Lecture 2

Bounded above sequence / Upper bound

$\{X(n)\}_{n=1}^{\infty}$ is said to be bounded above/ having upper bound if there exists some $M \in \mathbb{R}$ such that $x_n \leq M$.

Ex: $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ = {$1, \frac{1}{2}, \frac{1}{3}, \cdots$}

$$x_n \leq M, \quad M \in (1, \infty)$$

⇒ This is bounded above sequence with upper bound $M \in (1, \infty)$.

Ex: $\{n^2\}_{n=1}^{\infty}$ = {$1, 2^2, 3^2, \cdots$ is not bounded above} as $x_n \nleq M$ for $M \in \mathbb{R}$.

Bounded below sequence / lower bound

If there exists $m \in \mathbb{R} \quad \forall \quad n \geq n_1$ such that $m < x_n$ then the sequence $\{X(n)\}_{n=1}^{\infty}$ is said to be bounded below sequence.

Ex: $\{n\}_{n=1}^{\infty}$ is not bounded above sequence but we can choose $m \in (-\infty, 0)$ such that $m < x_n$.
⇒ it is a bounded below sequence.

Bounded sequence

If a sequence has both upper bound as well as lower bound then the sequence is called as a bounded sequence. Mathematically, for a bounded sequence, there exists m such that $|x_n| < m \quad \forall \quad n$.

Least upper bound (lub)

$$lub = inf/min\{M_i\}$$

where $M_i \rightarrow$ upper bound.

Ex: $\left\{\frac{1}{n}\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

$$M_i \in [1, \infty)$$

$$\Rightarrow lub\left\{\frac{1}{n}\right\} = 1$$

- Upper bound is not unique but least upper bound is always unique.

Greatest lower bound (glb)

$$glb = sup/max\{m_i\}$$

where $m_i \rightarrow$ lower bound.

Ex: $\left\{\frac{1}{n}\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

$$m_i \in (-\infty, 0]$$

$$\Rightarrow glb\left\{\frac{1}{n}\right\} = 0$$

- Lower bound is not unique but greatest lower bound is always unique.

> **Theorem.** Every convergent sequence is bounded sequence but converse need not be true.

Monotonically increasing sequence

A sequence is monotonically increasing if $x_n \leq x_{n+1} \quad \forall \quad n$.

Ex: $\left\{n\right\} = \{1, 2, 3, \cdots\}$ is monotonically increasing sequence as $x_n \leq x_{n+1}$ for all n.

Monotonically decreasing sequence

A sequence is monotonically decreasing sequence if

$$x_n \geq x_{n+1} \quad \forall \quad n$$

Ex: $\left\{\frac{1}{n}\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ is monotonically decreasing sequence.

- Constant sequence is both monotonically increasing as well as monotonically decreasing sequence.

Strictly monotonically increasing sequence

A sequence is strictly monotonically increasing sequence if

$$x_n < x_{n+1} \quad \forall \quad n$$

Strictly monotonically decreasing sequence

A sequence is monotonically decreasing sequence if

$$x_n > x_{n+1} \quad \forall \quad n$$

Monotonic convergence theorem

> 1. Monotonically increasing sequence which is bounded above is convergent.

Ex: $\left\{x_n = 1 - \frac{1}{n}\right\} = \{0, \frac{1}{2}, \frac{2}{3}, \cdots\}$

$$\lim_{n \to \infty} 1 - \frac{1}{n} = 1$$

This is convergent to lub.

> 2. Monotonically decreasing sequence which is bounded below is convergent.

Ex: $\left\{\frac{1}{n}\right\} = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

This is convergent to glb.

Subsequence

If $\{x_n\}_{n=1}^{\infty}$ is an infinite sequence then $\{b_i\}$ is a subsequence of $\{x_n\}$ if there exists an increasing sequence $\{n_i\}$ of $\mathbb{N}$ such that $\left\{b_i = x_{n_i}\right\}$.

Ex: $\left\{x_n = \frac{1}{n}\right\}_{n=1}^{\infty}$ and $\{n_i\} = 1, 4, 5, 7, \cdots$

Then we obtain subsequence $\{b_i\} = \left\{1, \frac{1}{4}, \frac{1}{5}, \cdots\right\}$.

> **Theorem.** Every subsequence of convergent sequence is convergent but converse need not be true.

$\{x_n\} \longrightarrow$ convergent then $\{x_{n_k}\}$ is also convergent.

Squeeze principle / Sandwich theorem

For sequences $\{a_n\}, \{b_n\}, \{c_n\}$ if

$$a_n \le b_n \le c_n$$

$$\&$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$$

then $\lim_{n\to\infty} b_n$ also equals L.

Thus if a_n and c_n converges to a same value then b_n also converges to that same value.

Cauchy sequence

$\{x_n\}_{n=1}^{\infty}$ is said to be a cauchy sequence if $\forall \quad \epsilon > 0, \quad \exists \quad n_1 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon \quad \forall \quad n, m \geq n_1$.

Ex: $\left\{\frac{1}{n}\right\}$
we can choose $\epsilon = 0.1$ and $n_1 = 2$. So for $n \geq 3, m \geq 4$,

$$x_n - x_m = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} < \epsilon$$

⇒ $\left\{\frac{1}{n}\right\}$ is a cauchy sequence.

> **Theorem.** Every convergent sequence is a cauchy sequence but converse need not be true.

#semester-1 #mathematics #real-analysis