = \begin{Bmatrix} \frac{1}{1-r}; & \text{if } |r| < 1. \\ \infty; & \text{if } |r| > 1. \end{Bmatrix}.$$

- Consider an infinite sum $\sum U_n$ to be convergent.
  $s_1 = u_1$
  $s_2 = u_1 + u_2$
  $\vdots$
  $s_{n-1} = u_1 + u_2 + \cdots + u_{n-1}$
  $s_n = u_1 + u_2 + \cdots + u_{n-1} + u_n$

  $$s_n - s_{n-1} = \sum_{k=1}^{n} U_k - \sum_{k=1}^{n} U_{k-1} = U_n$$

  ⇒ $s_n - s_{n-1} = u_n$
  ⇒ $\lim_{n \to \infty} s_n - s_{n-1} = U_n = 0$

> Thus if $\sum U_n$ is convergent then $\lim_{n\to\infty} U_n = 0$ but the converse may not be true.

⇒ The necessary condition for $\sum U_n$ to be convergent is $\lim_{n\to\infty} U_n = 0$

Ex: $\sum_{n=1}^{\infty} n$

$$\lim_{n\to\infty} \text{does not exist}$$

$$\Rightarrow \text{it cannot be convergent}$$

Comparision test

If $\sum U_n, \sum V_n$, $u_n \geq 0$ & $v_n \geq 0$ satisfies $u_n \leq v_n \quad \forall \quad n$ then,

1. If $\sum V_n$ is convergent then, $\sum U_n$ is convergent.
2. If $\sum U_n$ is divergent then, $\sum V_n$ is divergent.

Proof 1. If $\sum V_n$ is convergent ⇒ $\{s_n = v_1 + v_2 + \cdots + v_n\}_{n=1}^{\infty}$ is convergent. We need to show that partial sum of $\sum U_n$ is convergent.

$t_1 = u_1 \leq s_1 = v_1$
$t_2 = u_1 + u_2 \leq v_1 + v_2 = s_2$
$t_3 \leq s_3$
$\vdots$

since $\{t_n\}_{n=1}^{\infty}$ is monotonically increasing sequence,
$t_n \leq s_n$

as $n \to \infty, \lim_{n\to\infty} s_n = L$

⇒ $\{t_n\}_{n=1}^{\infty}$ is bounded above sequence hence by monotonics convergence theorem, it is convergent.

⇒ $\sum U_n$ is convergent

Proof 2. when $\{U_n\}$ is divergent,

$$t_n \leq s_n$$

$$\lim_{n\to\infty} t_n \longrightarrow \infty$$

⇒ s_n is not bounded abouve.
⇒ $\sum V_n$ is divergent.

Comparision test (limit form)

If $\sum U_n, \sum V_n$ is a positive term series such that $\lim_{n\to\infty} \frac{U_n}{V_n} = L(\neq 0)$ then $\sum U_n$ and $\sum V_n$ both converges or diverges together.

Ex: $\sum U_n = \sum \frac{1}{n}$ is divergent while $\sum V_n = \sum \frac{1}{n^2}$ is convergent.

$$\lim_{n\to\infty} \frac{U_n}{V_n} = \lim_{n\to\infty} \frac{1}{n^2} \times n = 0$$

- Assume $\sum U_n, \sum V_n$ such that $\lim_{n\to\infty} \frac{U_n}{V_n} = L(\neq 0)$.

⇒ for $\epsilon > 0, \quad \exists \quad n_1$ such that

$$\left|\frac{U_n}{V_n} - L\right| < \epsilon \quad \forall \quad n \geq n_1$$

$$\Rightarrow L - \epsilon < \frac{U_n}{V_n} < L + \epsilon \quad \forall \quad n \geq n_1$$

$$\Rightarrow (L - \epsilon)V_n < U_n \quad \forall \quad n \geq n_1$$

if $\sum U_n$ is convergent, by comparision test, $\sum V_n$ is also convergent.

if $\sum U_n$ is divergent we use the condition $\frac{U_n}{V_n} < L + \epsilon$ to obtain

$$U_n < V_n(L + \epsilon) \quad \forall \quad n \geq n_1$$

By comparision test, $\sum V_n$ is also divergent.

Similarly we can proove that if $\sum V_n$ is convergent, $\sum U_n$ is also convergent by using the condition $U_n < V_n(L + \epsilon)$ for comparision test.

#semester-1 #mathematics #real-analysis