Lecture 4

Properties of Eigenvalues & Eigenvectors cont...

8. The eigenvectors corresponding to distinct eigen value are linearly independent.
9. All eigenvalues of a real symmetric matrix are real.
10. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

> Vectors X_1, X_2 are said to be orthogonal in n - dimensional vector space if $X_1^T X_2 = 0$ (works like dot product)

Proofs:

property 8.

Assume that $\{X_1, X_2\}$ be linearly dependent vectors with eogenvalues λ_1, λ_2 respectively and $\lambda_1 \neq \lambda_2$

Therefore, $c_1 X_1 + c_2 X_2 = 0$ ⇒ $X_2 = \frac{-c_1}{c_2} X_1 = k X_1$ ofr some scalar k.

$$AX_2 = \lambda_2 X_2$$

$$A(cX_1) = \lambda_2 X_2$$

$$cAX_1 = \lambda_2 X_2$$

$$c\lambda_1 X_1 = \lambda_2 X_2$$

$$\Rightarrow \lambda_1 X_2 = \lambda_2 X_2$$

But $X_2 \neq 0$ ⇒ $\lambda_1 = \lambda_2$. This contradicts our assumption. Thus eigen vectors $\{X_1, X_2\}$ are linearly independent.

property 9.

If A is a symmetric real matrix (⇒ $A = A^T$ & $A = \overline{A}$)

$$AX = \lambda X$$

pre-multiplying with $\bar{X^T}$ both the sides.

$$\overline{X^T} AX = \overline{X^T} \lambda X \qquad \cdots \qquad (1)$$

taking conjugate on both sides,

$$\overline{\overline{X^T}AX} = \overline{\lambda\overline{X^T}X}$$

$$\overline{\overline{X^T}}\,\overline{AX} = \overline{\lambda}\,\overline{\overline{X^T}}\,\overline{X}$$

$$X^T A\overline{X} = \overline{\lambda}X^T\overline{X}$$

take transpose both sides,

$$\overline{X^T}A^T X = \overline{\lambda}\overline{X^T}X$$

But since $A = A^T$,

$$\overline{X^T}AX = \overline{\lambda}\overline{X^T}X \qquad \cdots \qquad (2)$$

Equating (1) & (2),

$$\overline{X^T}\lambda X = \overline{\lambda}\overline{X^T}X$$

$$\Rightarrow (\lambda - \overline{\lambda})(\overline{X}X) = 0$$

if we take $X = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ then $\overline{X^T} = [\overline{z_1} \quad \overline{z_2}]$.

⇒ $\overline{X^T}X = \overline{z_1}z_1 + \overline{z_2}z_2 = |z_1|^2 + |z_2|^2 \neq 0$
⇒ $\lambda = \overline{\lambda}$ we represents that all eigen values of real symmetric matrix are real.

property 10.

$$AX_1 = \lambda_1 X_1$$

pre-multiplying with X_2^T,

$$X_2^T A X_1 = X_2^T \lambda_1 X_1$$

take transpose,

$$(X_2^T A X_1)^T = (\lambda_1 X_2^T X_1)^T$$

$$X_1^T A^T X_2 = \lambda_1 X_1^T X_2$$

Using the fact that A is symmetric ($A = A^T$),

$$\Rightarrow X_1^T A X_2 = \lambda_1 X_1^T X_2 \qquad \cdots \qquad (1)$$

Similarly using $AX_2 = \lambda_2 X_2$ we get,

$$X_1^T A X_2 = \lambda_2 X_1^T X_2 \qquad \cdots \qquad (2)$$

Equating (1) & (2)

$$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$$

But since $\lambda_1 \neq \lambda_2$, $X_1^T X_2 = 0$.

⇒ X_1 is *orthogonal* to X_2.

Q. If $A = I_{3\times 3}$,

$$|A - \lambda I| = 0$$

$$\Rightarrow (1-\lambda)^3 = 0 \Rightarrow \lambda = 1, 1, 1$$

How many linearly independent eigenvectors do we get?

Ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda = 1, 1, 1$$

When $\lambda = 1$,

$$(A - 1 \cdot I)X = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

We get $x_2 = 0$ & $x_3 = 0$ ⇒ $X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$.

We cannot find other non-zero eigen vectors which are linearly independent with $X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$.

Ex: $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$|A - \lambda I| = 0$$

$$\Rightarrow \lambda = 1, 1, 1$$

When $\lambda = 1$,

$$(A - 1 \cdot I)X = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

We get $x_1 = k_1$, $x_2 = 0$ & $x_3 = k_3$
⇒ $X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ when $k_1 = 1$ & $k_3 = 0$ and $X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ when $k_1 = 0$ & $k_3 = 1$
Thus we get two linearly independent vectors for $\lambda = 1$.

Property: If λ is an eigenvalues of multiplicity m of a square matrix A of order n, then the number of linearly independent eigenvectors associated with λ is given by

$$p = n - r$$

where, r = rank$(A - \lambda I)$.

#semester-1 #mathematics #matrices