# **Properties of Eigenvalues & Eigenvectors cont...**

- 8. The eigenvectors corresponding to distinct eigen value are linearly independent.
- 9. All eigenvalues of a real symmetric matrix are real.
- 10. The eigen vectors corresponding to distinct eigen values of a real symmetric matrix are orthogonal.

Vectors  $X_1, X_2$  are said to be orthogonal in n - dimensional vector space if  $X_1^T X_2 = 0$  (works like dot product)

# **Proofs:**

## **property 8.**

Assume that  $\{X_1, X_2\}$  be linearly dependent vectors with eogenvalues  $\lambda_1, \lambda_2$  respectively and  $\lambda_1 \neq \lambda_2$ **Lecture 4.**<br> **Properties of Eigenvalues & Eigenvectors cont...**<br>
8. The eigenvectors corresponding to distinct eigen value are linearly independent.<br>
9. All eigenvalues of a real symmetric matrix are real.<br>
10. The eigen **Lecture 4**<br> **Properties of Eigenvalues & Eigenvectors cont...**<br>
8. The eigenvectors corresponding to distinct edgen value are finedly independent.<br>
10. The eigenvectors corresponding to distinct edgen value are finedly i

Therefore,  $c_1X_1 + c_2X_2 = 0 \Rightarrow X_2 = \frac{-c_1}{c_2}X_1 = kX_1$  ofr some scalar k.

$$
AX_2 = \lambda_2 X_2
$$
  
\n
$$
A(cX_1) = \lambda_2 X_2
$$
  
\n
$$
cAX_1 = \lambda_2 X_2
$$
  
\n
$$
c\lambda_1 X_1 = \lambda_2 X_2
$$
  
\n
$$
\Rightarrow \lambda_1 X_2 = \lambda_2 X_2
$$

But  $X_2 \neq 0 \Rightarrow \lambda_1 = \lambda_2$ . This contradicts our assumption. Thus eigen vectors  $\{X_1, X_2\}$  are linearly independent.

### **property 9.**

If  $A$  is a symmetric real matrix ( $\Rightarrow A = A^T$  &  $A = \overline{A}$  )

$$
AX=\lambda X
$$

pre-multiplying with  $\bar{X^T}$  both the sides.

$$
\overline{X^T}AX = \overline{X^T}\lambda X \qquad \cdots \qquad (1)
$$

taking conjugate on both sides,

taking conjugate on both sides,  
\n
$$
\frac{\overline{\overline{X}^T A X}}{\overline{X}^T A X} = \overline{\lambda \overline{X}^T X}
$$
\n
$$
\frac{\overline{\overline{X}^T A X}}{\overline{X}^T A X} = \overline{\lambda \overline{X}^T X}
$$
\ntake transpose both sides,  
\n
$$
\overline{X^T A^T X} = \overline{\lambda \overline{X}^T X}
$$
\nBut since  $A = A^T$ ,  
\n
$$
\overline{X^T A X} = \overline{\lambda \overline{X}^T X} \qquad \cdots \qquad (2)
$$
\nEquating (1) & (2),  
\n
$$
\overline{X^T \lambda X} = \overline{\lambda \overline{X}^T X}
$$

take transpose both sides,

$$
\overline{X^T}A^TX=\overline{\lambda}\overline{X^T}X
$$

But since  $A = A^T$ ,

$$
\overline{X^T}AX = \overline{\lambda}\overline{X^T}X \qquad \cdots \qquad (2)
$$

Equating (1) & (2),

$$
\overline{X^T} \lambda X = \overline{\lambda} \overline{X^T} X
$$

$$
\Rightarrow (\lambda - \overline{\lambda})(\overline{X}X) = 0
$$

if we take  $X = \begin{bmatrix} z_1 \ z \end{bmatrix}$  then  $\overline{X^T} = [\overline{z_1} \quad \overline{z_2}].$  $\overline{z}_2$  $\overline{X^T} = [\overline{z_1} \quad \overline{z_2}]$  .

$$
\Rightarrow \overline{X^T}X = \overline{z_1}z_1 + \overline{z_2}z_2 = |z_1|^2 + |z_2|^2 \neq 0
$$
  
\n
$$
\Rightarrow \lambda = \overline{\lambda}
$$
 we represents that all eigen values of real symmetric matrix are real.

