Lecture 3

- If X is an eigen vector corresponding to λ, $AX = \lambda X$, How will cX transform?

$$A(cX) = c(AX) = c(\lambda X) = \lambda(cX)$$

  ⇒ If X is eigenvector corresponding to λ then cX will also be an eigenvector corresponding to the same λ.

  Thus eigen vector corresponding to an eigen value is **not unique**.

- If let say X_1 is an eigen vector corresponding to λ_1 and X_2 is another eigen vector corresponding to λ_2, $\lambda_1 \neq \lambda_2$ ⇒ X_2 cannnot be eigen vector corresponding to λ_1.

  Proof: Suppose X_2 is an eigen vector corresponding to λ_1.

$$\Rightarrow AX_2 = \lambda_1 X_2$$

  also,

$$AX_2 = \lambda_2 X_2$$

  ⇒ $\lambda_1 X_2 = \lambda_2 X_2$. But since $X \neq 0$, we get $\lambda_1 = \lambda_2$ which is not true. Thus our assumption is wrong. X_2 cannnot be eigen vector corresponding to λ_1.

  Ex: $X \neq 0$, $\rightarrow \lambda X$ where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$.
  To get eigen values, $|A - \lambda I| = 0$

  ⇒ $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = 0$
  Solving the determinant we get $\lambda = 1, 2, 3$
  When $\lambda = 1$,

$$(A - 1 \cdot I)X = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} X = 0$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

We get the equations $x_2 + x_3 = 0$ & $2x_1 + 2x_3 = 0$. Thus we get infinite eigen vectors of the form $X = \begin{pmatrix} -k \\ -k \\ k \end{pmatrix}$.

When $\lambda = 2$,

$$(A - 2 \cdot I)X = 0$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix} X = 0$$

We get the equations $-x_1 = 0$ & $x_3 = 0$ & $2x_1 + x_3 = 0$. Thus we get infinite eigen vectors of the form $X = \begin{pmatrix} 0 \\ k \\ 0 \end{pmatrix}$.

When $\lambda = 3$,

$$(A - 3 \cdot I)X = 0$$

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{pmatrix} X = 0$$

We get the equations $-2x_1 = 0$ & $-x_2 + x_3 = 0$ & $2x_1 = 0$. Thus we get infinite eigen vectors of the form $X = \begin{pmatrix} 0 \\ k \\ k \end{pmatrix}$.

Properties of Eigenvalues & Eigenvectors

1. If λ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.
2. If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA.
3. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A then $\lambda_1^p, \lambda_2^p, \cdots, \lambda_n^p$ will be eigen values of A^p.
4. If λ is an eigen value of A, them $\lambda - k$ will be an eigen value of $A - kI$.
5. A and A^T have the same eigen value.
6. Product of fthe eigen values is equal to |A|.
7. Sum of eigen values is equal to Trace(A).

Proofs:

property 1.

$$AX = \lambda X$$

$$A^{-1}(AX) = A^{-1}(\lambda X)$$

$$X = \lambda(A^{-1}X)$$

$$\frac{1}{\lambda}X = A^{-1}X$$

⇒ $\frac{1}{\lambda}$ is eigenvalue for A^{-1} provided that A^{-1} exists.

corollary: If any one of eigen value of A is zero, then A^{-1} does not exist.

property 3.

If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigen values of A.
⇒ $AX_i = \lambda_i X_i$ $\forall$ $i \in$ {1, 2, ... , n}.

$$A(AX_i) = A(\lambda_i X_i) = A^2 X_i = \lambda_i(AX_i) = \lambda_i^2 X_i$$

Therefore λ^2 will be eigenvalues for A^2. Hence in general λ_i^p are eigen values for A^p.

property 5.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

$|A^T - \lambda I| = 0$ will heve the same determinant hence same characteristic equation hence, same eigenvalue λ.

property 6. & 7.

$A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

$$|A - \lambda I| = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} = 0$$

$$(a_{11} - \lambda)(a_22 - \lambda) - a_21a_12 = 0$$

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{21}a_{12} = 0$$

suppose λ_1, λ_2 are roots of a second degree polynomial,

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = 0$$

Comparing the equations, we get sum of eigenvalues = Trace(A) & product of eigenvalues = |A|.

Linearly dependent & independent sets

If for a set $\{v_1, v_2, \cdots, v_n\}$, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$, $\alpha_i = 0 \; \forall$ i.
⇒ the set is linearly independent.

If for a set $\{v_1, v_2, \cdots, v_n\}$, $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$, but for some i, $\alpha_i \neq 0$.
⇒ v_i will be linearly dependent on other vectors & the set is linearly dependent.

Ex: Identify $\mathbb{R}^3$ → {(1, 0, 0), (0, 1, 0), (0, 0, 1)} as linearly dependent or independent set.

$$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (0,0,0)$$

$$\Rightarrow (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$$

$$\Rightarrow \alpha_i = 0 \forall i$$

This implies $\mathbb{R}^3$ is a linearly independent set.

#semester-1 #mathematics #matrices