Lecture 9

Quadratic form

> Any homogeneous polynomial of degree 2 in 'n' variable is called quadratic form.

$f(x_1, x_2, x_3) =$

$$a_1x_1 + a_2x_2 + a_3x_3 = d_1$$

$$b_1x_1 + b_2x_2 + b_3x_3 = d_2$$

$$c_1x_1 + c_2x_2 + c_3x_3 = d_3$$

is called system of linear equations which can be represented in the matrix form as AX = c where, $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that (x_1, x_2, x_3) maps to (c_1, c_2, c_3).

Thus if $X \in \mathbb{R}^2$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ then,

$$Q = a_1x_1^2 + a_2x_2^2 + a_3x_1x_2$$

is a quadratic polynomial in x_1, x_2.

if $X \in \mathbb{R}^3$,

$$Q = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{13}x_1x_3 + a_{23}x_2x_3 + a_{12}x_1x_2$$

is quadratic form in three variables.

Thus if $X \in \mathbb{R}^n$,

$$Q = a_{11}x_1^2 + a_{22}x_2^2 + \cdots + a_{nn}x_n^2 + a_{12}x_1x_2 + \cdots + a_{n1}x_nx_1 + \cdots$$

$$\Rightarrow Q = \sum_{i=1, j=1}^{n} a_{ij}x_ix_j$$

is called quadratic form in n variables.

Classification of the quadratic form

1. **Positive definite quadratic form :** (Q > 0 $\forall$ x $\neq$ 0, Q = 0 iff X = 0)
2. **Negative definite quadratic form :** (Q < 0 $\forall$ x $\neq$ 0, Q = 0 iff X = 0)
3. **Positive semi-definite quadratic form :** (Q $\geq$ 0, but there exists X $\neq$ 0 which makes Q = 0)
4. **Negative semi-definite quadratic form :** (Q $\leq$ 0, but there exists X $\neq$ 0 which makes Q = 0)
5. **Indefinite quadratic form :** ($Q \in \mathbb{R}$)

Quadratic form in matrix notation

If $X \in \mathbb{R}^2$,

$$[x \quad y] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= [a_{11}x + a_{21}y \quad a_{12}x + a_{22}y] \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= a_{11}x^2 + a_{22}y^2 + a_{21}yx + a_{12}xy$$

Thus the quadratic form $Q = a_{11}x^2 + a_{22}y^2 + a_{21}yx + a_{12}xy$ can be represented as $[x \quad y] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ in the matrix form.

Ex: $Q = 2x^2 + 3y^2 + 5z^2 + 2xy + 4yz + 2zx$

$$\Rightarrow [x \quad y \quad z] \begin{bmatrix} 2 & 2 & 0 \\ 0 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- We can also write the quadratic form as

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j = X^TAX$$

$$= [x_1 \quad x_2 \quad \cdots \quad x_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- We always write $Q = X^TAX$, with A as a symmetric matrix. So,

$$Q = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}x_ix_j = X^TBX$$

$$= X^T \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix} X$$

where $b_{ij} = \frac{a_{ij}+a_{ji}}{2}$ such that B becomes symmetric matrix.

Ex: If $Q = 2x^2 + 4y^2 - 8z^2 - 2yz + 8zx - 3xy$

$$Q = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -3/2 & 4 \\ -3/2 & 4 & -1 \\ 4 & -1 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X^T A X$$

where $A^T = A$.

#semester-1 #mathematics #matrices