#### **property 10.**

$$
AX_1=\lambda_1X_1
$$

pre-multiplying with  $X_2^T$ ,

$$
X_2^T A X_1 = X_2^T \lambda_1 X_1
$$

take transpose,

$$
(X_2^T A X_1)^T = (\lambda_1 X_2^T X_1)^T
$$

$$
X_1^T A^T X_2 = \lambda_1 X_1^T X_2
$$

Using the fact that A is symmetric  $(A = A<sup>T</sup>)$ ,

$$
\Rightarrow X_1^T A X_2 = \lambda_1 X_1^T X_2 \qquad \cdots \qquad (1)
$$

Similarly using  $AX_2 = \lambda_2 X_2$  we get,

$$
X_1^T A X_2 = \lambda_2 X_1^T X_2 \qquad \cdots \qquad (2)
$$

Equating (1) & (2)

$$
(\lambda_1-\lambda_2)X_1^TX_2=0
$$

But since  $\lambda_1 \neq \lambda_2$ ,  $X_1^T X_2 = 0$ .

 $\Rightarrow$   $X_1$  is *orthogonal* to  $X_2$ .

Q. If  $A = I_{3 \times 3}$ 

$$
|A - \lambda I| = 0
$$
  

$$
\Rightarrow (1 - \lambda)^3 = 0 \Rightarrow \lambda = 1, 1, 1
$$

How many linearly independent eigenvectors do we get?

Ex:  $A = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ .  $\|$  $1 \quad 1 \quad 0$ 0 1 1  $0 \quad 0 \quad 1$  $\mathbf{L}$  $|A - \lambda I| = 0$  $\Rightarrow \lambda = 1, 1, 1$ 

When  $\lambda = 1$ ,

$$
(A - 1 \cdot I)X = 0
$$
  
\n
$$
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0
$$
  
\n= 0 &  $x_3 = 0 \Rightarrow X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$ .  
\nfind other non-zero eigen vectors which are l  
\n
$$
\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$
  
\n
$$
|A - \lambda I| = 0
$$
  
\n
$$
\Rightarrow \lambda = 1, 1, 1
$$
  
\n
$$
(A - 1 \cdot I)X = 0
$$

We get  $x_2 = 0$  &  $x_3 = 0 \Rightarrow X = \begin{bmatrix} 0 \end{bmatrix}$ .  $\overline{0}$ 

We cannot find other non-zero eigen vectors which are linearly independent with  $\sqrt{k}$ 

$$
\begin{aligned}\n\begin{bmatrix}\n0 & 0 & 1\n\end{bmatrix} & |A - \lambda I| &= 0 \\
\Rightarrow \lambda &= 1, 1, 1\n\end{aligned}
$$
\nWhen  $\lambda = 1$ ,

\n
$$
(A - 1 \cdot I)X =
$$
\n
$$
\begin{pmatrix}\n0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0\n\end{pmatrix}\n\begin{pmatrix}\nx_1 \\
x_2 \\
x_3\n\end{pmatrix}
$$
\nWe get  $x_2 = 0$  &  $x_3 = 0 \Rightarrow X = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$ .

\nWe cannot find other non-zero eigen vectors which

\n
$$
X = \begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}.
$$
\nEx:  $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$ 

\n
$$
|A - \lambda I| = 0
$$
\n
$$
\Rightarrow \lambda = 1, 1, 1
$$
\nWhen  $\lambda = 1$ ,

\n
$$
(A - 1 \cdot I)X =
$$

When  $\lambda = 1$ ,

$$
(A-1\cdot I)X=0
$$

$$
\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0
$$

We get  $x_1 = k_1$ ,  $x_2 = 0$  &  $x_3 = k_3$  $\Rightarrow X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  when  $k_1 = 1$  &  $k_3 = 0$  and  $X_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$  when  $k_1 = 0$  & ∖(annen mundernet)<br>annen mundernet / ( ( ( rs ty = n ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | ( ) | (  $\binom{0}{0}$  o  $\binom{1}{1}$  fo m ass  $-$ /<br>
wh<br>
a ciε<br>
α ciε  $\begin{bmatrix} 1 \ 0 \ 0 \end{bmatrix}$ when  $k_1 = 1$  &  $k_3 = 0$  and  $X_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  $\begin{bmatrix} 0 \ 0 \ 1 \end{bmatrix}$ when  $k_1 = 0$  &  $k_3 = 1$ 

Thus we get two linearly independent vectors for  $\lambda = 1$ .

**Property:** If  $\lambda$  is an eigenvalues of multiplicity m of a square matrix A of order n, then the number of linearly independent eigenvectors associated with  $\lambda$  is given by  $\chi$ :  $\frac{1}{2}$ <br> $\chi$ :  $\frac{1}{2}$ <br> $\chi$ :  $\frac{1}{2}$ /<br>et t<br>line<br>rar ──1∃  $\setminus$ l is f<br>y n<br>a a  $n$ )<br>0 i 0<br>∴sc − r

 $p = n - r$ 

where,  $r = rank(A - \lambda I)$ .

#semester-1 #mathematics #matrices