Lecture 2

Mapping of a square matrix $A_{n,n} : \mathbb{R}^n \to \mathbb{R}^n$ is transformation of the shape in same dimensional vector space.

What if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and we need a_i, eigen values and non trivial solution of eigen vectors (X $\neq$ 0)?

For this we will solve AX = λX.

$$\Rightarrow AX - \lambda X = 0$$

$$\Rightarrow AX - \lambda IX = 0$$

$$\Rightarrow (A - \lambda I)X = 0$$

To get non-trivial solutions, |A - λI| = 0. Solving the determinant this gives us a polynomial in λ, $P_n(\lambda) = 0$

$$P_n(\lambda) = \lambda^n - c_1\lambda^{n-1} + \cdots (-1)^n c_n = 0$$

$P_n(\lambda) = 0$ is called the characteristic equation.

For example, $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}.$

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

$$|A - \lambda I| = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12} = 0$$

$$\Rightarrow P_2(\lambda) = \lambda^2 - (a_{11} + a_{22})\lambda - a_{21}a_{12} + a_{11}a_{22} = 0$$

Let say this characteristic equation has $\lambda_1, \lambda_2, \cdots, \lambda_n$ as roots of $P_n(\lambda) = 0$. So, all $\lambda_i \in \mathbb{R}$ will be considered as Eigenvalues.

All X_i for which $(A - \lambda_i I)X = 0$ has non-trivial solution will be called **eigenvectors** corresponding to **eigenvalues** λ_i.

Ex: $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and we need to find eigenvalues & eigenvectors.

Solution: $(A - \lambda I)X = 0$ so, $\begin{bmatrix} 5 - \lambda & 4 \\ 1 & 2 - \lambda \end{bmatrix} X = 0$

$$\Rightarrow |A - \lambda I| = \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(2-\lambda) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 6)(\lambda - 1) = 0$$

Thus eigenvalues are $\lambda = 1, 6$.
For $\lambda = 6$, $(A - 6I)X = 0$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} X = 0$$

From this we get the two equations $-x + 4y = 0$ and $x - 4y = 0$ which are one and the same. Thus we have infinite solutions for eigenvectors of the form X = (x, y) = (4k, k).
Similarly for $\lambda = 1$ we also have infinite solutions of the form (k, -k).

Spectrum & spectrum radius

- $\{\lambda_1, \lambda_2\}$ = {6, 1} is called *spectrum* of A. In other words, the set of eigenvalues of a square matrix A is called the spectrum.
- The spectral radius of a square matrix is the **largest absolute value of its eigenvalues**. $\rho(A)$ = max $\{|\lambda_i|, i = 1, 2, \cdots, n\}$ is called the *spectrum radius* of A.

#semester-1 #mathematics #matrices