Lecture 7

Theorem. If A & B are similar matrices, they will have same eigenvalues.
Proof. Since A & B are similar matrices, $B = P^{-1}AP$.

$$|B - \lambda I|$$
$$= |P^{-1}AP - \lambda I|$$
$$= |P^{-1}AP - \lambda P^{-1}IP|$$
$$= |P^{-1}(A - \lambda I)P|$$
$$= |P^{-1}||A - \lambda I||P|$$
$$\Rightarrow |B - \lambda I| = |A - \lambda I|$$

Hence A & B have the same characteristic equation hence the same eigen values.

- If A is similar to diagonal matrix D then, A is said to be *diagonalizable*.

  $$P^{-1}AP = D$$

  where P is called *model matrix* and $P^{-1}AP$ is called *similarity transformation*.

Ex: Verify if $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable.

$$|A - \lambda I| = 0$$
$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$
$$= (\lambda - 1)(\lambda - 2)^2 = 0$$

Thus we get $\lambda = 1, 2, 2$.

for $\lambda = 1, \longrightarrow X_1$.
for $\lambda = 2, \longrightarrow X_2$.

X_1 & X_2 are linearly independent eigen vectors but will the third eigenvalue (which has repeated roots) give linearly independent eigenvectors?

To find this we can use this property: If λ is an eigenvalues of multiplicity m of a square matrix A of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r$$

where, r = rank$(A - \lambda I)$.

$$r = \text{rank}(A - 2 \cdot I)$$

$$A - 2 \cdot I = \begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{pmatrix}$$

$R_3 \rightarrow R_3 - R_1$

$$\begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{pmatrix}$$

$R_3 \rightarrow R_3 + 2R_2$

$$\begin{pmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

⇒ rank r = 2.
⇒ p = 3 - 2 = 1. So we only have 1 linearly independent eigenvector for $\lambda = 2$.
⇒ A is not diagonalizable.

Ex: $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

Solving this we get the characteristic equation $P_3(\lambda) = \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$ and thus eigenvalues $\lambda = 5, -3, -3$.

for $\lambda = 5, \longrightarrow X_1$.
for $\lambda = -3, \longrightarrow X_2$.

X_1 & X_2 are linearly independent eigen vectors but we need to check if the repeated eigenvalue $\lambda = -3$ will give some other linearly independent eigenvector.

when $\lambda = 5$,

$$(A - 5 \times I)X = \begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} X = 0$$

Solving using Gauss elimination, we get $X_1 = \begin{bmatrix} -k \\ -2k \\ k \end{bmatrix} = k \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

when $\lambda = -3$,

$$(A - 3 \times I)X = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} X = 0$$

Observe that all the equations are scalar multiples of the equation $x_1 + 2x_2 - 3x_3 = 0$. This is like 1 equation with 3 unknowns.

Thus $X_2 = \begin{bmatrix} -2k_2 + 3k_3 \\ k_2 \\ k_3 \end{bmatrix}$. Taking $k_3 = 0, k_2 = 1$ gives

$$X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

and if we take $k_2 = 0, k_3 = 1$, we get

$$X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

We get 3 independent eigen vectors X_1, X_2, X_3.

⇒ We can construct $P = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$ and the matrix A is diagonalizable.

$$P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We can now find $D = P^{-1}AP$ which will have $a_{ij} = 0 \forall i \neq j$ and diagonal elements equal to the eigen values of A.

- Even this process of diagonalizing a matrix is time consuming since finding P^{-1} is difficult. This is easier when A is real symmetric matrix.

  proof. We know that for a real symmetrix matrix, all eigenvalues are real and if $\lambda_i \neq \lambda_j$, then corresponding eigenvectors are orthogonal ($X_i^T X_j = 0$).

  For orthogonal matrix A, $A^T A = I \Rightarrow A^{-1} = A^T$. So if we can make P to be an orthogonal matrix, finding its inverse would become very easy.

Thus when A is a real symmetric matrix of order n and having n linearly independent eigen vectors then it is always possible to construct an invertible matrix P that is orthogonal.

So if $P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ X_1 & X_2 & \cdots & X_3 \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$, is it orthogonal?

It may not be. To make it orthogonal we can take its coloums as *normalised eigenvectors* (dividing eigenvectors by its length)

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \frac{X_1}{|X_1|} & \frac{X_2}{|X_2|} & \cdots & \frac{X_n}{|X_n|} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

This vector P is always orthogonal in case of A to be a real symmetric matrix.

#semester-1 #mathematics #matrices