Lecture 5

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

$A_{n\times n}$, $P_n(\lambda) = \lambda^2 + a_1\lambda^{n-1} + \cdots + a_n = 0$ According to *cayley-hamilton theorem*,

$$A^n + a_1A^{n-1} + \cdots + a_nI = 0$$

Example: $A_{2\times 2}$, $P_2(\lambda) = \lambda^2 + a_1\lambda + a_2 = 0.$

$$A^2 + a_1A + a_2 = 0$$

Using this theorem, we can find Inverse of A or higher powers of A.

for A^3, premultiply by A

$$A^3 + a_1A^2 + a_2A = 0$$

$$\Rightarrow A^3 = -a_1A^2 - a_2A$$

for A^{-1}, premultiply by A^{-1}

$$A + a_1I + a_2A^{-1} = 0$$

$$A^{-1} = \frac{1}{a_2}[-A - a_1I]$$

provided that $a_2 \neq 0.$

We can also calculate higher powers of A

$$\lambda^n = (\lambda^2 + a_1\lambda + a_2)Q_{n-2}(\lambda) + a\lambda + b$$

$$A^n = (A^2 + a_1A + a_2)Q(A) + aA + b$$

$$\Rightarrow A^n = aA + b$$

$$\lambda = \lambda_1, \qquad \lambda_1^n = 0 + a\lambda_1 + b \qquad \cdots \qquad (1)$$

$$\lambda = \lambda_2, \qquad \lambda_2^n = 0 + a\lambda_2 + b \qquad \cdots \qquad (2)$$

Solve eq^ns (1) & (2) to get a, b and finally A^n.

Verifying Cayley - Hamilton Theorem

$A = \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 7-\lambda & 3 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(7-\lambda)(6-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - 13\lambda + 36 = 0 \qquad \cdots \qquad (1)$$

Solving the characteristic equation we the the eigenvalues $\lambda = 4, 9$.

According to cayley - hamilton theorem,

$$A^2 - 13A + 36 = 0$$

$$= \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} - 13 \begin{bmatrix} 7 & 3 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix}$$

$$\begin{bmatrix} 55 & 39 \\ 26 & 42 \end{bmatrix} - \begin{bmatrix} 91 & 39 \\ 26 & 78 \end{bmatrix} + \begin{bmatrix} 36 & 0 \\ 0 & 36 \end{bmatrix} = 0$$

⇒ $A^2 - 13A + 36 = 0$.

To find A^3,

$$A^3 - 13A^2 + 36I = 0$$

$$\Rightarrow A^3 = 13A^2 - 36A$$

To find A^{-1},

$$A - 13I + 36A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{36}[13I - A]$$

for A^n, $\lambda^n = (\lambda^2 - 13\lambda + 36)a_{n-2}(\lambda) + aA + b$
put $\lambda = 4, 9$

$$4^n = 4a + b \qquad \cdots \qquad (2)$$

$$9^n = 9a + b \qquad \cdots \qquad (3)$$

Solving equations (1) & (2), we get

$$a = \frac{9^n - 4^n}{5}$$

$$b = \frac{9 \cdot 4^n - 4 \cdot 9^n}{5}$$

$$\Rightarrow A^n = \left[\frac{9^n - 4^n}{5}\right] A + \left[\frac{9 \cdot 4^n - 4 \cdot 9^n}{5}\right]$$

Similar Matrix

Matrices A & B are called similar if there exists some invertible matrix P such that $A = P^{-1}BP$ ⇒ $PA = BP$.

#semester-1 #mathematics #matrices