Lecture 6

Theorem. A square matrix A with all different eigen values may be diagonalised by a similarity transformation.

$$D = P^{-1}AP$$

where P is the matrix whose columns are eigen vectors of A. The diagonal matrix D has its diagonal elements as the eigen values of A.

$$or$$

A square matrix A of order n is diagonalizable iff it has n linearly independent eigen vectors.

Lets take A, B such that $P^{-1}AP = B$ ⇒ $AP = PB$ where P is a matrix whose columns are eigen vectors of A.

Matrix A has 'n' linearly independent eigen vectors $\{X_1, X_2, \cdots, X_n\}$.

$$P = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ X_1 & X_2 & \cdots & X_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$AP = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ X_1 & X_2 & \cdots & X_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ AX_1 & AX_2 & \cdots & AX_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

Using $AX_i = \lambda_i X_i$,

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \lambda_1 X_1 & \lambda_2 X_2 & \cdots & \lambda_n X_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$$

$$= \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ X_1 & X_2 & \cdots & X_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$= P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

⇒ AP = PD or $P^{-1}AP = D$, where A, D are similar matrix.

$$\Rightarrow P^{-1}A^nP = D^n$$

We can find A^n easily by $A^n = PD^nP^{-1}$. We can also find

$$P(A) = A^n + a_1A^{n-1} + \cdots + a_nI$$

by using $A^n = PD^nP^{-1}$.

$$= PD^nP^{-1} + a_1PD^{n-1}P^{-1} + \cdots + a_nPIP^{-1}$$

$$= P(D^nP^{-1} + a_1D^{n-1}P^{-1} + \cdots + a_nIP^{-1})$$

$$= P(D^n + a_1D^{n-1} + \cdots + a_nI)P^{-1}$$

$$= P(P(D))P^{-1}$$

This is better as calculating P(D) is easier as compared to P(A).

Ex: Let $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & -1 \\ -2 & 1-\lambda & 2 \\ 0 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)[(1-\lambda)(2-\lambda) - 2] + 2[(2-\lambda) + 1] = 0$$

$$= (3-\lambda)[\lambda^2 - 3\lambda + 2] = 0$$

We get $\lambda = 1, 2, 3$.

When $\lambda = 1$,

$$(A - 1 \times I)X = 0$$

$$\begin{pmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} X = 0$$

Use Gauss elimination (eliminate a_{31}, a_{21} & then a_{32}) (use elementary row operations: $R_2 \to R_2 + R_1$ and then $R_3 \to R_3 - R_2$) to get,

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} X = 0$$

we get, $x_3 = k$, $2x_1 + x_2 - x_3 = 0$ & $x_2 + x_3 = 0$.

$$\Rightarrow X_1 = k \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

When $\lambda = 2$,

$$(A - 2 \times I) = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{pmatrix} X = 0$$

Using $R_2 \to R_2 + 2R_1$ and $R_3 \to R_3 - R_2$, we get

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} X = 0$$

$$\Rightarrow x_3 = k, x_2 = 0$$

$$x_1 + x_2 - x_3 = 0$$

Solving this we obtain $X_2 = k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

When $\lambda = 3$,

$$(A - 3 \times I) = 0$$

$$\begin{pmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{pmatrix} X = 0$$

we get $x_3 = k, x_2 - x_3 = 0$ & $-2x_1 - 2x_2 + 2x_3 = 0$

$$\Rightarrow x_3 = k, x_2 = k, x_1 = 0$$

$$X_3 = k \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Now we verify AP = PD to get the diagonal matrix D.

$$D = P^{-1}AP$$

$$P^{-1} = \frac{Adj(P)}{|P|} = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Thus we get correct matrix D as the diagonal elements are eigen values and non-diagonal elements are zero.

#semester-1 #mathematics #matrices